Computers and Mathematics with Applications 64 (2012) 3436–3443

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of solutions for a fractional boundary value problem via the Mountain Pass method and an iterative technique☆

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ARTICLE INFO

Keywords:
Fractional differential equation
Solution
Boundary value problem
Critical point
Iterative technique

ABSTRACT

In this paper, we consider the existence of solution to the following fractional boundary value problem

$$\begin{cases} \frac{d}{dx}(p\,{}_0D_x^{-\beta}(u'(x)) + q\,{}_xD_1^{-\beta}(u'(x))) + f(x, u(x)) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where the constants $\beta \in (0, 1)$, ${}_0D_x^{-\beta}$ and ${}_xD_1^{-\beta}$ denote left and right Riemann–Liouville fractional integrals of order β respectively, $0 < p = 1 - q < 1$ and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Due to the general assumption on the constants p and q, the problem does not have a variational structure. Despite that, here we study it performing variational methods, combining with an iterative technique, and give an existence criteria of solution for the problem under suitable assumptions. The results extend the results in [F. Jiao, Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, Comput. Math. Appl. 62 (2011) 1181–1199].

1. Introduction

Fractional calculus have applications in many areas including fluid flow, electrical networks, probability and statistics, viscoelasticity, chemical physics and signal processing, and so on; see [1–8] and references therein. Fractional differential operators have got attention from many researchers which is mainly due to its application as a model for physical phenomena exhibiting anomalous diffusion.

In this paper, we investigate the solvability of the following fractional boundary value problem

$$\begin{cases} \frac{d}{dx}\left(p\,{}_0D_x^{-\beta}(u'(x)) + q\,{}_xD_1^{-\beta}(u'(x))\right) + f(x, u(x)) = 0, \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where the constant $\beta \in (0, 1)$, $0 < p = 1 - q < 1$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, ${}_0D_x^{-\beta}$ and ${}_xD_1^{-\beta}$ denote left and right Riemann–Liouville fractional integrals of order β respectively and are defined by

$${}_0D_x^{-\beta}u = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}u(s)ds, \qquad {}_xD_1^{-\beta}u = \frac{1}{\Gamma(\beta)}\int_x^1 (s-x)^{\beta-1}u(s)ds.$$

☆ Supported by NSF of China (10801065), FRFCU (lzujbky-2011-43, lzujbky-2012-k25) and SRF for ROCS, SEM.
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doi:10.1016/j.camwa.2012.02.023

Our interest in studying problem (1.1) comes from the fractional advection–dispersion equation, it describes nonsymmetric transition and can be a steady state for advection and nonsymmetric fractional dispersion equation; see [2,3,9].

Very recently in [10], in the special case of $p = q = \frac{1}{2}$, for problem (1.1), Jiao and Zhou study the existence of the problem by establishing corresponding variational structure in some suitable fractional space and applying the least action principle and Mountain Pass theorem.

For problem (1.1), since the appearance of left and right Riemann–Liouville fractional integral, it is difficult to find the equivalent integral equation corresponding to (1.1), so it seems that fixed point theorems could not be applied to this problem. Due to the general assumption $0 < p = 1 - q < 1$ on the constant p and q, problem (1.1) is not variational, we cannot find some functional such that its critical point is the solution corresponding to problem (1.1), so the well-developed critical point theory is of no avail for, at least, a direct attack to problem (1.1) above.

In recent years, De Figueiredo et al. [11] (see also [12]) considered the existence of solution for semilinear elliptic equation with the nonlinearity depending on the gradient of the solution. The approach used in these papers consists of associating with the problem a family of semilinear elliptic problems with no dependence of the gradient of the solution. This family of problems is variational, by applying Mountain Pass theorem, they obtained a sequence of solutions and proved that the weak limit of the sequence is a solution of the problem.

Motivated by the papers [11,12], in this paper, we attempt to use Mountain Pass theorem and iterative technique to study the existence of solution of problem (1.1). In order to use variational methods, we consider a family of fractional boundary value problem with variational structure, that is, for each $w \in H_0^\alpha(0, 1)$ (which will be defined in Section 2), we discuss the following problem

$$\begin{cases} \dfrac{d}{dx}(q\,{}_0D_x^{-\beta}(u'(x)) + q\,{}_xD_1^{-\beta}(u'(x))) + (p-q)\,{}_0D_x^{-\beta}(w'(x)) + f(x, u(x)) = 0, \quad x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

so we can solve problem (1.2) by variational methods. Then, for every $w \in H_0^\alpha(0, 1)$, we find a solution $u_w \in H_0^\alpha(0, 1)$ with some bounds. Next, by iterative technique, we get the existence of solution of (1.1) under suitable assumption.

