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A selection of open problems

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ABSTRACT

This is a collection of open problems which touch on Neil Hindman's mathematics and were collected in conjunction with the Conference on Ramsey Theory and Topological Algebra in his honor.

1. Introduction

Most of the problems collected below were presented at the Open Problems session held at the Conference on Ramsey Theory and Topological Algebra in honor of Neil Hindman. In addition we have included some problems which were submitted by those friends and admirers of Neil Hindman who were not able to attend the conference. The problems are diverse and address quite a wide variety of topics. Rather than attempting to arrange the problems thematically we decided to list them alphabetically by name of presenter. Practically all of the problems are of the type Neil Hindman would like to see advanced. We wish our friend and collaborator Neil Hindman many more productive years of doing, teaching, and disseminating mathematics.

2. Vitaly Bergelson

Minimal idempotents and multiple recurrence

Let $E \subseteq \mathbb{N}$ be a set of positive upper density $\bar{d}(E) = \limsup \frac{|E \cap \{1,2,\ldots,N\}|}{N}$. The ergodic proof of Szemerédi's theorem [47] due to Furstenberg [24] not only establishes the existence of arbitrarily long arithmetic progressions in E, but actually

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implies that for every $k \in \mathbb{N}$ the set

$$\left\{d \in \mathbb{Z}\colon \exists a\colon \left\{a, a+d, \ldots, a+(k-1)d\right\} \subseteq E\right\}$$

is syndetic (that is, has bounded gaps). As far as we know, this result has no purely combinatorial proof. In ergodic language this fact can be formulated thusly: If $(X, \mathcal{B}, \mu, T)$ is an invertible probability measure preserving system, then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, the set

$$R_A = \left\{n \in \mathbb{Z}\colon \mu\left(A \cap T^{-n}A \cap \cdots \cap T^{-(k-1)n}A\right) > 0\right\}$$

is syndetic.

One is naturally interested in knowing whether the family of sets of the form R_A, where $\mu(A) > 0$, has the filter property and whether the fact that the sets of multiple returns are large holds true for measure preserving actions of general groups. It was proved in [26] that the above set R_A is actually an IP*-set. That is, it has a non-trivial intersection with every set of the form $\mathrm{FS}(x_i)_{i=1}^{\infty}$, where $(x_i)_{i=1}^{\infty}$ is an increasing sequence of integers, and

$$\mathrm{FS}(x_i)_{i=1}^{\infty} = \{x_{i_1} + x_{i_2} + \cdots + x_{i_n}\colon n \in \mathbb{N} \text{ and } i_1 < i_2 < \cdots < i_n\}.$$

By utilizing Hindman's finite sums theorem [30] it is not hard to see that the intersection of any finite family of IP*-sets is an IP*-set. This, in turn, implies that the family of sets of the form R_A has the filter property. Actually, the results obtained in [26] apply to general (countable, discrete) abelian groups. In particular, it is proved in [26] that if $T_1, \ldots, T_k$ are commuting invertible measure preserving transformations of the probability space $(X, \mathcal{B}, \mu)$ then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists a constant $c > 0$ such that the set

$$\left\{n \in \mathbb{Z}\colon \mu\left(A \cap T_1^n A \cap T_2^n A \cap \cdots \cap T_k^n A\right) > c\right\}$$

is IP*.

Suppose now that G is a countable discrete group and that $(T_g^{(1)})_{g\in G}, \ldots, (T_g^{(k)})_{g\in G}$ are k commuting measure preserving actions of G on a probability measure space $(X, \mathcal{B}, \mu)$. "Commuting" here means that for any $i \neq j$, $i, j \in \{1, \ldots, k\}$, and any $h, g \in G$ one has $T_g^{(i)} T_h^{(j)} = T_h^{(i)} T_g^{(j)}$. The sets

$$R_{G,k,A} = \left\{g \in G\colon \mu\left(A \cap T_g^{(1)}A \cap T_g^{(1)}T_g^{(2)}A \cap \cdots \cap T_g^{(1)}T_g^{(2)} \cdots T_g^{(k)}A\right) > 0\right\}$$

form the analogue of the sets of multiple recurrence R_A considered above (the lack of space does not allow us to explain here why it is these sets rather than sets of the form $\{g \in G\colon\ \mu(A \cap T_g^{(1)}A \cap T_g^{(2)}A \cap \cdots \cap T_g^{(k)}A) > 0\}$ that naturally form the "non-commutative" version of sets R_A).

