Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 429 (2008) 331–345
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On the invertibility of "rectangular" bi-infinite matrices and applications in time–frequency analysis

Götz E. Pfander *

School of Engineering and Science, Jacobs University Bremen, 28759 Bremen, Germany

Received 9 May 2007; accepted 24 February 2008

Submitted by H. Schneider

Abstract

Finite dimensional matrices with more columns than rows have no left inverses while those with more rows than columns have no right inverses. We give generalizations of these simple facts to bi-infinite matrices. Our results are then used to obtain density results for p-frames of time–frequency molecules in modulation spaces and identifiability results for operators with bandlimited Kohn–Nirenberg symbols.

Keywords: Slanted/skew diagonal matrices; Gabor frames; Operator identification; Kohn–Nirenberg symbol; Spreading function

1. Introduction

Matrices in $\mathbb{C}^{m\times n}$ are not invertible if $m \neq n$. To generalize this basic fact from finite dimensional linear algebra to bi-infinite matrices, we first associate the quadratic shape $\square$ of $M \in \mathbb{C}^{n\times n}$ to bi-infinite matrices decaying away from their diagonals, more precisely, by matrices $M = (m_{j'j})_{j',j\in\mathbb{Z}^d}$ with $m_{j'j}$ small for $|\|j'\|_\infty - \|j\|_\infty|$ large. The rectangular shape $\sqsubset\!\sqsupset$ of $M \in \mathbb{C}^{m\times n}$, $m < n$, is then taken to correspond to bi-infinite matrices decaying off those wedges that are situated between two slanted diagonals of slope less than one and which are open "to the left and to the right". In short, for $\lambda > 1$ we assume $m_{j'j}$ small for $\lambda\|j'\|_\infty - \|j\|_\infty$ positive and large. In this case, we use the symbol $\bowtie$. Similarly, the case $\square$, that is, $M \in \mathbb{C}^{m\times n}$, $m > n$,

* Tel.: +49 421 2003211; fax: +49 421 200493211.
E-mail address: g.pfander@jacobs-university.de

0024-3795/$ - see front matter
doi:10.1016/j.laa.2008.02.025

corresponds to those bi-infinite matrices that are adjoints of the ⧓ matrices described above. That is, the bi-infinite case ⧗ is described by: for $\lambda < 1$ assume $m_{j'j}$ small for $-\lambda\|j'\|_\infty + \|j\|_\infty$ positive and large. In both cases, $\lambda \neq 1$ corresponds to $\frac{n}{m} \neq 1$ in the theory of finite dimensional matrices.

Throughout, we consider bi-infinite matrices that act on weighted l^p spaces, $1 \leqslant p \leqslant \infty$. To illustrate our main result we shall first state its simplest case.

Theorem 1.1. *Let* $M = (m_{j'j}) : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ *and let* $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *satisfy* $w(x) = \mathrm{o}(x^{-1-\delta})$, $\delta > 0$.

1. *If* $|m_{j'j}| < w(\lambda|j'| - |j|)$ *for* $\lambda|j'| - |j| > 0$ *and* $\lambda > 1$, *then* M *has no bounded left inverses.*
2. *If* $|m_{j'j}| < w(-\lambda|j'| + |j|)$ *for* $-\lambda|j'| + |j| > 0$ *and* $\lambda < 1$, *then* M *has no bounded right inverses.*

Note that slanted matrices as covered in [1] and in the wavelets literature [2,3,4,5] are defined through their decay off slanted diagonals, that is, $|m_{j',j}|$ is small for $\|\lambda j' - j\|_\infty$ large. Since $\|\lambda j' - j\|_\infty \geqslant |\,\lambda\|j'\|_\infty - \|j\|_\infty|$, our results in Section 2 apply to slanted matrices as well.

After stating and proving our main results as Theorems 2.1 and 2.2 in Section 2, we shall illustrate their usefulness in Section 3 by giving applications of these results in the area of time–frequency analysis. First, Theorem 2.2 is used to obtain elementary proofs of density theorems for Banach frames of time–frequency molecules, in particular of Gabor systems, in so-called modulation spaces [6,7]. Second, we discuss how Theorem 2.1 has been used to obtain necessary conditions on the identifiability of pseudodifferential operators that are characterized by a band-limitation of the operators' Kohn–Nirenberg symbols [8,9,10]. The background on time–frequency analysis that is used in Section 3 is given in Section 3.1.

2. Non-invertibility of "rectangular" bi-infinite matrices

Let $l_s^p(\mathbb{Z}^d)$, $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, be the weighted l^p-space with norm $\|\{x_j\}\|_{l_s^p} = \|\{(1 + \|j\|_\infty)^s x_j\}\|_{l^p}$, where $\|\{x_j\}\|_p = \left(\sum_j |x_j|^p\right)^{\frac{1}{p}}$ for $p < \infty$ and $\|\{x_j\}\|_\infty = \sup_j\{|x_j|\}$.

Theorem 2.1. *Let* $1 \leqslant p_1, q_1, p_2 \leqslant \infty$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, *and* $M = (m_{j'j}) : l^{p_1}(\mathbb{Z}^d) \to l^{p_2}(\mathbb{Z}^d)$. *If a function* $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *exists with* $w(x) = \mathrm{o}\left(x^{-(\frac{1}{q_1}+\frac{1}{p_2})d - r_1 - r_2 - \delta}\right)$ *and*

$$|m_{j'j}| \leqslant w(\lambda\|j'\|_\infty - \|j\|_\infty)(1 + \|j\|_\infty)^{r_1}(1 + \|j'\|_\infty)^{r_2}, \quad \lambda\|j'\|_\infty - \|j\|_\infty > K_0,$$

for some constants $\lambda, K_0, r_1, r_2, \delta$ *with* $\lambda, K_0 > 1$, $\delta \geqslant 0$, $r_1 + \delta > 0$, *and* $\frac{d}{p_2} + r_1 + r_2 + \delta > 0$, *then* M *has no bounded left inverses.*

Theorem 2.1 is easily generalized:

Theorem 2.2. *Let* $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$, $r_1, r_2, s_1, s_2 \in \mathbb{R}$, *and* $M = (m_{j'j}) : l_{s_1}^{p_1}(\mathbb{Z}^d) \to l_{s_2}^{p_2}(\mathbb{Z}^d)$.

1. *If there is a* $\delta \geqslant 0$ *with* $r_1 - s_1 + \delta > 0$ *and* $\frac{d}{p_2} + r_1 + r_2 - s_1 + s_2 + \delta > 0$, *and if* $\lambda > 1$, $K_0 > 0$, *and a function* $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *exist with* $w(x) = \mathrm{o}\left(x^{-\left(\frac{1}{q_1}+\frac{1}{p_2}\right)d - r_1 - r_2 + s_1 - s_2 - \delta}\right)$ *and*

$$|m_{j'j}| \leqslant w(\lambda\|j'\|_\infty - \|j\|_\infty)(1 + \|j\|_\infty)^{r_1}(1 + \|j'\|_\infty)^{r_2}, \quad \lambda\|j'\|_\infty - \|j\|_\infty > K_0,$$

then M *has no bounded left inverses.*

2. *If there is a* $\delta \geqslant 0$ *with* $r_2 - s_2 + \delta > 0$ *and* $\frac{d}{p_1} + r_1 + r_2 + s_1 - s_2 + \delta > 0$ *and if* $0 < \lambda < 1$, $K_0 > 0$ *and a function* $w : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ *exist with* $w(x) = \mathrm{o}\left(x^{-\left(\frac{1}{p_1}+\frac{1}{q_2}\right)d - r_1 - r_2 - s_1 + s_2 + \delta}\right)$ *and*

$$|m_{j'j}| \leqslant w(-\lambda\|j'\|_\infty + \|j\|_\infty)(1 + \|j\|_\infty)^{r_1}(1 + \|j'\|_\infty)^{r_2}, \quad -\lambda\|j'\|_\infty + \|j\|_\infty > K_0,$$

$\lambda, K_0 > 0$, *then* M *has no bounded right inverses.*

Clearly, Theorem 1.1 is equal to Theorem 2.2 for $r_1 = r_2 = s_1 = s_2 = 0$, $p_1 = q_1 = p_2 = q_2 = 2$, and $d = 1$.