This paper is organized as follows. In Section 2, some preliminaries are presented, the assumption on the problem and the main result are listed. Section 3 is devoted to give the proof of our main result.

2. Preliminaries and main result

To apply critical point theory to study the existence of solutions for problem (1.1), we shall state some basic notations and results which will be used in the proof of our main results.

For $\alpha > 0$, we define the space $J_{L,0}^\alpha(0, 1)$ or $J_{R,0}^\alpha(0, 1)$ [9, Defintion 2.5] as the completion of $C_0^\infty((0, 1))$ under the norm

$$\|u\|_{J_L^\alpha} = \left(\int_0^1 |u(x)|^2 dx + \int_0^1 |{}_0D_x^\alpha u|^2 dx\right)^{\frac{1}{2}},$$

or

$$\|u\|_{J_R^\alpha} = \left(\int_0^1 |u(x)|^2 dx + \int_0^1 |{}_xD_1^\alpha u|^2 dx\right)^{\frac{1}{2}},$$

where ${}_0D_x^\alpha u$ and ${}_xD_1^\alpha u$ denote left and right Riemann–Liouville fractional derivative of order α respectively and are defined by

$${}_0D_x^\alpha u = \frac{d^n}{dx^n}\,{}_0D_x^{\alpha-n} u \quad \text{and} \quad {}_xD_1^\alpha u = (-1)^n \frac{d^n}{dx^n}\,{}_xD_1^{\alpha-n} u,$$

where $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$, $n = \alpha$, if $\alpha \in \mathbb{N}$. For more properties of fractional operators, we refer to [6,7].

For $0 < \alpha < 1$, the fractional Sobolev space $H_0^\alpha(0, 1)$ defines as the completion of $C_0^\infty((0, 1))$ under the norm

$$\|u\| = \left(\int_0^1 |u(x)|^2 dx + \int_0^1 \int_0^1 \frac{|u(x) - u(y)|^2}{|x - y|^{1+2\alpha}} dx dy\right)^{\frac{1}{2}}.$$

For $\frac{1}{2} < \alpha < 1$, by embedding theorem, we know $H_0^\alpha(0, 1) \hookrightarrow C([0, 1])$ is compact and if $u \in H_0^\alpha(0, 1)$, then $u(0) = u(1) = 0$.

From [9, Theorem 2.13], we know for $\alpha > 0$, if $\alpha - \frac{1}{2} \notin \mathbb{N}$, then the spaces $J_{L,0}^\alpha(0, 1)$, $J_{R,0}^\alpha(0, 1)$ and $H_0^\alpha(0, 1)$ are equal and have equivalent norms. In view of the definition of $J_{L,0}^\alpha(0, 1)$, we know that $J_{L,0}^\alpha(0, 1)$ is reflexive, thus $H_0^\alpha(0, 1)$ is a reflexive space.

For the space $J_{L,0}^\alpha(0, 1)$, we have the following results.

Lemma 2.1. *If $u \in J_{L,0}^\alpha(0, 1)$, then ${}_0D_x^\alpha u$ exists a.e. in* $[0, 1]$.

Proof. Assume $u_m \in C_0^\infty((0,1))$ and $\|u_m - u\|_{J_L^\alpha} \to 0$ as $m \to \infty$. We let n denote the smallest integer which is greater than or equal to α.

Since ${}_0D_x^{\alpha-n}$ is a bounded linear operator from $L^2(0,1)$ to $L^2(0,1)$ [6] (see also [10]), in view of $u_m \to u$ in $L^2(0,1)$ as $m \to \infty$, we know ${}_0D_x^{\alpha-n}u_m \to {}_0D_x^{\alpha-n}u$ as $m \to \infty$, from $\|u_m - u_l\|_{J_L^\alpha} \to 0$ as $m, l \to \infty$, we get $\|{}_0D_x^{\alpha-n}(u_m - u_l)\|_{H^n(0,1)} \to 0$ as $m, l \to \infty$, so there exists $v \in H^n(0,1)$ such that ${}_0D_x^{\alpha-n}u_m \to v$ in $H^n(0,1)$. Hence ${}_0D_x^{\alpha-n}u = v$, and so ${}_0D_x^\alpha u$ exists a.e. in $(0,1)$. □