Conjecture 2.1. *Let G be a countable group, let $(X, \mathcal{B}, \mu)$ be a probability space, let $k \in \mathbb{N}$ and suppose that $(T_g^{(1)})_{g\in G}, \ldots, (T_g^{(k)})_{g\in G}$ are μ-preserving actions of G which commute in the sense that for $i \neq j$ and any $g, h \in G$ one has $T_g^{(i)} T_h^{(j)} = T_h^{(i)} T_g^{(j)}$. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $\lambda > 0$ such that the set*

$$R_{G,k,A} = \left\{g \in G\colon \mu\left(A \cap T_g^{(1)}A \cap T_g^{(1)}T_g^{(2)}A \cap \cdots \cap T_g^{(1)}T_g^{(2)} \cdots T_g^{(k)}A\right) > \lambda\right\}$$

is a member of any minimal idempotent in βG.

Remarks. 1. The first non-trivial case of this conjecture, corresponding to $k = 2$, was recently established in [12].

2. As it was mentioned above, when G is abelian the sets $R_{G,k,A}$ are IP*-sets (see [26]). It is not hard to see that this is equivalent to the fact that $R_{G,k,A}$ is a member of any idempotent in βG. There are, however, some good reasons to believe that for non-commutative G this is not always the case. See [7] and the discussion at the end of Section 3 in [12].

3. See [8] and [10] for results (and counterexamples) addressing the issues of multiple recurrence for possibly non-commuting $\mathbb{Z}$-actions.

Non-commutative Rado theory

In his paper on the equation $x^m + y^m \equiv z^m \bmod p$, I. Schur [44] has shown that for any finite coloring $\mathbb{N} = \bigcup_{i=1}^r C_i$ one of the C_i contains a triple x, y, z such that $x + y = z$. This fact has many proofs some of which, including the original proof in [44], are totally elementary. (See for example [28,25,3].)

Schur's result as well as van der Waerden's theorem are special cases of a general theorem due to Rado [43], which establishes the necessary and sufficient conditions for partition regularity of systems of linear equations over $\mathbb{Z}$. (For various "abelian" extensions of Rado theory see [5,6], and [36, Ch. 15].)

Suppose now that we are given a (countable, discrete) group G, and let $G = \bigcup_{i=1}^r C_i$ be a finite coloring. Assuming that G is sufficiently non-commutative one would like to know if one of the C_i contains x, y, z_1, z_2 such that $xy \neq yx$ and $xy = z_1$, $yx = z_2$. "Sufficiently non-commutative" here means that there are negligibly few elements which commute with many elements of G. This can be formalized as follows. Let $C(g)$ denote the centralizer of G and let H be the subgroup of G

consisting of all elements of G such that $[G : C(g)] < \infty$. (See [11, Theorem 3] for a proof that it is indeed a subgroup.) We will say that G has property SNC (for Sufficiently Non-Commutative) if $[G : H] = \infty$.

Plenty of countable groups have property SNC, for example the Heisenberg group or the group of finite permutations of $\mathbb{N}$.

Question 2.2. Is it true that a countable group G has property SNC if and only if it has no abelian subgroups of finite index?

It was proved in [11] that if G is an amenable group with property SNC, then for any finite coloring $G = \bigcup_{i=1}^{r} C_i$ one of the C_i contains x, y, z_1, z_2 such that $xy \neq yx$ and $xy = z_1$, $yx = z_2$. Somewhat surprisingly, the proof of this fact utilizes a non-trivial ergodic theorem obtained in [13] and, moreover, does not extend to non-amenable groups.

Conjecture 2.3. *If G is a countable abelian group with property SNC then for any finite partition $G = \bigcup_{i=1}^{r} C_i$ one of the C_i contains x, y, z_1, z_2 such that $xy \neq yx$ and $xy = z_1$, $yx = z_2$.*

Remarks. 1. Besides the above mentioned "amenable" result proved in [11], very little is known about the validity of this conjecture. For example it is not known whether it holds true in the case when G is a free group.

2. If true, the above conjecture could form a starting point for development of a non-commutative Rado theory. For a result in this direction in the framework of nilpotent groups see [9, Theorem 5.5].