Proof of Theorem 2.1. We begin with the case $p_1 > 1$ and $p_2 < \infty$ and show that if $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies $w(x) = \mathrm{o}\left(x^{-\left(\frac{1}{q_1}+\frac{1}{p_2}\right)d - r_1 - r_2 - \delta}\right)$, $\delta \geqslant 0$, $r_1 + \delta > 0$, and $\frac{d}{p_2} + r_1 + r_2 + \delta > 0$, then

$$A_{K_1} = K_1^{p_2 r_1} \sum_{K \geqslant K_1} K^{p_2 r_2 + d - 1} \left(\sum_{k \geqslant K} k^{d-1} w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \to 0 \quad \text{as } K_1 \to \infty. \tag{1}$$

To this end, we set $\tilde{w}(x) = \sup_{y \leqslant x} w(y) \in \mathrm{o}\left(x^{-\left(\frac{1}{q_1}+\frac{1}{p_2}\right)d - r_1 - r_2 - \delta}\right)$ and choose $v \in C_0(\mathbb{R}^+)$ with $\tilde{w}(x) \leqslant v(x) x^{-\left(\frac{1}{q_1}+\frac{1}{p_2}\right)d - r_1 - r_2 - \delta}$. Then

$$\begin{aligned}
&\sum_{K \geqslant K_1 + 2} K^{p_2 r_2 + d - 1} \left(\sum_{k \geqslant K} k^{d-1} w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\quad \leqslant \sum_{K \geqslant K_1 + 1} K^{p_2 r_2 + d - 1} \left(\sum_{k \geqslant K+1} k^{d-1} \tilde{w}(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\quad \leqslant \int_{K_1}^{\infty} x^{p_2 r_2 + d - 1} \left(\int_x^{\infty} y^{d-1} \tilde{w}(y)^{q_1} \mathrm{d}y \right)^{\frac{p_2}{q_1}} \mathrm{d}x \\
&\quad \leqslant \int_{K_1}^{\infty} x^{p_2 r_2 + d - 1} \left(\int_x^{\infty} v(y)^{q_1} y^{-1 - \frac{q_1}{p_2} d - q_1 r_2 - q_1 r_1 - q_1 \delta} \mathrm{d}y \right)^{\frac{p_2}{q_1}} \mathrm{d}x
\end{aligned}$$

$$\leqslant \frac{\|v|_{[K_1,\infty)}\|_\infty^{p_2}}{\frac{q_1}{p_2}d+q_1r_2+q_1r_1+q_1\delta}\int_{K_1}^\infty x^{p_2r_2+d-1}x^{-d-p_2r_2-p_2r_1-p_2\delta}\mathrm{d}x$$

$$\leqslant \frac{\|v|_{[K_1,\infty)}\|_\infty^{p_2}}{(r_1+\delta)(q_1d+p_2q_1r_2+p_2q_1r_1+p_2q_1\delta)}K_1^{-p_2r_1-p_2\delta}=\mathrm{o}(K_1^{-p_2r_1}),$$

since $\|v|_{[K_1,\infty)}\|_\infty\to 0$ as $K_1\to\infty$ and (1) follows.

To show that $\inf_{x\in l_0(\mathbb{Z}^d)}\left\{\frac{\|Mx\|_{l^{p_2}}}{\|x\|_{l^{p_1}}}\right\}=0$ we fix $\epsilon>0$ and note that (1) provides us with a $K_1>K_0$ satisfying $A_{K_1}\leqslant (2^d d)^{-\frac{p_2}{q_1}-1}2^{-p_2r_2}\left(\frac{\lambda-1}{\lambda}\right)^{p_2r_1}\epsilon^{p_2}$.

Set $N=\left\lceil\frac{\lambda(K_1+1)}{\lambda-1}\right\rceil$ and $\widetilde{N}=\lceil\frac{N}{\lambda}\rceil+K_1$. Then $\frac{\lambda(K_1+1)}{\lambda-1}\leqslant N\leqslant\frac{\lambda(K_1+2)}{\lambda-1}$ implies $\lambda N\geqslant\lambda K_1+\lambda+N$ and $N\geqslant K_1+\frac{N}{\lambda}+1>K_1+\lceil\frac{N}{\lambda}\rceil=\widetilde{N}$. Therefore, $(2\widetilde{N}+1)^d<(2N+1)^d$ and the matrix $\widetilde{M}=(m_{j'j})_{\|j'\|_\infty\leqslant\widetilde{N},\|j\|\leqslant N}:\mathbb{C}^{(2N+1)^d}\longrightarrow\mathbb{C}^{(2\widetilde{N}+1)^d}$ has a nontrivial kernel. We now choose $\tilde{x}\in\mathbb{C}^{(2N+1)^d}$ with $\|\tilde{x}\|_{p_1}=1$ and $\widetilde{M}\tilde{x}=0$. We define $x\in l_0(\mathbb{Z}^2)$ according to $x_j=\tilde{x}_j$ if $\|j\|_\infty\leqslant N$ and $x_j=0$ otherwise.

By construction we have $\|x\|_{l^{p_1}}=1$ and $(Mx)_{j'}=0$ for $\|j'\|_\infty\leqslant\widetilde{N}$. To estimate $(Mx)_{j'}$ for $\|j'\|_\infty>\widetilde{N}$, we fix $K>K_1$ and one of the $2d(2(\lceil\frac{N}{\lambda}\rceil+K))^{d-1}$ indices $j'\in\mathbb{Z}^d$ with $\|j'\|_\infty=\lceil\frac{N}{\lambda}\rceil+K$. We have $\|\lambda j'\|_\infty\geqslant N+K\lambda$ and $\lambda\|j'\|_\infty-\|j\|_\infty\geqslant K\lambda\geqslant K$ for all $j\in\mathbb{Z}^d$ with $\|j\|_\infty\leqslant N$. Therefore,

$$\begin{aligned}|(Mx)_{j'}|^{q_1}&=\left|\sum_{\|j\|_\infty\leqslant N}m_{j'j}x_j\right|^{q_1}\leqslant\|x\|_{p_1}^{q_1}\sum_{\|j\|_\infty\leqslant N}|m_{j'j}|^{q_1}\\&\leqslant(1+\|j'\|_\infty)^{q_1r_2}\sum_{\|j\|_\infty\leqslant N}(1+\|j\|_\infty)^{q_1r_1}w(\lambda\|j'\|_\infty-\|j\|_\infty)^{q_1}\\&\leqslant(1+\|j'\|_\infty)^{q_1r_2}(N{+}1)^{q_1r_1}\sum_{\|j\|_\infty\geqslant K}w(\|j\|_\infty)^{q_1}\\&=2^d d(1+\|j'\|_\infty)^{q_1r_2}(N{+}1)^{q_1r_1}\sum_{k\geqslant K}k^{d-1}w(k)^{q_1}.\end{aligned}$$

Finally, we compute

$$\begin{aligned}\|Mx\|_{l^{p_2}}^{p_2}&=\sum_{j'\in\mathbb{Z}^d}|(Mx)_{j'}|^{p_2}=\sum_{\|j'\|_\infty\geqslant\lceil\frac{N}{\lambda}\rceil+K_1}|(Mx)_{j'}|^{p_2}\\&\leqslant(2^d d)^{\frac{p_2}{q_1}}\sum_{\|j'\|_\infty\geqslant\lceil\frac{N}{\lambda}\rceil+K_1}(1+\|j'\|_\infty)^{p_2r_2}(N{+}1)^{p_2r_1}\left(\sum_{k\geqslant\|j'\|_\infty}k^{d-1}w(k)^{q_1}\right)^{\frac{p_2}{q_1}}\\&\leqslant(2^d d)^{\frac{p_2}{q_1}}(N{+}1)^{p_2r_1}\sum_{K\geqslant\lceil\frac{N}{\lambda}\rceil+K_1}2d(2K)^{d-1}(K+1)^{p_2r_2}\left(\sum_{k\geqslant K}k^{d-1}w(k)^{q_1}\right)^{\frac{p_2}{q_1}}\\&\leqslant(2^d d)^{\frac{p_2}{q_1}+1}2^{p_2r_2}\left(\frac{\lambda(K_1+2)}{\lambda-1}+1\right)^{p_2r_1}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{K \geqslant \lceil \frac{N}{\lambda} \rceil + K_1} K^{p_2 r_2 + d - 1} \left(\sum_{k \geqslant K} k^{d-1} w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leqslant (2^d d)^{\frac{p_2}{q_1}+1} 2^{p_2 r_2} \left(\frac{\lambda}{\lambda - 1} \right)^{p_2 r_1} (K_1 + 3)^{p_2 r_1} \\
&\times \sum_{K \geqslant \lceil \frac{N}{\lambda} \rceil + K_1} K^{p_2 r_2 + d - 1} \left(\sum_{k \geqslant K} k^{d-1} w(k)^{q_1} \right)^{\frac{p_2}{q_1}} \\
&\leqslant \epsilon^{p_2},
\end{aligned}
$$

that is, $\|Mx\|_{l^{p_2}} \leqslant \epsilon$. Since ϵ was chosen arbitrarily and $\|x\|_{l^{p_1}} = 1$, we have $\inf_{x \in l_0(\mathbb{Z}^2)} \left\{ \frac{\|Mx\|_{l^{p_2}}}{\|x\|_{l^{p_1}}} \right\} = 0$. Consequently, M is not bounded below and M has no bounded left inverses.