Lemma 2.2. *If* $\frac{1}{2} < \alpha < 1$ *and* $u \in H_0^\alpha(0,1)$, *then we have*

$$\|u\|_\infty \le \frac{1}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}\|{}_0D_x^\alpha u\|_{L^2}, \tag{2.1}$$

$$\|u\|_{L^2} \le \frac{1}{\Gamma(\alpha+1)}\|{}_0D_x^\alpha u\|_{L^2}, \tag{2.2}$$

$$|\cos(\pi\alpha)|\,\|{}_0D_x^\alpha u\|_{L^2}^2 \le -\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx \le \frac{1}{|\cos(\pi\alpha)|}\|{}_0D_x^\alpha u\|_{L^2}^2, \tag{2.3}$$

$$\int_0^1 |{}_xD_1^\alpha u|^2 dx \le \frac{1}{|\cos(\pi\alpha)|^2}\|{}_0D_x^\alpha u\|_{L^2}^2. \tag{2.4}$$

Proof. For $u \in C_0^\infty((0,1))$, by a similar proof of Propositions 3.2 and 4.1 in [10], we know the inequalities (2.1)–(2.4) hold. By density, we know the conclusions are satisfied. □

From (2.2), we know the space $H_0^\alpha(0,1)$ has an equivalent norm $\|{}_0D_x^\alpha u\|_{L^2}$. So, hereafter, we denote

$$\|u\|_\alpha = \|{}_0D_x^\alpha u\|_{L^2}$$

as a norm in $H_0^\alpha(0,1)$.

Definition 2.3. Let $\alpha = 1 - \frac{\beta}{2}$. A function $u \in H_0^\alpha(0,1)$ is called a weak solution of (1.1) if

$$p\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha v dx + q\int_0^1 {}_0D_x^\alpha v \cdot {}_xD_1^\alpha u dx + \int_0^1 f(x,u)v dx = 0, \tag{2.5}$$

for every $v \in H_0^\alpha(0,1)$.

Definition 2.4. Let $\alpha = 1 - \frac{\beta}{2}$. A function $u \in H_0^\alpha(0,1)$ is called a solution of (1.1) if $p_0D_x^{2\alpha-1}u - q_xD_1^{2\alpha-1}u$ is derivable in $x \in (0,1)$ and

$$\frac{d}{dx}(p\,{}_0D_x^{2\alpha-1}u - q\,{}_xD_1^{2\alpha-1}u) + f(x,u) = 0, \quad x \in (0,1).$$

Remark 2.5. From Lemma 2.1, we know if $u \in H_0^\alpha(0,1)$, then ${}_0D_x^\alpha u$ and ${}_xD_1^\alpha u$ exist a.e. in $(0,1)$ and belong to $L^2(0,1)$. Hence, ${}_0D_x^{2\alpha-1}u$ and ${}_xD_1^{2\alpha-1}u$ exist a.e. in $(0,1)$.

For given $w \in H_0^\alpha(0,1)$, we define functional I_w on $H_0^\alpha(0,1)$ as

$$I_w(u) = -q\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx - \int_0^1 F(x,u)dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u dx, \quad u \in H_0^\alpha(0,1),$$

where $F(x,u) = \int_0^u f(x,s)ds$. Clearly, by the continuous assumption on f, we have $I_w \in C^1(H_0^\alpha(0,1), \mathbb{R})$ and

$$I_w'(u)v = -q\int_0^1 ({}_0D_x^\alpha u \cdot {}_xD_1^\alpha v + {}_0D_x^\alpha v \cdot {}_xD_1^\alpha u)dx - \int_0^1 f(x,u)v dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha v dx, \tag{2.6}$$

for $u, v \in H_0^\alpha(0,1)$.

Lemma 2.6. *If* $u \in H_0^\alpha(0,1)$ *is a weak solution of* (1.1), *then* u *is a solution of* (1.1).

Proof. We can take $v \in C_0^\infty((0,1))$ in (2.5), similar to the argument of Theorem 4.2 in [10], we obtain

$$\int_0^1 \left(p\,{}_0D_x^{2\alpha-1}u - q\,{}_xD_1^{2\alpha-1}u + \int_0^x f(s,u(s))ds\right) v'(x)dx = 0,$$

for every $v \in C_0^\infty((0, 1))$. So, there exist constant C, such that

$$p\, {}_0D_x^{2\alpha-1}u - q\, {}_xD_1^{2\alpha-1}u + \int_0^x f(s, u(s))ds = C,$$

and then

$$\frac{d}{dx}(p\, {}_0D_x^{2\alpha-1}u - q\, {}_xD_1^{2\alpha-1}u) + f(x, u) = 0, \quad x \in (0, 1). \quad \square$$

We impose f the following condition.