3. Andreas Blass[1]

Consider the torus $T = \mathbb{R}^2/\mathbb{Z}^2$. For any irrational number α, the line in $\mathbb{R}^2$ with equation $y = \alpha x$ projects to a curve $L(\alpha)$ dense in T. Topologize $L(\alpha)$ as a subspace of T. Under what conditions on α and β are $L(\alpha)$ and $L(\beta)$ homeomorphic?

An easy sufficient condition is that $\alpha = \frac{a\beta+b}{c\beta+d}$ for some integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant ± 1. In this case $L(\alpha)$ and $L(\beta)$ are related by an auto-homeomorphism of T.

I do not know any counterexample to the idea that this sufficient condition is also necessary. On the other hand, I do not know any counterexample to the idea that all the spaces $L(\alpha)$, for all irrational α, are homeomorphic.

4. W.W. Comfort[2]

Conventions. (1) All groups here are infinite and abelian. (2) $\mathcal{S}(G)$ is the set of point-separating subgroups of $\mathrm{Hom}(G, \mathbb{T})$. (3) The topology induced by $A \in \mathcal{S}(G)$ on G is denoted $\mathcal{T}_A$. (4) The symbol $G^\#$ abbreviates $(G, \mathcal{T}_{\mathrm{Hom}(G,\mathbb{T})})$. (5) The expression $X =_{\mathrm{top}} Y$ indicates that X and Y are homeomorphic topological spaces.

It is easily seen for each group G that $\mathrm{Hom}(G, \mathbb{T})$ itself separates points of G (i.e., $\mathrm{Hom}(G, \mathbb{T}) \in \mathcal{S}(G)$), so $G^\#$ is a Hausdorff topological group. Observing that, van Douwen [22] asked this bold and beautiful question: Given groups G and H with $|G| = |H|$, must $G^\# =_{\mathrm{top}} H^\#$? That question was answered in the negative, independently and approximately simultaneously, by Kunen [37] and by Dikranjan and Watson [21]. (The examples may take the form $G = \bigoplus_\kappa \mathbb{Z}(2)$, $H = \bigoplus_\kappa \mathbb{Z}(3)$, with $\kappa = \omega$ in [37], $\kappa > 2^{2^{\mathfrak{c}}}$ in [21].) But there are results in the positive direction, such as these two.

(1) Quite early, Trigos-Arrieta [48] showed that for every infinite κ there exist non-isomorphic G and H for which $|G| = |H| = \kappa$ and $G^\# =_{\mathrm{top}} H^\#$, a result later strengthened and substantially extended by J.E. Hart and Kunen [29].
(2) $\mathbb{Q}^\# =_{\mathrm{top}} ((\mathbb{Q}/\mathbb{Z}) \times \mathbb{Z})^\# = (\mathbb{Q}/\mathbb{Z})^\# \times \mathbb{Z}^\#$ [18].

These results and others (see the references below, and the many papers cited by those authors) stimulated by van Douwen's paper leave many specific questions unresolved. Of these the following, cited with other questions in such works as [17,18,20,27], is in this author's opinion the most tantalizing and the most elegant in its apparent simplicity.

Question 4.1. Are the topological spaces $\mathbb{Z}^\#$ and $(\mathbb{Z} \times \mathbb{Z})^\# = \mathbb{Z}^\# \times \mathbb{Z}^\#$ homeomorphic?

(a) It is clear for each G and each $A \in \mathcal{S}(G)$ that the evaluation isomorphism $e_A : G \to \mathbb{T}^A$ (defined by $(e_A(x))_h = h(x) \in \mathbb{T}$ for $x \in G$, $h \in A$) is an isomorphism of G onto $(G, \mathcal{T}_A) \subseteq \mathbb{T}^A$, an inclusion of topological groups. Thus topologically and algebraically one has

$$\mathbb{Z}^\# \subseteq \mathbb{T}^{\mathrm{Hom}(\mathbb{Z},\mathbb{T})} = \mathbb{T}^{\mathfrak{c}} \quad \text{and} \quad (\mathbb{Z} \times \mathbb{Z})^\# \subseteq \mathbb{T}^{\mathrm{Hom}(\mathbb{Z}\times\mathbb{Z},\mathbb{T})} = \mathbb{T}^{\mathfrak{c}},$$

relations which apparently afford many tools for establishing or refuting the (possible) homeomorphism $\mathbb{Z}^\# =_{\mathrm{top}} (\mathbb{Z} \times \mathbb{Z})^\#$.