The cases $p_1 = 1$ and/or $p_2 = \infty$ are proven similarly. □

Proof of Theorem 2.2. *Part 1.* Let $M = (m_{j'j}) : l_{s_1}^{p_1}(\mathbb{Z}^d) \to l_{s_2}^{p_2}(\mathbb{Z}^d)$ satisfy the hypothesis of Theorem 2.2, *part 1*. Suppose that $M = (m_{j'j}) : l_{s_1}^{p_1}(\mathbb{Z}^d) \to l_{s_2}^{p_2}(\mathbb{Z}^d)$ has a bounded left inverse. This clearly implies that

$$\widetilde{M} = (\tilde{m}_{j'j}) = (m_{j'j}(1 + \|j'\|_\infty)^{s_2}(1 + \|j\|_\infty)^{-s_1}) : l^{p_1}(\mathbb{Z}^d) \to l^{p_2}(\mathbb{Z}^d)$$

has a bounded left inverse. But this contradicts Theorem 2.1 because for $\lambda\|j'\|_\infty - \|j\|_\infty > K_0$ we have

$$
\begin{aligned}
|\tilde{m}_{j'j}| &= |m_{j'j}(1 + \|j'\|_\infty)^{s_2}(1 + \|j\|_\infty)^{-s_1})| \\
&\leqslant w(\lambda\|j'\|_\infty - \|j\|_\infty)(1 + \|j\|_\infty)^{r_1 - s_1}(1 + \|j'\|_\infty)^{r_2 + s_2}
\end{aligned}
$$

with $\delta \geqslant 0$, $r_1 - s_1 + \delta > 0$, $\frac{d}{p_2} + r_1 + r_2 - s_1 + s_2 + \delta > 0$, and $w(x) = \mathrm{o}\left(x^{-\left(\frac{1}{q_1} + \frac{1}{p_2}\right)d - r_1 - r_2 + s_1 - s_2 - \delta} \right)$.

Part 2. The matrix $M : l_{s_1}^{p_1}(\mathbb{Z}^d) \to l_{s_2}^{p_2}(\mathbb{Z}^d)$ has a bounded right inverse if and only if its adjoint $M^* : l_{s_2}^{p_2}(\mathbb{Z}^d) \to l_{s_1}^{p_1}(\mathbb{Z}^d)$ has a bounded left inverse. The conditions on M in Theorem 2.2, *part 2*, are equivalent to the conditions on M^* in Theorem 2.2, *part 1*. The result follows. □

3. Applications

Before formulating applications of Theorems 2.1 and 2.2, we give a brief account of the concepts from time–frequency analysis that are relevant to this section. For additional background on time–frequency analysis and, in particular, Gabor frames see [11].

3.1. Time–frequency analysis and Gabor frames

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is given by $\widehat{f}(\gamma) = \int f(x)\mathrm{e}^{-2\pi \mathrm{i} x \cdot \gamma} \mathrm{d}x$, $\gamma \in \widehat{\mathbb{R}}^d$, where $\widehat{\mathbb{R}}^d$ is the dual group of $\mathbb{R}^d$. Aside from notation, $\widehat{\mathbb{R}}^d = \mathbb{R}^d$. The Fourier transform can be extended to act unitarily on $L^2(\mathbb{R}^d)$ and isomorphically on the dual space of the Frechet space consisting of Schwarz class functions $\mathscr{S}(\mathbb{R}^d)$, that is, on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d) \supset \mathscr{S}(\mathbb{R}^d)$.

The *translation operator* $T_y : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, is given by $(T_y f)(x) = f(x - y)$, $x \in \mathbb{R}^d$, and the *modulation operator* $M_\xi : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ is given by $(M_\xi f)(x) = \mathrm{e}^{2\pi\mathrm{i}x\xi} f(x)$, $x \in \mathbb{R}^d$. Both extend isomorphically to $\mathscr{S}'(\mathbb{R}^d)$ and so does their composition, the so-called *time–frequency shift operator* $\pi(z) = \pi(y,\xi) = T_y M_\xi$, $z = (y,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Note that the adjoint operator $\pi(z)^*$ of $\pi(z) = \pi(y,\xi)$ is $\pi(z)^* = \mathrm{e}^{2\pi\mathrm{i}y\xi}\pi(-z)$.

The *short-time Fourier transform* $V_g f$ of $f \in L^2(\mathbb{R}^d) \subseteq \mathscr{S}'(\mathbb{R}^d)$ with respect to a window function $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ is given by

$$V_g f(z) = \langle f, \pi(z)g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x-y)}\mathrm{e}^{-2\pi\mathrm{i}(x-y)\cdot\xi}\mathrm{d}x, \quad z = (y,\xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

We have $V_g f \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ and $\|V_g f\|_{L^2} = \|f\|_{L^2}\|g\|_{L^2}$.

A central goal in Gabor analysis is to find $g \in L^2(\mathbb{R}^d)$ and *full rank lattices* $\Lambda = A\mathbb{Z}^{2d} \subset \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, $A \in \mathbb{R}^{2d\times 2d}$ full rank, which allow the discretization of the formula $\|V_g f\|_{L^2} = \|f\|_{L^2}\|g\|_{L^2}$. That is, for which $g \in L^2(\mathbb{R}^d)$ and for which full rank lattices Λ does $A, B > 0$ exist with

$$A\|f\|_{L^2}^2 \leqslant \sum_{z\in\Lambda} |V_g f(z)|^2 \leqslant B\|f\|_{L^2}^2, \quad f \in L^2(\mathbb{R}^d)? \tag{2}$$

If (2) is satisfied, then $(g,\Lambda) = \{\pi(z)g\}_{z\in\Lambda}$ is called *Gabor frame* for the Hilbert space $L^2(\mathbb{R}^d)$. More recently, the question posed above has also been considered for less structured or even unstructured sequences Γ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ in place of full rank lattice Λ [12–17].

To generalize the Hilbert space frame concept (2) to Banach spaces, we adopt the definition of p-frames from [18].

Definition 3.1. The Banach space valued sequence $\{g_j\}_{j\in\mathbb{Z}^d} \subseteq X'$, $d \in \mathbb{N}$, is an l_s^p-frame for the Banach space X, $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, if the analysis operator $C_{\mathscr{F}} : X \longrightarrow l_s^p(\mathbb{Z}^d)$, $f \mapsto \{\langle f, g_j\rangle\}_j$, is bounded and bounded below, that is, if $A, B > 0$ exist with

$$A\|f\|_X \leqslant \|\{\langle f, g_j\rangle\}\|_{l_s^p} \leqslant B\|f\|_X, \quad f \in X. \tag{3}$$

Note that in the Hilbert space setting $X = L^2(\mathbb{R}^d)$ and $l_s^p(\mathbb{Z}^{2d}) = l^2(\mathbb{Z}^{2d})$ discussed above, (2) necessitates that $C_{\mathscr{F}}$ has a bounded left inverse, while in Banach spaces condition (3) alone does not guarantee the existence of a bounded left inverse. Therefore, the existence of a bounded left inverse of $C_{\mathscr{F}}$ is included in the definition of the standard generalization of frames to Banach frames in Banach spaces [19,20,21].

Analogous to Definition 3.1 we state a generalization of the Riesz basis concept in the Banach space setting.

Definition 3.2. A sequence $\{g_j\}_{j\in\mathbb{Z}^d} \subseteq X$, $d \in \mathbb{N}$, is called l_s^p-Riesz basis in the Banach space X, $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, if the synthesis operator $D_{\{g_j\}_j} : l_s^p(\mathbb{Z}^{2d}) \longrightarrow X$, $\{c_j\}_j \mapsto \sum_j c_j g_j$, is bounded and bounded below, that is, if $A, B > 0$ exist with

$$A\|\{c_j\}_j\|_{l_s^p} \leqslant \left\|\sum_j c_j g_j\right\|_X \leqslant B\|\{c_j\}_j\|_{l_s^p}, \quad \{c_j\}_j \in l_s^p(\mathbb{Z}^d). \tag{4}$$

Note that for any $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, l^p_s-Riesz bases form unconditional bases for their closed linear span. This follows directly from (4) and Definition 12.3.1 and Lemma 12.3.6 in [11].

The Banach spaces of interest to us are so-called modulation spaces which we shall describe now (see also [11,22,23]).