(H1) For every $R > 0$, there is $M(R) > 0$ such that

$$|f(x, \xi_1) - f(x, \xi_2)| \le M(R)|\xi_1 - \xi_2| \quad \text{for all } x \in [0, 1], \xi_1, \xi_2 \in \mathbb{R};$$

(H2) $\lim_{\xi \to 0} \frac{f(x,\xi)}{\xi} = 0$ uniformly with respect to $x \in [0, 1]$;

(H3) there exist $\mu > 2$ and $M \ge 0$ such that

$$0 < \mu F(x, \xi) \le \xi f(x, \xi), \quad x \in [0, 1], \xi \in \mathbb{R}, |\xi| \ge M,$$

where $F(x, \xi) = \int_0^\xi f(x, s)ds$;

(H4) there are positive constant c and $s > 1$ such that

$$|f(x, \xi)| \le c(1 + |\xi|^s), \quad x \in [0, 1], \xi \in \mathbb{R}.$$

We note that (H3) implies there are $c_0 \ge 0$, $c_1, c_2 > 0$ such that

$$F(x, \xi) \le \frac{1}{\mu} f(x, \xi)\xi + c_0 \quad \text{for } x \in [0, 1], \xi \in \mathbb{R}, \tag{2.7}$$

$$F(x, \xi) \ge c_1|\xi|^\mu - c_2 \quad \text{for } x \in [0, 1], \xi \in \mathbb{R}, \tag{2.8}$$

and assumptions (H2) and (H4) yield that for any $\delta > 0$, there exists $c_3(\delta) > 0$ such that

$$|F(x, \xi)| \le \delta|\xi|^2 + c_3(\delta)|\xi|^{s+1}. \tag{2.9}$$

In the following, for $R > 0$, we denote by L_R as

$$L_R = \sup\left\{ \frac{|f(x, \xi_1) - f(x, \xi_2)|}{|\xi_1 - \xi_2|}, t \in [0, 1], |\xi_i| \le R, i = 1, 2, \xi_1 \ne \xi_2 \right\}. \tag{2.10}$$

For problem (1.1), since the symmetric position of the constants p and q lie in, without loss of generality, we can assume that $p \ge q$.

For convenience, hereafter, we denote $\alpha = 1 - \frac{\beta}{2}$,

$$a = \frac{q(\mu - 2)}{\mu}|\cos(\pi\alpha)|, \qquad b = \frac{(p - q)(\mu - 1)}{\mu|\cos(\pi\alpha)|}, \tag{2.11}$$

where μ is given in (H3). Assume that $a^2 > b^2 + b$, we take

$$\varepsilon_1 = \frac{a - \sqrt{a^2 - b(b + 1)}}{2(1 + b)}, \qquad \varepsilon_2 = \frac{a + \sqrt{a^2 - b(b + 1)}}{2(1 + b)}. \tag{2.12}$$

Let us fix a function $\varphi \in H_0^\alpha(0, 1)$ with $\|\varphi\|_\alpha = 1$, for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, we define

$$\bar{t} = \bar{t}(\varepsilon) = \left[\frac{2}{c_1\mu\|\varphi\|_{L^\mu}^\mu} \left(\frac{q}{|\cos(\pi\alpha)|} + \frac{(p - q)^2}{4\varepsilon|\cos(\pi\alpha)|^2} \right) \right]^{\frac{1}{\mu-2}},$$

$$C(\varepsilon) = \left(\frac{q}{|\cos(\pi\alpha)|} + \frac{(p - q)^2}{4\varepsilon|\cos(\pi\alpha)|^2} \right) \bar{t}^2 - c_1\|\varphi\|_{L^\mu}^\mu \bar{t}^\mu + c_2 + c_0, \tag{2.13}$$

$$R_1 = R_1(\varepsilon) = \left(\frac{4\varepsilon C(\varepsilon)}{4a\varepsilon - 4(b + 1)\varepsilon^2 - b} \right)^{\frac{1}{2}}, \qquad R_2 = R_2(\varepsilon) = \frac{R_1}{\Gamma(\alpha)(2\alpha - 1)^{\frac{1}{2}}}, \tag{2.14}$$

where c_0, c_1, c_2 are given in (2.7) and (2.8).