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(b) The groups $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ do admit many pairwise homeomorphic non-discrete group topologies. It is known [19] for each $A \in \mathcal{S}(G)$ that $w(G, \mathcal{T}_A) = |A|$ and that the elements of A are exactly the $\mathcal{T}_A$-continuous homomorphisms from G to $\mathbb{T}$, so distinct elements of $\mathcal{S}(G)$ induce distinct group topologies on G. When $|G| = \omega$ and $A \in \mathcal{S}(G)$ is countable, the space $(G, \mathcal{T}_A)$ is a countably infinite metrizable space without isolated points, hence according to a familiar theorem of Sierpiński [45] is homeomorphic to $\mathbb{Q}$ in its usual topology. Thus both $G = \mathbb{Z}$ and $G = \mathbb{Z} \times \mathbb{Z}$ admit $\mathfrak{c}$-many distinct metrizable topological group topologies (of the form $\mathcal{T}_A$) [15], each satisfying $(G, \mathcal{T}_A) =_{\text{top}} \mathbb{Q}$.

(c) The observations in (a) and (b) shed no direct light on the question asked above, since when $|G| = \omega$ the space $G^\#$ is a separable space with weight $w(G^\#) = |\text{Hom}(G, \mathbb{T})| = \mathfrak{c}$, hence is not metrizable.

Remark 4.2. For a list and discussion of many fresh questions which have arisen after the solution [37,21] of van Douwen's general question [22], see §6 of [20]. See also Trigos-Arrieta [49] for a proof, answering related questions of Markov [38] and van Douwen [22], of this theorem: When $|G| > \omega$, the topological space $G^\#$ is not normal.

5. Alexander Fish[3]

The following question is well known (see for example [4, Question 11]) and seems to be wide open.

Is it true that for a finite partition of $\mathbb{N} = \bigcup_{i=1}^{k} C_i$ there exists $1 \leqslant i \leqslant k$ such that C_i contains a Pythagorean triple, i.e., $\{x, y, z\}$ such that $x^2 + y^2 = z^2$?

Recall that a set $S \subset \mathbb{N}$ is called *normal* if $1_S \in \{0, 1\}^{\mathbb{N}}$ is a normal infinite binary sequence, i.e., every finite binary word v appears in 1_S with the frequency $\frac{1}{2^{|v|}}$, where $|v|$ denotes the length of v.

The following problem is motivated by the above question but hopefully is easier.

Question 5.1. Is it true that for any normal set $S \subset \mathbb{N}$ there exists a Pythagorean triple which lies in S, i.e., there exist $x, y, z \in S$ such that $x^2 + y^2 = z^2$?

Remark. It was shown in [23] that for any normal set $S \subset \mathbb{N}$ there exist $x, y \in S$ such that $x^2 + y^2$ is a square.

6. Neil Hindman[4]

The following theorem was proved by Dona Strauss in [46], answering a question of Eric van Douwen.

Theorem 6.1. *If $\phi : \beta\mathbb{N} \to \mathbb{N}^*$ is a continuous homomorphism, then $\phi[\beta\mathbb{N}]$ is finite and $|\phi[\mathbb{N}^*]| = 1$.*

This raised the following natural question, which has been open ever since.

Question 6.2. Is there any non-trivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$?

By [36, Corollary 10.20], the above question is equivalent to each of the following two questions.

Question 6.3. Do there exist $p \neq q$ in $\mathbb{N}^*$ such that $p + q = q + p = p + p = q + q = q$?

Question 6.4. Is there a finite subsemigroup of $(\mathbb{N}^*, +)$ whose members are not all idempotent?

Switching to a Ramsey theoretic question we have:

Problem 6.5. Prove that whenever $\mathbb{N}$ is finitely colored there exist arbitrarily large finite sequences with all of their finite sums and finite products in one color.

This is stated as a problem rather than a question, because I am certain that it is a fact. The reader may recall that fact has only been proved (by Ron Graham in [31]) for sequences of length two and then only for two colors. But I am absolutely certain that it is a fact.

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7. Bruce Landman[5]

Let $w(s,t)$ be the 2-color van der Waerden number, i.e., $w(s,t)$ is the least positive integer n such that every (red, blue)-coloring of $\{1,2,\ldots,n\}$ admits either an s-term red arithmetic progression or a t-term blue arithmetic progression.