Clearly, $V_g f(z) = \langle f, \pi(z)g\rangle$, $z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, is well defined whenever $g \in \mathscr{S}(\mathbb{R}^d)$ and $f \in \mathscr{S}'(\mathbb{R}^d)$ (or vice versa). This together with $\|V_g f\|_{L^2} = \|g\|_{L^2}\|f\|_{L^2}$ in the L^2-case motivates the following. We let $g = \mathfrak{g} \in \mathscr{S}(\mathbb{R}^d)$ be an L^2-normalized Gaussian, that is, $\mathfrak{g}(x) = 2^{\frac{d}{4}} \mathrm{e}^{-\pi\|x\|_2^2}$, $x \in \mathbb{R}^d$, and define the *modulation space* $M^p_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, by

$$M^p_s(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : V_{\mathfrak{g}} f \in L^p_s(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)\}.$$

We equip $M^p_s(\mathbb{R}^d)$ with the Banach space norm

$$\|f\|_{M^p_s} = \|V_{\mathfrak{g}} f\|_{L^p_s} = \left(\int |(1+\|z\|)^s V_{\mathfrak{g}} f(z)|^p \mathrm{d}z\right)^{\frac{1}{p}} < \infty, \quad 1 \leqslant p < \infty,$$

with the usual adjustment for $p = \infty$. The discussion above shows that $M^2_0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ with identical norms.

Example 3.3. For $\lambda < 1$, $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l^2-frame for $L^2(\mathbb{R}^d)$ [24,25]. Since $\mathfrak{g} \in \mathscr{S}(\mathbb{R}^d) \subset M^1_t(\mathbb{R}^d)$ for all $t \geqslant 0$, Theorem 20 in [17] implies that $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l^p_s-frames for $M^p_s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$. The so-called Wexler–Raz identity implies that for $\lambda > 1$, $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l^2-Riesz basis in $L^2(\mathbb{R}^d)$ [11,26]. Hence, $D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})} : l^2(\mathbb{Z}^{2d}) \longrightarrow L^2(\mathbb{R}^d)$ has a bounded left inverse of the form $C_{(\tilde{\mathfrak{g}},\lambda\mathbb{Z}^{2d})}$ where the so-called dual function $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ satisfies $\tilde{\mathfrak{g}} \in \mathscr{S}(\mathbb{R}^d)$ [27]. The operator $C_{(\tilde{\mathfrak{g}},\lambda\mathbb{Z}^{2d})}$ is a bounded operator mapping $M^p_s(\mathbb{R}^d)$ to $l^p_s(\mathbb{Z}^{2d})$. This implies that $D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ has a left inverse and $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l^p_s-Riesz basis in $M^p_s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$.

3.2. Density results for Gabor type l^p_s-frames in modulation spaces

One of the central results in Gabor analysis is the fact that (g, Λ), $g \in L^2(\mathbb{R}^d)$, cannot be a frame for $L^2(\mathbb{R}^d)$ if the measure of a fundamental domain of the full rank lattice Λ is larger than 1 (see [28] and references within, in particular [29,30]). This result was first generalized to less structured or even unstructured sequences Γ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ in [12,13,14]. They require an alternative definition of density [15,28,31].

Definition 3.4. Let $Q_R = [-R, R]^{2d} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and let Γ be a sequence of points in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then

$$D^-(\Gamma) = \liminf_{R\to\infty} \inf_{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \frac{|\Gamma \cap Q_R + z|}{(2R)^{2d}} \quad \text{and} \quad D^+(\Gamma) = \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \frac{|\Gamma \cap Q_R + z|}{(2R)^{2d}}$$

are called lower and upper Beurling density of Γ. If $D^+(\Gamma) = D^-(\Gamma)$, then Γ is said to have uniform density $D(\Gamma) = D^+(\Gamma) = D^-(\Gamma)$.

Remark 3.5. The density of a sequence Γ does not always equal the density of its range set. For example, the density of the sequence

$$\{\ldots, -2, -2, -1, -1, 0, 0, 1, 1, 2, 2, 3, 3, \ldots\}$$

in $\mathbb{R}$ is 2, while the density of the range of the sequence, namely of $\mathbb{Z}$, is 1.

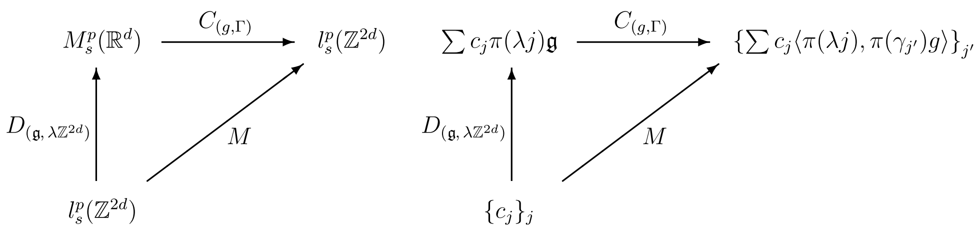

Fig. 1. Sketch of the proof of Theorem 3.6. We choose $\lambda > 1$ so that $(\mathfrak{g}, \lambda\mathbb{Z}^{2d})$ is an l_s^p-Riesz basis in $M_s^p(\mathbb{R}^d)$. Consequently, $D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ is bounded below. Theorem 2.2 applied to $M = C_{(g,\Gamma)} \circ D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ shows that M is not bounded below. This implies that $C_{(g,\Gamma)}$ is not bounded below and, therefore, does not have a bounded left inverse.

In [32], it was shown that if (g, Γ), $g \in L^2(\mathbb{R}^d)$ and $\Gamma \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, is an l^2-frame for $L^2(\mathbb{R}^d) = M_0^2(\mathbb{R}^d)$, then $1 \leqslant D^-(\Gamma) \leqslant D^+(\Gamma) < \infty$, a result that has recently been refined by Theorems 3 and 5 in [16]. For l_s^p-frames for $M_s^p(\mathbb{R}^d)$, Theorem 2.2 implies the following:

Theorem 3.6. *Let* $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, *and* $g \in M_{2d}^\infty$ *if* $s < 0$ *and* $p \neq \infty$ *and* $g \in M_{2d+\delta}^\infty$, $\delta > s, 0$, *else. If* (g, Γ) *is an* l_s^p*-frame for* $M_s^p(\mathbb{R}^d)$, *then* $D^+(\Gamma) \geqslant 1$.

Proof. See Fig. 1 for an illustration of the following arguments. Let Γ be given with $D^+(\Gamma) < 1$. We choose $\lambda > 1$ with $1 > \lambda^{-4d} > D^+(\Gamma)$ and $R_0 > 0$ with

$$|\Gamma \cap Q_R| < \sup_{z\in\mathbb{R}^d\times\widehat{\mathbb{R}}^d} |\Gamma \cap Q_R + z| < \lambda^{-4d}(2R)^{2d}, \quad R > R_0.$$

Since $D^+(\Gamma) < \infty$, the sequence Γ has no accumulation points and we can enumerate the sequence Γ by $\mathbb{Z}^{2d}$ so that $\|\gamma_{j'}\|_\infty \geqslant \|\gamma_{j''}\|_\infty$ implies $\|j'\|_\infty \geqslant \|j''\|_\infty$ for $j', j'' \in \mathbb{Z}^{2d}$.

Now observe that if $\gamma_{j'} \in Q_R$ for $R > R_0$, then $\gamma_{j''} \in Q_R$ for all j'' with $\|j''\|_\infty < \|j'\|_\infty$, a condition which is satisfied by $(2(\|j'\|_\infty - 1) + 1)^{2d} = (2\|j'\|_\infty - 1)^{2d}$ indices. As Q_R contains at most $\lambda^{-4d}(2R)^{2d}$ points, we conclude that

$$\gamma_{j'} \notin Q_R \quad \text{if } (2\|j'\|_\infty - 1)^{2d} \geqslant \lambda^{-4d}(2R)^{2d}, \quad R > R_0. \tag{5}$$

Solving the inequality in (5) for R, we obtain in particular that

$$\gamma_{j'} \notin Q_{\lambda^2\|j'\|_\infty - \frac{\lambda^2}{2}} \quad \text{for } \lambda^2\|j'\|_\infty - \frac{\lambda^2}{2} > R_0. \tag{6}$$

We have

$$C_{(g,\Gamma)} \circ D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})} : l_s^p(\mathbb{Z}^{2d}) \longrightarrow l_s^p(\mathbb{Z}^{2d}),$$

$$\{c_j\}_j \mapsto \left\{\sum_j c_j\langle\pi(\lambda j)h, \pi(\gamma_{j'})g\rangle\right\} = M\{c_j\}_j,$$

with $M = (m_{j'j})$ and $|m_{j'j}| = |\langle\pi(\lambda j)h, \pi(\gamma_{j'})g\rangle| = |V_g h(\gamma_{j'} - \lambda j)|$.