The main result of this paper is the following.

Theorem 2.7. *Assume that* (H1)–(H4) *hold and* $p \geq q$, $a^2 > b^2 + b$. *If there exist* $\delta \in \left(0, \frac{q|\cos(\pi\alpha)|(\Gamma(\alpha+1))^2}{2}\right)$ *and* $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ *such that*

$$\left(\frac{2(p-q)R_1}{q|\cos(\pi\alpha)|^2}\right)^{s-1} < \frac{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^{s+1}}{c_3(\delta)}\left(\frac{q|\cos(\pi\alpha)|}{2} - \frac{\delta}{(\Gamma(\alpha+1))^2}\right), \tag{2.15}$$

and

$$L_{R_2} < \left(2q|\cos(\pi\alpha)| - \frac{p-q}{|\cos(\pi\alpha)|}\right)(\Gamma(\alpha+1))^2, \tag{2.16}$$

then problem (1.1) *has a nontrivial solution.*

3. The proof of main result

In this section, we give the proof of Theorem 2.7 by Mountain Pass theorem [13,14] and iterative technique.

Proof of Theorem 2.7. Let R_1 be as in (2.14). Let us fix $w \in H_0^\alpha(0,1)$ with $\|w\|_\alpha \leq R_1$.

In order to prove Theorem 2.7, we proceed by three steps.

Step 1: For given $w \in H_0^\alpha(0,1)$ with $\|w\|_\alpha \leq R_1$, we prove that I_w has a nontrivial critical point in $H_0^\alpha(0,1)$ by the Mountain Pass theorem.

In order to apply Mountain Pass theorem, we first show that there exist $\rho, \beta_1 > 0$ such that $I_w(u) \geq \beta_1$ for $u \in \{u \in H_0^\alpha(0,1) : \|u\|_\alpha = \rho\}$.

In fact, by (2.9), Hölder's inequality, (2.3), (2.2), (2.1) and (2.4), we have

$$\begin{aligned}
I_w(u) &= -q\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx - \int_0^1 F(x,u)dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u dx \\
&\geq -q\int_0^1 {}_0D_x^\alpha u \cdot {}_xD_1^\alpha u dx - \delta\int_0^1 |u|^2 dx - c_3(\delta)\int_0^1 |u|^{s+1}dx - (p-q)\|{}_0D_x^\alpha w\|_{L^2} \cdot \|{}_xD_1^\alpha u\|_{L^2} \\
&\geq \left(q|\cos(\pi\alpha)| - \frac{\delta}{(\Gamma(\alpha+1))^2}\right)\|u\|_\alpha^2 - \frac{c_3(\delta)\|u\|_\alpha^{s+1}}{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^{s+1}} - \frac{(p-q)R_1}{|\cos(\pi\alpha)|}\|u\|_\alpha \\
&= \left(\frac{q|\cos(\pi\alpha)|}{2} - \frac{\delta}{(\Gamma(\alpha+1))^2} - \frac{c_3(\delta)\|u\|_\alpha^{s-1}}{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^{s+1}}\right)\|u\|_\alpha^2 + \left(\frac{q|\cos(\pi\alpha)|}{2}\|u\|_\alpha - \frac{(p-q)R_1}{|\cos(\pi\alpha)|}\right)\|u\|_\alpha.
\end{aligned}$$

By the assumption (2.15), we can choose $\rho > 0$ such that

$$\frac{q|\cos(\pi\alpha)|}{2} - \frac{\delta}{(\Gamma(\alpha+1))^2} > \frac{c_3(\delta)}{(\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}})^{s+1}}\rho^{s-1}$$

and

$$\frac{q|\cos(\pi\alpha)|}{2}\rho > \frac{(p-q)R_1}{|\cos(\pi\alpha)|}.$$

Hence, now, let $u \in H_0^\alpha(0,1)$ with $\|u\|_\alpha = \rho$, we know that there exists $\beta_1 > 0$ such that for $\|u\|_\alpha = \rho$, $I_w(u) \geq \beta_1$ uniformly for $w \in H_0^\alpha(0,1)$ with $\|w\| \leq R_1$.