Problem 7.1. Prove that for all $s \geqslant t \geqslant 2$, $w(s,t) \geqslant w(s+1,t-1)$.

Or, a slightly weaker result:

Problem 7.2. Prove that for all positive integers m,

$$\max\bigl\{w(s,t)\colon\ s+t=2m\bigr\} = w(m,m).$$

Or, more generally (using r colors rather than 2)

Problem 7.3. Prove that over the hyperplane $x_1+x_2+\cdots+x_k=b$, the maximum of $w(x_1,x_2,\ldots,x_r)$ occurs when $|x_i-x_j| \leqslant 1$ for all i and j.

Another related question:

Question 7.4. For a 2-coloring $f:\{1,2,\ldots,n\} \to \{\text{red}, \text{blue}\}$, denote by R and B, the set of red members and blue members of $\{1,2,\ldots,n\}$ respectively. Is it true that for all positive integers s, there is a 2-coloring of $\{1,2,\ldots,w(s,s)-1\}$ with no monochromatic s-term arithmetic progression and such that $||R|-|B|| \leqslant 1$? (In other words for every s, is there is a "balanced" coloring of $\{1,2,\ldots,w(s,s)-1\}$ that avoids s-term monochromatic arithmetic progressions?)

8. Imre Leader[6]

Singletons and pairs without Hindman's theorem

Suppose that we colour the natural numbers with finitely many colours. Must there exist a sequence $x_1, x_2, \ldots$ such that all the x_i and all the $x_i + x_j$ $(i \neq j)$ are the same colour? The answer is, of course, yes, by Hindman's theorem. But is there a proof of this assertion that does *not* go via Hindman's theorem?

To make this a well-posed question, let us write as usual $FS(x_1,x_2,\ldots)$ for the set of all finite sums (without repetition) of $x_1, x_2, \ldots,$ and $FS_{\leqslant 2}(x_1,x_2,\ldots)$ for the set of all sums (without repetition) of at most two of the x_i.

Question 8.1. Does there exist a set S of natural numbers such that whenever S is finitely coloured there is a monochromatic set of the form $FS_{\leqslant 2}(x_1,x_2,\ldots)$, and yet S contains no set of the form $FS(y_1,y_2,\ldots)$?

Note that if we have such a set S then to show that it has the required partition property we would have to have a proof of the form mentioned at the start. This question, with some variants, first appeared in [33]. Interestingly, a more general question appears in [39], with a conjectured answer of 'yes', and another more general question appears in [32], with a conjectured answer of 'no'!

Infinite sums of Ramsey games

The *Ramsey game* is played on the edges of a K_N (a complete graph on N vertices). Two players take it in turn to claim edges, and the first player to complete a K_s wins the game. If the game ends with neither player having a K_s, then the game is a draw. By Ramsey's theorem, this game cannot end in a draw if N is large enough (say N is at least the Ramsey number $R(s,s)$), and in that case it is easy to check (by 'strategy-stealing') that the game must be a first-player win.

Now, what happens if we play not on one copy of K_N but on two disjoint copies? Again, this cannot end in a draw (for N at least $R(s,s)$), and so this too is a first-player win. Of course, the same applies to any finite disjoint union of copies of K_N. But what happens for infinitely many copies?

Question 8.2. Is the Ramsey game played on an infinite disjoint union of copies of K_N a first-player win (for N sufficiently large)?

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Note that if at the end of the game ('at time ω') neither player has occupied a K_s, then the game is a draw – there are no 'transfinite' moves.

It seems impossible that the second player could 'slow down' the first player by playing in more and more copies of K_N, but proving this seems remarkably elusive. A similar question is asked in [2], where we play on an infinite complete board K_ω – the question above is designed to be a 'locally finite' version of that question.

9. Amir Maleki[7]

References for the background material presented here are [14] and [36].

Let l_∞ be the space of all complex valued and bounded functions defined on the set of natural numbers $\mathbb{N}$. Then l_∞ is a commutative Banach algebra with identity.

Let $\beta\mathbb{N}$ be the set of all nonzero multiplicative linear functionals on l_∞. It is well known that $\beta\mathbb{N}$ with the Gelfand topology is a compact Hausdorff space and is a model for the Stone–Čech compactification of the discrete space $\mathbb{N}$. Furthermore for each n in $\mathbb{N}$ the mapping $\hat{n}: l_\infty \to \mathbb{C}$ defined by $\hat{n}(x) = x(n)$ is the imbedding of $\mathbb{N}$ into $\beta\mathbb{N}$. Now let $B(l_\infty)$ be the set of all bounded linear operators from l_∞ to itself. So $B(l_\infty)$ is the complex Banach algebra of all bounded linear operators with operator norm.