Using (6) we observe that for $\|j'\|_\infty > \frac{R_0}{\lambda^2} + \frac{1}{2}$,

$$\|\gamma_{j'} - \lambda j\|_\infty \geqslant \lambda^2\|j'\|_\infty - \frac{\lambda^2}{2} - \|\lambda j\|_\infty = \lambda\left(\lambda\|j'\|_\infty - \|j\|_\infty - \frac{\lambda}{2}\right),$$

and so

$$|m_{j'j}| = |\langle \pi(\lambda j)\mathfrak{g}, \pi(\gamma_{j'})g\rangle| = |V_g\mathfrak{g}(\gamma_{j'} - \lambda j)| \leqslant w(\lambda\|j'\|_\infty - \|j\|_\infty),$$

where

$$w(\|z\|) = (1 + \|z\|)^{-2d-\delta} \sup_{\tilde{z}}((1 + \|\tilde{z}\|)^{2d+\delta}|V_g\mathfrak{g}(\tilde{z})|), \quad z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

Theorem 2.2 now implies that $C_{(g,\Gamma)} \circ D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ is not bounded below. Since $D_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ is bounded below, we conclude that $C_{(g,\Gamma)}$ is not bounded below. This completes the proof. □

Note that the last lines in the proof of Theorem 3.6 can be modified to apply to time–frequency molecules which we shall consider in the following.

Definition 3.7. A sequence $\{g_{j'}\}_{j'}$ of functions consists of time–frequency molecules that are (v, r_1, r_2)-localized at $\Gamma = \{\gamma_{j'}\}_{j'}$ if

$$|V_\mathfrak{g} g_{j'}(z)| \leqslant (1 + \|z\|_\infty)^{r_1}(1 + \|j'\|_\infty)^{r_2} w(\|z - \gamma_{j'}\|_\infty), \quad w = \mathrm{o}(x^{-v}).$$

If $r_1 = r_2 = 0$, then we simply speak of time–frequency molecules that are v-localized at Γ.

Note that if $\{g_{j'}\}_{j'} \subseteq (M_s^p(\mathbb{R}^d))'$ is (v, r_1, r_2)-localized, then, by definition, $\{g_{j'}\}_{j'} \subseteq M^\infty_{v-r_1}(\mathbb{R}^d)$, and, consequently, if $v - r_1 > 2d$ we have $\{g_{j'}\}_{j'} \subseteq M^1(\mathbb{R}^d)$, a fact which we take into consideration when stating the hypotheses of Theorems 3.8 and 3.9

Related concepts of localization were introduced in [1,17,15,16] to obtain density results and to describe the time–frequency localization of dual frames of irregular Gabor frames, see also Remark 3.11.

Theorem 3.8. *If* $\{g_{j'}\}_{j'} \subseteq (M_s^p(\mathbb{R}^d))' \cap M^\infty_{v-r_1}$, $1 \leqslant p \leqslant \infty$, $s \in \mathbb{R}$, *is an* l_s^p*-frame for* $M_s^p(\mathbb{R}^d)$ *which is* (v, r_1, r_2)*-localized at* $\Gamma = \{\gamma_{j'}\}_{j'}$ *with* $\delta - s, v - r_1 - r_2 - 2d - \delta, r_1 + \frac{2d}{p} + \delta > 0$, *and* $\delta \geqslant 0$, *then* $D^+(\Gamma) \geqslant 1$.

Note that Theorem 9 in [16] states that if $\{g_{j'}\}$ is an l^2-frame for $L^2(\mathbb{R}^d)$ which consists of time–frequency molecules $d + \delta$-localized at Γ, $\delta > 0$, then necessarily $1 \leqslant D^-(\Gamma)$. Below, we show that the heuristics underlying the proof of Theorem 2.1 can be used to obtain some of the density results given above with $D^+(\Gamma)$ being replaced by the less restrictive $D^-(\Gamma)$.

Theorem 3.9. *If* $\{g_{j'}\}_{j'} \subseteq M^1(\mathbb{R}^d)$ *is an* l^p*-frame for* $M^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, *which is* $2d + \delta$*-localized at* $\Gamma = \{\gamma_{j'}\}_{j'}$ *with* $D^+(\Gamma) < \infty$ *and* $\delta > 0$, *then* $D^-(\Gamma) \geqslant 1$.

Proof. Suppose that $\{g_{j'}\}_{j'}$ is an l_s^p-frame for $M^p(\mathbb{R}^d)$ which is $2d + \delta$-localized at $\Gamma = \{\gamma_{j'}\}_{j'}$, $D^-(\Gamma) < 1$. For $z_0 \in \mathbb{R}^{2d}$ and $0 < \alpha_3 < 1$ chosen below, we shall consider the Gabor system $\{\pi(\alpha_3^{-1} j + z_0)\mathfrak{g}\}_{j\in\mathbb{Z}^{2d}}$ which is an l^p-Riesz basis for $M^p(\mathbb{R}^d)$ (see Example 3.3). We shall show that $\{g_{j'}\}$ is not an l^p-frame by arguing that

$$\inf_{x \in l^p(\mathbb{Z}^d)} \frac{\|C_{\{g_{j'}\}} \circ D_{\{\pi(\alpha_3^{-1} j + z_0)\mathfrak{g}\}} x\|_{l^p}}{\|x\|_{l^p}} = 0.$$

To this end, fix $\epsilon > 0$. We first assume $1 < p < \infty$.

Since $D^+(\Gamma) < \infty$, there exist $\alpha_1 \geqslant 1$ and $\widetilde{R_0} \geqslant 1$ with $\infty > \alpha_1^{2d} > D^+(\Gamma) \geqslant 0$ and

$$|\Gamma \cap Q_R + z| \leqslant \alpha_1^{2d}(2R)^{2d}, \quad z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ R \geqslant \widetilde{R_0}.$$

Further, we can pick $\alpha_2, \alpha_3 > \frac{1}{2}$ with $D^-(\Gamma) < \alpha_2^{2d} < \alpha_3^{2d} < 1$, and select $n_0 \in \mathbb{N}$ with

$$\alpha_2 + \alpha_1 \left(\left(1 + \frac{1}{n_0}\right)^{2d} - 1 \right)^{-2d} < \alpha_3 \left(1 - \frac{1}{2n_0}\right)^{2d}.$$

We now choose a monotonically decreasing function w with $w(x) = \mathrm{o}(x^{-2d-\delta})$ and $|V_{\mathfrak{g}} g_{j'}(z)| \leqslant w(\|z - \gamma_{j'}\|_\infty)$. As demonstrated in the proof of Theorem 2.1, $w(x) = \mathrm{o}(x^{-2d-\delta})$, $\delta > 0$, allows us to pick $\widetilde{K}_2$ such that for all $K_2 \geqslant \widetilde{K}_2$

$$(2^{2d}2d)^{\frac{p}{q}+1} \sum_{K \geqslant K_2} K^{2d-1} \left(\sum_{k \geqslant \frac{\alpha_3}{2\alpha_1} K} k^{2d-1} w(k)^q \right)^{\frac{p}{q}} < \epsilon^p.$$

Also, there exist R_0, $N_0 = \lceil \alpha_3 R_0 \rceil$ such that

- there exists $z_0 \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with $|Q_{R_0} + z_0 \cap \Gamma| \leqslant \alpha_2^{2d}(2R_0)^{2d}$;
- $R_0 \geqslant \widetilde{R_0} n_0$; $N_0 \geqslant n_0, \frac{\alpha_1}{\alpha_2}\widetilde{R_0}$;
- $(5\frac{\alpha_1}{\alpha_3}R_0)^{2d} w(\frac{R_0}{n_0} - 2) < \epsilon$;
- $K_1 = N_0 - 1 - \lceil \alpha_2 N_0 \rceil > 1$;
- $K_2 = 2(\frac{\alpha_1}{\alpha_3}N_0 - \lceil \alpha_2 N_0 \rceil) \geqslant \widetilde{K}_2, K_1$.

The sequence Γ has no accumulation point since $D^+(\Gamma) < \infty$. This implies that we can choose an enumeration of the sequence Γ by $\mathbb{Z}^{2d}$ with $\|j'\|_\infty \leqslant \|j''\|_\infty$ if $\|\gamma_{j'} - z_0\|_\infty \leqslant \|\gamma_{j''} - z_0\|_\infty$, $j', j'' \in \mathbb{Z}^{2d}$. As mentioned earlier, we set $\mathfrak{g}_j = \pi(\alpha_3^{-1} j + z_0)\mathfrak{g}$ for $j \in \mathbb{Z}^{2d}$ and $M = (m_{j'j}) = (\langle g_{j'}, \mathfrak{g}_j \rangle)$.

The matrix $\widetilde{M} = (m_{j'j})_{\|j'\|_\infty \leqslant N_0 - 1, \|j\| \leqslant N_0} : \mathbb{C}^{(2N_0+1)^d} \to \mathbb{C}^{(2N_0-1)^d}$ has a nontrivial kernel, so we may choose $\tilde{x} \in \mathbb{C}^{(2N_0+1)^d}$ with $\|\tilde{x}\|_p = 1$ and $\widetilde{M}\tilde{x} = 0$. Define as before $x \in l_0(\mathbb{Z}^2)$ according to $x_j = \tilde{x}_j$ if $\|j\|_\infty \leqslant N_0$ and $x_j = 0$ otherwise.