For given $\varphi \in H_0^\alpha(0,1)$ with $\|\varphi\|_\alpha = 1$, by (2.3), (2.4) and (2.8), for $t > 0$, we obtain that

$$\begin{aligned}
I_w(t\varphi) &= -qt^2\int_0^1 {}_0D_x^\alpha \varphi \cdot {}_xD_1^\alpha \varphi dx - \int_0^1 F(x,t\varphi)dx - (p-q)t\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha \varphi dx \\
&\leq \frac{qt^2}{|\cos(\pi\alpha)|} - c_1 t^\mu \int_0^1 |\varphi|^\mu dx + c_2 + t(p-q)\|w\|_\alpha \|{}_xD_1^\alpha \varphi\|_{L^2} \\
&\leq \frac{qt^2}{|\cos(\pi\alpha)|} - c_1 t^\mu \int_0^1 |\varphi|^\mu dx + \frac{t(p-q)}{|\cos(\pi\alpha)|}\|w\|_\alpha + c_2 \\
&\to -\infty, \quad \text{as } t \to \infty.
\end{aligned}$$

Thus, there exists $t_0 > 0$ such that $e = t_0\varphi \in H_0^\alpha(0,1)$ satisfies $\|e\|_\alpha > \rho$ and $I_w(e) < 0$.

In order to apply Mountain Pass theorem to derive I_w has a critical point in $H_0^\alpha(0,1)$, now it suffice to show I_w satisfies P.S. condition.

In fact, let $\{u_n\} \subset H_0^\alpha(0,1)$, such that $|I_w(u_n)| \le K$ for some positive constant K and $I'_w(u_n) \to 0$ as $n \to \infty$. Then, using the condition (H3) and (2.6), we have

$$\begin{aligned} K &\ge I_w(u_n) \\ &= -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - \int_0^1 F(x,u_n)dx - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u_n dx \\ &\ge -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - \frac{1}{\mu}\int_0^1 u_n f(x,u_n)dx - c_0 - (p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u_n dx \\ &= \left(\frac{2}{\mu}-1\right) q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx + \frac{1}{\mu}I'_w(u_n)u_n + \left(\frac{1}{\mu}-1\right)(p-q)\int_0^1 {}_0D_x^\alpha w \cdot {}_xD_1^\alpha u_n dx - c_0 \\ &\ge \frac{q(\mu-2)}{\mu}|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \frac{1}{\mu}\|I'_w(u_n)\|\|u_n\|_\alpha - \frac{(\mu-1)(p-q)}{\mu|\cos(\pi\alpha)|}\|u_n\|_\alpha\|w\|_\alpha - c_0. \end{aligned}$$

Combining with $I'_w(u_n) \to 0$ as $n \to \infty$, we have $\{u_n\}$ is bounded in $H_0^\alpha(0,1)$. Therefore, without loss of generality, we can assume that $u_n \rightharpoonup u$ in $H_0^\alpha(0,1)$ and $u_n \to u$ in $C([0,1])$.

It follows from (H1) that

$$\int_0^1 (f(x,u_n) - f(x,u))(u_n - u)dx \to 0 \quad \text{as } n \to \infty,$$

and observe that

$$\begin{aligned} 2q|\cos(\pi\alpha)|\|u_n-u\|_\alpha^2 &\le -2q\int_0^1 {}_0D_x^\alpha (u_n-u) \cdot {}_xD_1^\alpha (u_n-u)dx \\ &= (I'_w(u_n) - I'_w(u))(u_n-u) + \int_0^1 (f(x,u_n)-f(x,u))(u_n-u)dx, \\ &\quad (I'_w(u_n) - I'_w(u))(u_n-u) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, $u_n \to u$ in $H_0^\alpha(0,1)$.

Therefore, by Mountain Pass theorem, I_w has a nontrivial critical point u_w in $H_0^\alpha(0,1)$ with

$$I_w(u_w) = \inf_{g\in\Gamma}\max_{u\in g([0,1])} I_w(u) \ge \beta_1 > 0,$$

where $\Gamma = \{g \in C([0,1], H_0^\alpha) | g(0) = 0, g(1) = e\}$.

Step 2: We construct iterative sequence $\{u_n\}$ and estimate its norm in $H_0^\alpha(0,1)$.

For $u_1 \equiv 0$, by Step 1, we know I_{u_1} has a nontrivial critical point u_2. If we can prove $\|u_2\|_\alpha \le R_1$, then, by Step 1, we get I_{u_2} has a critical point u_3. So, in order to obtain iterative sequence $\{u_n\}$, we need prove that if we assume that for $\|u_{n-1}\|_\alpha \le R_1$, then u_n, the nontrivial critical point of $I_{u_{n-1}}$ obtained by Step 1, satisfies $\|u_n\|_\alpha \le R_1$.