For each $\xi \in \beta\mathbb{N}$ define $T_\xi : l_\infty \to l_\infty$ by $(T_\xi x)(n) = \xi(L_n x)$, $x \in l_\infty$, $n \in \mathbb{N}$, where L_n is a mapping from l_∞ to itself defined by $(L_n x)(t) = x(n+t)$. One can show that for each ξ in $\beta\mathbb{N}$, T_ξ is in $B(l_\infty)$ and $T_{\hat{n}} = L_n$. Also one can show that the mapping ψ defined from $\beta\mathbb{N}$ into $B(l_\infty)$ given by $\psi(\xi) = T_\xi$ is one-to-one. Now for ξ and η in $\beta\mathbb{N}$ define $\xi + \eta$ by $(\xi + \eta)(x) = \xi(T_\eta x)$. Then $\xi + \eta$ is in $\beta\mathbb{N}$. In particular $\hat{n} + \hat{m} = \widehat{n+m}$ where n and m are in $\mathbb{N}$. Furthermore $(\beta\mathbb{N}, +)$ becomes a compact right topological semigroup and the mapping ψ is an algebraic imbedding of $\beta\mathbb{N}$ into the Banach algebra $B(l_\infty)$.

Now let ξ be in $\beta\mathbb{N}$ such that $\xi + \xi \neq \xi$ and for all n in $\mathbb{N}$, $\xi \neq \hat{n}$. The spectrum of T_ξ denoted by $\sigma(T_\xi)$ is the set of all complex numbers λ such that the operator $\lambda I - T_\xi$, where I is the identity mapping, has no inverse in $B(l_\infty)$.

Question 9.1. What is the spectrum of T_ξ?

10. Igor Protasov[8]

Question 10.1. Let G be a group and let $G = A_1 \cup \cdots \cup A_n$ be a finite partition of G. Do there exist a subset $K \subseteq G$, $|K| \leqslant n$, and $i \in \{1, 2, \ldots, n\}$ such that $G = KA_iA_i^{-1}$?

This is so if either G is amenable or $n = 2$ [1, Theorems 12.7 and 12.8].

A subset S of a group G is called *very thick* if, for every subset $F \subseteq G$, $|F| < |G|$, there exists $g \in G$ such that $gF \subseteq S$. Every infinite group G of regular cardinality can be partitioned in $|G|$ very thick subsets [42, Theorem 5.3.6], [16, Theorem 2.4].

Question 10.2. Can every group of singular cardinality be partitioned into at least two very thick subsets?

Let G be a group and let κ be a cardinal. A subset S of G is called κ-large if there exists a subset $F \subseteq G$, $|F| < \kappa$, such that $G = FS = SF$. By [1, Theorem 3.12], every infinite group G can be partitioned into countably many $\aleph_0$-large subsets.

Question 10.3. Let G be an infinite group and let κ be an infinite cardinal such that $\kappa \leqslant |G|$. Can G be partitioned into κ κ-large subsets?

This is so if G is abelian [41, Theorem 9.3.4].

Question 10.4. Let G be a free abelian group of rank $\aleph_2$. Can G be partitioned into $\aleph_2$ $\aleph_1$-large subsets?

Let G be an infinite group and let κ be an infinite cardinal such that $\kappa \leqslant |G|$. A subset S of G is called κ-small if $S \setminus FSF$ is κ-large for each $F \subseteq G$ such that $|F| < \kappa$. By [40, Theorem 3.1], for every uncountable abelian group G, there exists a $|G|$-small subset which is not $\aleph_0$-small.

Question 10.5. Let G be an infinite abelian group and let κ and κ' be infinite cardinals such that $\kappa < \kappa' \leqslant |G|$. Does there exist a κ'-small but not κ-small subset of G?

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11. Dona Strauss[9]

At the conference in Oxford, I mentioned some open problems about the algebra of $\beta\mathbb{N}$ which sound tantalizingly simple, but have remained open for several decades.