To estimate the contributions of each $|(Mx)_{j'}|$, $j' \in \mathbb{Z}^{2d}$, to $\|Mx\|_{l^p}$, we consider three cases.

Case 1. $\|j'\|_\infty \leqslant \lceil \alpha_2 N_0 \rceil + K_1 = N_0 - 1$. This implies $(Mx)_{j'} = 0$ by construction.

Case 2. $\lceil \alpha_2 N_0 \rceil + K_1 < \|j'\|_\infty \leqslant \lceil \alpha_2 N_0 \rceil + K_2$. Observe that the set $Q_{R_0 + \frac{R_0}{n_0}} + z_0 \setminus Q_{R_0} + z_0$ consists of a finite number of hypercubes of width $\frac{R_0}{n_0} \geqslant \widetilde{R_0}$. This allows us to estimate

$$\begin{aligned} |Q_{R_0 + \frac{R_0}{n_0}} + z_0 \cap \Gamma| &\leqslant \alpha_2^{2d}(2R_0)^{2d} + \alpha_1^{2d}\left(\left(2\left(R_0 + \frac{R_0}{n_0}\right)\right)^{2d} - (2R_0)^{2d} \right) \\ &\leqslant (2R_0)^{2d}\left(\alpha_2^{2d} + \alpha_1^{2d}\left(\left(1 + \frac{1}{n_0}\right)^{2d} - 1 \right) \right) \end{aligned}$$

$$\leqslant (2\alpha_3^{-1}N_0)^{2d}\alpha_3^{2d}\left(1-\frac{1}{2n_0}\right)^{2d}$$
$$\leqslant \left(2N_0-\frac{2N_0}{2n_0}\right)^{2d} \leqslant (2N_0-1)^{2d}.$$

As in the proof of Theorem 3.6 we conclude that for any j' with $\|j'\|_\infty \geqslant N_0 = \lceil\alpha_2 N_0\rceil + K_1 + 1$, we have $\gamma_j' \notin Q_{R_0+\frac{R_0}{n_0}} + z_0$ and, therefore, for $\|j\|_\infty \leqslant N_0 = \lceil\alpha_3 R_0\rceil$ we have

$$\|\alpha_3^{-1}j + z_0 - \gamma_{j'}\|_\infty = \|(\gamma_{j'} - z_0) - \alpha_3^{-1}j\|_\infty \geqslant R_0 + \frac{R_0}{n_0} - \alpha_3^{-1}\lceil\alpha_3 R_0\rceil$$
$$\geqslant \frac{R_0}{n_0} - \alpha_3^{-1} \geqslant \frac{R_0}{n_0} - 2.$$

Hence,

$$|m_{j'j}| = |\langle g_{j'}, \mathfrak{g}_j\rangle| = |V_{\mathfrak{g}} g_{j'}(\alpha_3^{-1}j + z_0)| \leqslant w(\|\alpha_3^{-1}j + z_0 - \gamma_{j'}\|_\infty) \leqslant w\left(\frac{R_0}{n_0} - 2\right).$$

This gives

$$\begin{aligned}
&\|Mx|_{\{j':\lceil\alpha_2N_0\rceil+K_1<\|j'\|_\infty\leqslant\lceil\alpha_2N_0\rceil+K_2\}}\|_p^p \\
&= \sum_{\lceil\alpha_2N_0\rceil+K_1<\|j'\|_\infty\leqslant\lceil\alpha_2N_0\rceil+K_2}\left|\sum_{\|j\|_\infty\leqslant N_0} m_{j'j}x_j\right|^p \\
&\leqslant \sum_{\lceil\alpha_2N_0\rceil+K_1<\|j'\|_\infty\leqslant\lceil\alpha_2N_0\rceil+K_2}\left(\sum_{\|j\|_\infty\leqslant N_0}|m_{j'j}|^q\right)^{\frac{p}{q}}\|\tilde{x}\|_p^p \\
&\leqslant w\left(\frac{R_0}{n_0}-2\right)^p \sum_{\lceil\alpha_2N_0\rceil+K_1<\|j'\|_\infty\leqslant\lceil\alpha_2N_0\rceil+K_2}(2N_0+1)^{2d\frac{p}{q}}\sum_{\|j\|_\infty\leqslant N_0}|x_j|^p \\
&\leqslant w\left(\frac{R_0}{n_0}-2\right)^p (2\cdot 2\frac{\alpha_1}{\alpha_3}N_0+1)^{2d}(2N_0+1)^{2d\frac{p}{q}} \\
&\leqslant w\left(\frac{R_0}{n_0}-2\right)^p (5\frac{\alpha_1}{\alpha_3}R_0)^{2d(1+\frac{p}{q})} \leqslant \epsilon^p.
\end{aligned} \tag{7}$$

Case 3. $\lceil\alpha_2N_0\rceil + K_2 < \|j'\|_\infty$. We set $N = \|j'\|_\infty$ and obtain $\alpha_1^{-1}(N-\frac{1}{2}) \geqslant \alpha_1^{-1}(\lceil\alpha_2N_0\rceil + K_2 + 1 - \frac{1}{2}) \geqslant \frac{\alpha_2}{\alpha_1}N_0 \geqslant \widetilde{R_0}$, and, hence,

$$|\Gamma \cap Q_{\alpha_1^{-1}(N-\frac{1}{2})} + z_0| \leqslant \alpha_1^{2d}(2\alpha_1^{-1}(N-\frac{1}{2}))^{2d} = (2N-1)^{2d}.$$

As seen in the proof of Theorem 3.6, this implies $\gamma_{j'} \notin Q_{\alpha_1^{-1}(\|j'\|_\infty-\frac{1}{2})} + z_0$. Similarly as in Case 2., we fix j', K with $\|j'\|_\infty = \lceil\alpha_2N_0\rceil + K$, $K > K_2$, and conclude that for $\|j\|_\infty \leqslant N_0$,

$$\|\alpha_3^{-1}j + z_0 - \gamma_{j'}\|_\infty = \|(\gamma_{j'} - z_0) - \alpha_3^{-1}j\|_\infty \geqslant \alpha_1^{-1}(\|j'\|_\infty - \frac{1}{2}) - \alpha_3^{-1}\|j\|_\infty$$
$$\geqslant \frac{\alpha_3}{\alpha_1}\|j'\|_\infty - \|j\|_\infty - \frac{\alpha_3}{2\alpha_1}$$

$$\begin{aligned}&\geqslant \frac{\alpha_3}{\alpha_1}\lceil\alpha_2 N_0\rceil + 2\frac{\alpha_3}{2\alpha_1}K - N_0 - \frac{\alpha_3}{2\alpha_1}\\ &\geqslant \frac{\alpha_3}{2\alpha_1}\left(K - 2\left(\frac{\alpha_1}{\alpha_3}N_0 - \lceil\alpha_2 N_0\rceil\right) - 1\right) + \frac{\alpha_3}{2\alpha_1}K \geqslant \frac{\alpha_3}{2\alpha_1}K.\end{aligned}$$

Therefore,

$$\begin{aligned}|(Mx)_{j'}|^q &= \left|\sum_{\|j\|_\infty\leqslant N_0} m_{j'j}x_j\right|^q \leqslant \|x\|_p^q \sum_{\|j\|_\infty\leqslant N_0}|m_{j'j}|^q\\ &\leqslant \sum_{\|j\|_\infty\leqslant N_0} w\left(\frac{\alpha_3}{\alpha_1}\|j'\|_\infty - \|j\|_\infty - \frac{\alpha_3}{2\alpha_1}\right)^q\\ &\leqslant \sum_{\|j\|_\infty\geqslant\frac{\alpha_3}{2\alpha_1}K} w(\|j\|_\infty)^q = \sum_{k\geqslant\frac{\alpha_3}{2\alpha_1}K} 2(2d)(2k)^{2d-1}w(k)^q\\ &= 2^{2d}2d\sum_{k\geqslant\frac{\alpha_3}{2\alpha_1}K} k^{2d-1}w(k)^q.\end{aligned}$$