Indeed, by the Mountain Pass characterization of the critical level, (2.3) and (2.4) and Cauchy's inequality with positive constant ε, we obtain

$$\begin{aligned} |I_{u_{n-1}}(u_n)| &\le \max_{t\in[0,\infty)} I_{u_{n-1}}(t\varphi) \\ &\le \frac{qt^2}{|\cos(\pi\alpha)|} - c_1 t^\mu \int_0^1 |\varphi|^\mu dx + c_2 + t(p-q)\|u_{n-1}\|_\alpha \|{}_xD_1^\alpha \varphi\|_{L^2} \\ &\le \frac{qt^2}{|\cos(\pi\alpha)|} - c_1\|\varphi\|_{L^\mu}^\mu t^\mu + \frac{t(p-q)}{|\cos(\pi\alpha)|}\|u_{n-1}\|_\alpha + c_2 \\ &\le \left(\frac{q}{|\cos(\pi\alpha)|} + \frac{(p-q)^2}{4\varepsilon|\cos(\pi\alpha)|^2}\right)t^2 - c_1\|\varphi\|_{L^\mu}^\mu t^\mu + \varepsilon\|u_{n-1}\|_\alpha^2 + c_2. \end{aligned}$$

If we take

$$g(t) = \left(\frac{q}{|\cos(\pi\alpha)|} + \frac{(p-q)^2}{4\varepsilon|\cos(\pi\alpha)|^2}\right)t^2 - c_1\|\varphi\|_{L^\mu}^\mu t^\mu + \varepsilon\|u_{n-1}\|_\alpha^2 + c_2,$$

then by simple calculation, we get $g(t)$ has a maximum at $t = \bar{t}$, where

$$\bar{t} = \left[\frac{2}{c_1\mu\|\varphi\|_{L^\mu}^\mu}\left(\frac{q}{|\cos(\pi\alpha)|} + \frac{(p-q)^2}{4\varepsilon|\cos(\pi\alpha)|^2}\right)\right]^{\frac{1}{\mu-2}}.$$

Hence, using (2.7) and $I'_{u_{n-1}}(u_n)u_n = 0$, we have

$$\begin{aligned} g(\bar{t}) &\geq I_{u_{n-1}}(u_n) \\ &= -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - \int_0^1 F(x, u_n)dx - (p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx \\ &\geq -q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx - \frac{1}{\mu}\int_0^1 u_n f(x, u_n)dx - c_0 - (p-q)\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx \\ &= \left(\frac{2}{\mu} - 1\right) q\int_0^1 {}_0D_x^\alpha u_n \cdot {}_xD_1^\alpha u_n dx + \frac{1}{\mu} I'_{u_{n-1}}(u_n)u_n - \frac{(p-q)(\mu-1)}{\mu}\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx - c_0 \\ &\geq \frac{q(\mu-2)}{\mu}|\cos(\pi\alpha)|\|u_n\|_\alpha^2 - \frac{(p-q)(\mu-1)}{\mu}\int_0^1 {}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n dx - c_0. \end{aligned}$$

So,

$$\begin{aligned} \frac{q(\mu-2)}{\mu}|\cos(\pi\alpha)|\|u_n\|_\alpha^2 &\leq g(\bar{t}) + \frac{(p-q)(\mu-1)}{\mu}\int_0^1 |{}_0D_x^\alpha u_{n-1} \cdot {}_xD_1^\alpha u_n| dx + c_0 \\ &\leq g(\bar{t}) + \frac{(p-q)(\mu-1)}{\mu}\|{}_0D_x^\alpha u_{n-1}\|_{L^2}\|{}_xD_1^\alpha u_n\|_{L^2} + c_0 \\ &\leq g(\bar{t}) + \frac{(p-q)(\mu-1)}{\mu|\cos(\pi\alpha)|}\|u_{n-1}\|_\alpha\|u_n\|_\alpha + c_0 \\ &= \varepsilon\|u_{n-1}\|_\alpha^2 + \frac{(p-q)(\mu-1)}{\mu|\cos(\pi\alpha)|}\|u_{n-1}\|_\alpha\|u_n\|_\alpha + C(\varepsilon) \\ &\leq \varepsilon\|u_{n-1}\|_\alpha^2 + \frac{(p-q)(\mu-1)}{\mu|\cos(\pi\alpha)|}\left(\varepsilon\|u_n\|_\alpha^2 + \frac{1}{4\varepsilon}\|u_{n-1}\|_\alpha^2\right) + C(\varepsilon), \end{aligned}$$

where $C(\varepsilon)$ is given in (2.13). Therefore, it follows from the definitions of a and b given in (2.11) that

$$(a - b\varepsilon)\|u_n\|_\alpha^2 \leq \left(\varepsilon + \frac{b}{4\varepsilon}\right)\|u_{n-1}\|_\alpha^2 + C(\varepsilon).$$