Question 11.1. Is there an element $p \in (\beta\mathbb{N}, +)$ which is not idempotent, but for which $p + p$ is idempotent?

This is equivalent to asking whether $\beta\mathbb{N}$ contains any element p which generates a finite subsemigroup of $\beta\mathbb{N}$ containing more than one element. We know, because of Zelenyuk's theorem [36, Theorem 7.17], that $\beta\mathbb{N}$ contains no non-trivial finite groups; but we do not know whether it contains any finite semigroups whose elements are not all idempotent.

Question 11.2. Is there a strictly increasing sequence of principal left ideals in $\beta\mathbb{N}$?

Mary Ellen Rudin formulated this question in the 1970's and it has remained unanswered. It is not hard to construct infinite decreasing chains of principal left ideals or of idempotents in $\beta\mathbb{N}$; but no one has succeeded in constructing infinite increasing chains of either kind. This situation is reminiscent of a property of the ordinals; namely, that infinite chains obviously exist in one direction, but do not exist in the other.

Question 11.3. Is there a minimal idempotent in $\beta\mathbb{N}$ which is also maximal?

(The ordering of idempotents is defined by stating that $p \leqslant q$ if $pq = qp = p$.)

Question 11.4. Is there a ZFC proof that left maximal idempotents exist in $\beta\mathbb{N}$?

There is a MA proof that they exist [34, Theorem 4.1]. But there is no known proof in ZFC.

12. Yevhen Zelenyuk[10]

The semigroup $\mathbb{H}$ is defined as a closed subsemigroup of $\beta\mathbb{N}$ by

$$\mathbb{H} = \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}} 2^n\mathbb{N}.$$

In [35], it was shown that the structure group of $K(\mathbb{H})$ contains copies of the free group on 2^{2^ω} generators. (All maximal groups in the smallest ideal of a compact right topological semigroup are isomorphic. Any copy of this group is called the *structure group* of the smallest ideal.)

Question 12.1. Is the structure group of $K(\mathbb{H})$ itself a free semigroup?

A semigroup Q in $\mathbb{N}^*$ is *left saturated* if for every $p \in \beta\mathbb{N} \setminus Q$, $(p + Q) \cap Q = \emptyset$. Every finite left saturated semigroup in $\mathbb{N}^*$ is a projective in the category $\mathfrak{F}$ of finite semigroups, and under Martin's Axiom, every projective in $\mathfrak{F}$ has a left saturated copy in $\mathbb{N}^*$ [51]. Projectives in $\mathfrak{F}$ have been characterized in [50]. These are certain chains of rectangular semigroups. In [51], it was shown that every finite chain of idempotents has a left saturated ZFC-copy in $\mathbb{N}^*$.

Question 12.2. Which projectives in $\mathfrak{F}$ have left saturated ZFC-copies in $\mathbb{N}^*$? Is there anything different from chains of idempotents among such semigroups in $\mathbb{N}^*$?

Let $\kappa > \omega$ and let $G = \bigoplus_\kappa \mathbb{Z}_2$. For every $\alpha < \kappa$, let

$$A_\alpha = \left\{x \in G \setminus \{0\}\colon \min\operatorname{supp}(x) > \alpha\right\} \quad\text{and}\quad B_\alpha = \left\{x \in G \setminus \{0\}\colon \alpha \notin \operatorname{supp}(x)\right\}.$$

Define the semigroups $\mathbf{H}$ and $\mathbf{T}$ in βG by

$$\mathbf{H} = \bigcap_{\alpha<\kappa} c\ell_{\beta G} A_\alpha \quad\text{and}\quad \mathbf{T} = \bigcap_{\alpha<\kappa} c\ell_{\beta G} B_\alpha.$$

In [52], it was shown that finite groups in $\mathbf{H}$ are trivial.

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Question 12.3. Is there any non-trivial finite group in **T**?

Let $b\mathbb{Z}$ denote the Bohr compactification of the discrete group $\mathbb{Z}$ of integer numbers and let $\mathcal{T}$ be the topology on the additive semigroup $\mathbb{N}$ of natural numbers induced from $b\mathbb{Z}$.

Question 12.4. Is the largest semigroup compactification of $(\mathbb{N}, \mathcal{T})$ different from $b\mathbb{Z}$?

Question 12.5. Can every compact Hausdorff right topological semigroup be topologically and algebraically embedded into a compact Hausdorff right topological semigroup with a dense topological center?

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