Finally, we compute

$$\begin{aligned}&\sum_{\|j'\|_\infty>\lceil\alpha_2 N_0\rceil+K_2} |(Mx)_{j'}|^p\\ &\leqslant (2^{2d}2d)^{\frac{p}{q}} \sum_{\|j'\|_\infty\geqslant\lceil\alpha_2 N_0\rceil+K_2}\left(\sum_{k\geqslant\frac{\alpha_3}{2\alpha_1}\|j'\|_\infty} k^{2d-1}w(k)^q\right)^{\frac{p}{q}}\\ &\leqslant (2^{2d}2d)^{\frac{p}{q}} \sum_{K\geqslant\lceil\alpha_2 N_0\rceil+K_2} 2(2d)(2K)^{2d-1}\left(\sum_{k\geqslant\frac{\alpha_3}{2\alpha_1}K} k^{2d-1}w(k)^q\right)^{\frac{p}{q}}\\ &\leqslant (2^{2d}2d)^{\frac{p}{q}+1} \sum_{K\geqslant\lceil\alpha_2 N_0\rceil+K_2} K^{2d-1}\left(\sum_{k\geqslant\frac{\alpha_3}{2\alpha_1}K} k^{2d-1}w(k)^{q_2}\right)^{\frac{p}{q}} \leqslant \epsilon^p \qquad (8)\end{aligned}$$

by hypothesis. Clearly, (7) and (8) give $\|Mx\|_{l^p} \leqslant 2^{\frac{1}{p}}\epsilon$ which completes the proof for $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ are proven similarly. □

Remark 3.10. If $\{g_j\} = (g, \Gamma)$ and the analysis operator $C_{(g,\Gamma)}$ is bounded, then $D^+(\Gamma) < \infty$ follows [32]. If $\{g_j\}$ is only assumed to consist of Γ localized time–frequency molecules, then boundedness of $C_{\{g_j\}}$ does not imply $D^+(\Gamma) < \infty$. For example, consider $\{g_j\} = \{2^{-j}\mathfrak{g}\}_{j\in\mathbb{N}}$.

Remark 3.11. Theorem 9 in [16] implies that time–frequency molecules $\{g_j\}$ which are v-localized at $\Gamma = \{\gamma_j\}$, $v > d$, and which generate an l^2-frame for $L^2(\mathbb{R})$ satisfy $1 \leqslant D^-(\Gamma) \leqslant D^+(\Gamma)$. Further, Theorem 22 in [17] states that if $v > d$ is replaced by $v > 2d + s$, then being an l^2-frame for $L^2(\mathbb{R}^d)$ is equivalent to being an l^p_s-frame for $M^p_s(\mathbb{R}^d)$ for all $1 \leqslant p \leqslant \infty$ and all $s \geqslant 0$. This result alone implies neither Theorem 3.8 nor Theorem 3.9 as these only assume that $\{g_j\}$ is an l^p_s-frame for $M^p_s(\mathbb{R}^d)$ for some p and s. Under stronger conditions on v, [1] fills this gap. Namely, Theorem 3.1 and Example 3.1 in [1] show that if $v > (2d+1)^2 + 2d$ and if $\{g_j\}$ is for one p, $1 \leqslant p \leqslant \infty$, an l^p-frame for $M^p(\mathbb{R}^d)$ which is v-localized at $\Gamma = \{\gamma_j\}$, then $\{g_j\}$ is an l^p frame for $M^p(\mathbb{R}^d)$ for all p and, therefore, also for the well-studied case $p = 2$ [16]. This result implies Theorem 3.9 for $v > (2d+1)^2 + 2d$.

3.3. Identification of operators with bandlimited Kohn–Nirenberg symbols

A central goal in applied sciences is to identify a partially known operator H from a single input–output pair (g, Hg). We refer to an operator class $\mathscr{H}$ as identifiable, if there exists an element g in the domain of all $H \in \mathscr{H}$ that induces a map $\Phi_g : \mathscr{H} \longrightarrow Y$, $H \mapsto Hg$, which is bounded and bounded below as a map between Banach spaces.

In [8,9], special cases of Theorem 2.2 played crucial roles in showing that classes of pseudodifferential operators with Kohn–Nirenberg symbols bandlimited to a rectangular domain $\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]$ are not identifiable if $ab > 1$. The bandlimitation of a Kohn–Nirenberg symbol to a rectangular domain $\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]$ can be expressed by a corresponding support condition on the operators' so-called spreading function η_H.[1] Analogously, we consider here operators $H : D \longrightarrow M^p_s(\mathbb{R})$, $D \subseteq M^\infty(\mathbb{R})$, included in

$$\mathscr{H}^p_s\left(\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]\right) = \left\{H = \int_{\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]} \eta_H(z)\pi(z)\mathrm{d}z,\ \eta_H \in M^p_s(\mathbb{R} \times \widehat{\mathbb{R}})\right\} \tag{9}$$

and with norm $\|H\|_{\mathscr{H}^p_s} = \|\eta_H\|_{M^p_s}$. The integral in (9) is defined weakly using $\langle Hf, h\rangle = \langle \eta_H, V_h f\rangle$ [2] [9]. In [8] the following was shown.

Theorem 3.12. *There is $g \in M^\infty(\mathbb{R})$ with $\Phi_g : \mathscr{H}^2_0\left(\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]\right) \longrightarrow M^2_0(\mathbb{R})$ bounded and bounded below if and only if $ab \leqslant 1$.*

Note that $\mathscr{H}^1_0\left(\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]\right)$ consists of Hilbert–Schmidt operators and the norm $\|\cdot\|_{\mathscr{H}^2_0}$ is equivalent to the Hilbert–Schmidt space norm.

The main result in [9] is the following:

Theorem 3.13. *If $ab < 1$, then there is $g \in M^\infty(\mathbb{R})$ with $\Phi_g : \mathscr{H}^\infty_0\left(\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]\right) \longrightarrow M^\infty_0(\mathbb{R})$ bounded and bounded below, while if $ab > 1$ then no such $g \in M^\infty(\mathbb{R})$ exists.*

Here, we use the generality of Theorem 2.2 to obtain:

Theorem 3.14. *Let $1 \leqslant p \leqslant \infty$ and $s \in \mathbb{R}$. For $ab > 1$, no $g \in M^\infty(\mathbb{R})$ exists with $\Phi_g : \mathscr{H}^p_s\left(\left[-\frac{a}{2}, \frac{a}{2}\right] \times \left[-\frac{b}{2}, \frac{b}{2}\right]\right) \longrightarrow M^p_s(\mathbb{R})$ bounded and bounded below.*

[1] In fact, the spreading function of an operator is the symplectic Fourier transform of the operator's Kohn–Nirenberg symbol [8,10].

[2] Here, $\langle \cdot, \cdot \rangle$ is taken to be linear in the first component and conjugate linear in the second.

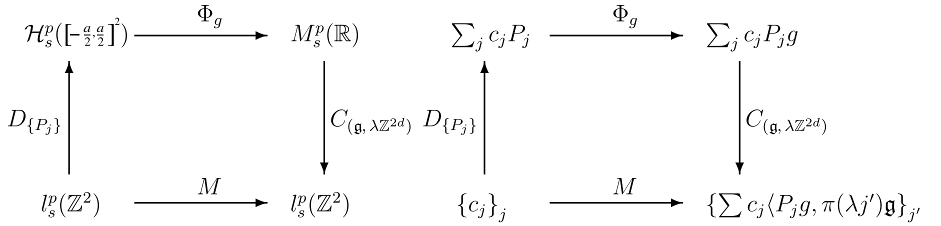

Fig. 2. Sketch of the proof of Theorem 3.14. We choose a structured operator family $\{P_j\} \subseteq \mathscr{H}_s^p$ so that the corresponding synthesis map $D_{\{P_j\}} : \{c_j\} \longrightarrow \sum c_j P_j$ has a bounded left inverse. Recall that $C_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})}$ has a bounded left inverse for $\lambda < 1$ as well. We then use Theorem 2.2 to show that for any $g \in M^\infty(\mathbb{R})$, the composition $M = C_{(\mathfrak{g},\lambda\mathbb{Z}^{2d})} \circ \phi_g \circ D_{\{P_j\}}$ has no bounded left inverses. This implies that $\phi_g : \mathscr{H}_s^p \longrightarrow M_s^p(\mathbb{R})$ also has no bounded left inverses.

Sketch of proof. We assume $a = b$ and $a^2 > 1$. The general case $ab > 1$ follows similarly. The goal is to show that for any $g \in M^\infty(\mathbb{R})$ which induces a bounded operator Φ_g: $\mathscr{H}_s^p\left(\left[-\frac{a}{2},\frac{a}{2}\right]^2\right) \longrightarrow M_s^p(\mathbb{R})$, the operator Φ_g is not bounded below (see Fig. 2).

To see this, we pick $\lambda > 1$ with $1 < \lambda^4 < a^2$ and define a prototype operator $P \in \mathscr{H}_s^p\left(\left[-\frac{a}{2},\frac{a}{2}\right]^2\right)$ via its spreading function $\eta_P(t,\nu) = \eta(t)\eta(\nu)$ where η is smooth, takes values in [0,1], and satisfies $\eta(t) = 1$ for $|t - a/2| \leqslant a/2\lambda$ and $\eta(t) = 0$ for $|t - a/2| \geqslant a/2$.