When $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, where $\varepsilon_1, \varepsilon_2$ are given in (2.12), we have

$$\varepsilon + \frac{b}{4\varepsilon} < a - b\varepsilon.$$

Then, we obtain

$$\begin{aligned} \|u_n\|_\alpha^2 &\leq \frac{\varepsilon + \frac{b}{4\varepsilon}}{a - b\varepsilon}\|u_{n-1}\|_\alpha^2 + \frac{C(\varepsilon)}{a - b\varepsilon} \\ &\leq \left(\frac{\varepsilon + \frac{b}{4\varepsilon}}{a - b\varepsilon}\right)^{n-1}\|u_1\|_\alpha^2 + \frac{C(\varepsilon)}{a - b\varepsilon}\sum_{k=0}^{n-2}\left(\frac{\varepsilon + \frac{b}{4\varepsilon}}{a - b\varepsilon}\right)^k \\ &\leq \|u_1\|_\alpha^2 + \frac{4\varepsilon C(\varepsilon)}{4a\varepsilon - 4(b+1)\varepsilon^2 - b}. \end{aligned}$$

Therefore, if we take $u_1 \equiv 0$ and let u_n be a nontrivial critical point of $I_{u_{n-1}}$ for $n = 2, 3, \ldots$, then, from the above argument, we get $\|u_n\|_\alpha \leq R_1$ and $I_{u_{n-1}}(u_n) \geq \beta_1 > 0$ for $n = 2, 3, \ldots$.

Step 3: We show the iterative sequence $\{u_n\}$ constructed in Step 2 is convergent to a nontrivial solution of (1.1).

We intend to prove $\{u_n\}$ is a Cauchy sequence in $H_0^\alpha(0, 1)$.

Indeed, since $\|u_n\|_\alpha \leq R_1$, in view of (2.1) and the definition of R_2, $\|u_n\|_\infty \leq R_2$. So by (2.3) and (2.6), $I'_{u_n}(u_{n+1})(u_{n+1} - u_n) = 0$, $I'_{u_{n-1}}(u_n)(u_{n+1} - u_n) = 0$, (H2), (2.2), we get

$$\begin{aligned} 2q|\cos(\pi\alpha)|\|u_{n+1} - u_n\|_\alpha^2 &\leq -2q\int_0^1 {}_0D_x^\alpha(u_{n+1} - u_n) \cdot {}_xD_1^\alpha(u_{n+1} - u_n)dx \\ &= (I'_{u_n}(u_{n+1}) - I'_{u_{n-1}}(u_n))(u_{n+1} - u_n) + \int_0^1 (f(x, u_{n+1}) - f(x, u_n))(u_{n+1} - u_n)dx \end{aligned}$$

$$+ (p-q)\int_0^1 {}_0D_x^\alpha(u_n-u_{n-1})\cdot {}_xD_1^\alpha(u_{n+1}-u_n)dx$$
$$\le \frac{L_{R_2}}{(\Gamma(\alpha+1))^2}\|u_{n+1}-u_n\|_\alpha^2+\frac{p-q}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\alpha.$$

Hence,

$$\left(2q|\cos(\pi\alpha)|-\frac{L_{R_2}}{(\Gamma(\alpha+1))^2}\right)\|u_{n+1}-u_n\|_\alpha\le\frac{p-q}{|\cos(\pi\alpha)|}\|u_n-u_{n-1}\|_\alpha.$$

By the assumption (2.16), we know

$$\frac{p-q}{|\cos(\pi\alpha)|}<2q|\cos(\pi\alpha)|-\frac{L_{R_2}}{(\Gamma(\alpha+1))^2}$$

and therefore $\{u_n\}$ is a Cauchy sequence in $H_0^\alpha(0,1)$. So, we can assume that $u_n\to u$ in $H_0^\alpha(0,1)$. In view of the definition of $\{u_n\}$, we know u is a weak solution and then by Lemma 2.6, it is a solution of (1.1). Since $I_{u_{n-1}}(u_n)\ge\beta_1>0$ for $n=2,3,\ldots,$ and the positive number β_1 does not depend on n, we know u is a nontrivial solution of (1.1). □

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