The collection of functions $\{M_{\frac{\lambda}{a}j}\eta_P\}_{j\in\mathbb{Z}^2}$ corresponds to the operator family $\{\pi(\frac{\lambda}{a}j)P\pi(\frac{\lambda}{a}j)^*\}_{j\in\mathbb{Z}^2}$ [9]. Further, it forms a Riesz basis for its closed linear span in $L^2(\mathbb{R}\times\widehat{\mathbb{R}})$ and, for $c > 0$ sufficiently large, the collection $\{\pi(\frac{\lambda}{a}j,\frac{1}{c}k)\eta_P\}_{j,k\in\mathbb{Z}^2}$ is a frame for $L^2(\mathbb{R}^2)$ [11,33]. Arguing as in Example 3.3, we obtain a bounded left inverse of $D_{\{M_{\frac{\lambda}{a}j}\eta_P\}} : l_s^p(\mathbb{Z}^2) \longrightarrow M_s^p(\mathbb{R}\times\widehat{\mathbb{R}})$, thereby showing that $D_{\{M_{\frac{\lambda}{a}j}\eta_P\}}$ and also the corresponding operator synthesis map $D_{\{P_j\}} : l_s^p(\mathbb{Z}^2) \longrightarrow \mathscr{H}_s^p(\mathbb{R}\times\widehat{\mathbb{R}})$ with $P_j = \pi(\frac{\lambda}{a}j)P\pi(\frac{\lambda}{a}j)^*$, $j \in \mathbb{Z}^2$, are bounded below.

For any fixed $g \in M^\infty(\mathbb{R})$ which induces a bounded map $\Phi_g : \mathscr{H}_s^p([-\frac{a}{2},\frac{a}{2}]^2) \longrightarrow M_s^p(\mathbb{R})$ we consider the operator

$$M = (m_{jj'}) = C_{(\mathfrak{g},\frac{\lambda^2}{a})} \circ \Phi_g \circ D_{\{P_j\}}: l_s^p(\mathbb{Z}^2) \longrightarrow l_s^p(\mathbb{Z}^2).$$

We have $|m_{jj'}| = |\langle \pi(\frac{\lambda}{a}j)P\pi(\frac{\lambda}{a}j)^*g, \pi(\frac{\lambda^2}{a}j')\mathfrak{g}\rangle| = |V_\mathfrak{g} P\pi(\frac{\lambda}{a}j)^*g(\frac{\lambda}{a}(\lambda j' - j))|$. In [8] it is shown that smoothness and compact support of η_P implies that there exist nonnegative functions d_1 and d_2 on $\mathbb{R}$, decaying rapidly at infinity, such that for all $g \in M^\infty(\mathbb{R})$, $|Pg(x)| \leqslant \|g\|_{M^\infty} d_1(x)$ and $|\widehat{Pg}(\xi)| \leqslant \|g\|_{M^\infty} d_2(\xi)$. This implies that $V_\mathfrak{g} P\pi(\frac{\lambda}{a}j)^*g$ decays rapidly and independently of j, a fact that allows us to apply Theorem 2.2 to show that M is not bounded below. Since $\frac{\lambda^2}{a} < 1$, Example 3.3 implies that $C_{(\mathfrak{g},\frac{\lambda^2}{a})}$ is bounded below. Also, $D_{\{P_j\}}$ is bounded below, implying that Φ_g cannot be bounded below. Since $g \in M^\infty(\mathbb{R})$ was chosen arbitrarily, the proof is complete. □

References

[1] A. Aldroubi, A. Baskakov, I. Krishtal, Slanted matrices, banach frames, and sampling, Preprint, 2007.

[2] A. Cavaretta, W. Dahmen, C. Micchelli, Stationary subdivision, Mem. Amer. Math. Soc. 93 (453) (1991).

[3] W. Dahmen, C. Micchelli, Banded matrices with banded inverses. II. Locally finite decomposition of spline spaces, Const. Approx. 9 (2–3) (1993) 263–281.

[4] M. Gasca, C.A. Micchelli, J.M. Peña, Banded matrices with banded inverses. III. p-Slanted matrices, in: Wavelets, images, and surface fitting, AK Peters, Wellesley, MA, 1994, pp. 245–268 (Chamonix–Mont–Blanc, 1993).
[5] C. Micchelli, Banded matrices with banded inverses, J. Comput. Appl. Math. 41 (3) (1992) 281–300.
[6] H. Feichtinger, K. Gröchenig, Gabor frames and time–frequency analysis of distributions, J. Funct. Anal. 146 (2) (1997) 464–495.
[7] H. Feichtinger, Atomic characterizations of modulation spaces through Gabor-type representations, in: Proc. Conf. Constructive Function Theory, Edmonton, July 1986, 1989, pp. 113–126.
[8] W. Kozek, G. Pfander, Identification of operators with bandlimited symbols, SIAM J. Math. Anal. 37 (3) (2006) 867–888.
[9] G. Pfander, D. Walnut, Operator identifcation and Feichtinger's algebra, Sampl. Theory Signal Image Process. 5 (2) (2006) 151–168.
[10] G. Pfander, D. Walnut, Measurement of time-variant channels, IEEE Trans. Info. Theory 52 (11) (2006) 4808–4820.
[11] K. Gröchenig, Foundations of Time–Frequency Analysis, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA, 2001.
[12] H. Landau, On the density of phase-space expansions, IEEE Trans. Inform. Theory 39 (4) (1993) 1152–1156.
[13] J. Ramanathan, T. Steger, Incompleteness of sparse coherent states, Appl. Comput. Harm. Anal. 2 (1995) 148–153.
[14] A. Janssen, A density theorem for time-continuous filter banks, in: J. Zeevi, R. Coifman (Eds.), Signal and Image Representation in Combined Spaces, Academic Press, New York, San Francisco, London, 1998.
[15] R. Balan, P. Casazza, C. Heil, Z. Landau, Density, overcompleteness, and localization of frames. I: theory, J. Fourier Anal. Appl. 12 (2) (2006) 105–143.
[16] R. Balan, P. Casazza, C. Heil, Z. Landau, Density, overcompleteness, and localization of frames. II: Gabor systems, J. Fourier Anal. Appl. 12 (3) (2006) 307–344.
[17] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl. 10 (2) (2004) 105–132.
[18] A. Aldroubi, Q. Sun, W. Tang, p-frames and shift invariant subspaces of L^p, J. Fourier Anal. Appl. 7 (1) (2001) 1–21.
[19] O. Christensen, An introduction to frames and Riesz bases, Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc., Boston, MA, 2003.
[20] K. Gröchenig, Describing functions: atomic decompositions versus frames, Monatsh. Math. 112 (3) (1991) 1–42.
[21] H. Feichtinger, G. Zimmermann, A Banach space of test functions for Gabor analysis, in: H. Feichtinger, T. Strohmer (Eds.), Gabor Analysis and Algorithms: Theory and Applications, Birkhäuser, Boston, MA, 1998, pp. 123–170.
[22] H. Feichtinger, Modulation spaces on locally compact abelian groups, Tech. Rep., Univ. Vienna, Dept. of Math., 1983.
[23] H. Feichtinger, K. Gröchenig, Gabor frames and time–frequency analysis of distributions, J. Funct. Anal. 146 (2) (1996) 464–495.
[24] Y. Lyubarskii, Frames in the Bargmann space of entire functions, Adv. Soviet Math. 429 (1992) 107–113.
[25] K. Seip, R. Wallstén, Density theorems for sampling and interpolation in the Bargmann–Fock space. II, J. Reine Angew. Math. 429 (1992) 107–113.
[26] J. Wexler, S. Raz, Discrete Gabor expansions, Signal Process. 21 (3) (1990) 207–221.
[27] A. Janssen, Duality and biorthogonality for Weyl–Heisenberg frames, J. Fourier Anal. Appl. 1 (4) (1995) 403–436.
[28] C. Heil, History and evolution of the density theorem for Gabor frames, J. Fourier Anal. Appl. 12 (2) (2007) 113–166.
[29] L. Baggett, Processing a radar signal and representations of the discrete Heisenberg group, Colloq. Math. 60/61 (1) (1990) 195–203.
[30] I. Daubechies, The wavelet transform time–frequency localization and signal analysis, IEEE Trans. Inform. Theory (1990) 961–1005.
[31] H. Landau, Necessary density conditions for sampling an interpolation of certain entire functions, Acta Math. 117 (1967) 37–52.
[32] O. Christensen, B. Deng, C. Heil, Density of Gabor frames, Appl. Comput. Harmon. Anal. 7 (3) (1999) 292–304.
[33] D. Walnut, Continuity properties of the Gabor frame operator, J. Math. Anal. Appl. 165 (2) (1992) 479–504.