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The nearest trapezoidal form of a generalized left right fuzzy number

S. Abbasbandy [a,*], M. Amirfakhrian [b]

[a] Department of Mathematics, Imam Khomeini International University, Ghazvin 34194, Iran
[b] Department of Mathematics, Islamic Azad University, Central Tehran Branch, 14168-94351, Iran

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Abstract

We propose a new approach to assigning distance between fuzzy numbers. A pseudo-metric on the set of fuzzy numbers and a metric on the set of trapezoidal fuzzy numbers are described. The regular reducing functions and the Hausdorff metric are used to define the metric. Using this metric, we can approximate an arbitrary generalized left right fuzzy number with a trapezoidal one. Finally, powers and multiplication of trapezoidal fuzzy numbers are approximated.

Keywords: Generalized *LR* fuzzy number; Value; Ambiguity

1. Introduction

In some applications of fuzzy logic such as control theory, it may be better to use fuzzy numbers with the same type. Obviously, if we use a defuzzification rule which replaces a fuzzy set by a single number, we generally loose too many important information. Also, interval approximations for fuzzy numbers are considered in [3,6], where a fuzzy computation problem is converted into interval arithmetic problem. But, in this case, we may loose the modal (the core with height 1) of fuzzy numbers. Even in some works such as [2,4,7,8], we solve an optimization problem to obtain the nearest trapezoidal fuzzy

* Corresponding author.
E-mail address: abbasbandy@yahoo.com (S. Abbasbandy).

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number which is related to an arbitrary fuzzy number however in these cases it is not guaranteed to have the same modal value (or interval).

In this work we use value and ambiguity of fuzzy numbers same as [4,11]. For almost all distances such as distances defined in [1,10,14], we can find two different triangular or trapezoidal fuzzy numbers $\widetilde{A}, \widetilde{B}$ which $\widetilde{A} \approx \widetilde{B}$ but actually $\widetilde{A} \neq \widetilde{B}$.

The structure of the present paper is as follows. In Section 2 the basic concepts of our work are introduced. In Section 3 we introduce a metric D on $TRF(\mathbb{R})$, the set of all trapezoidal fuzzy numbers, which is a pseudo-metric on $F(\mathbb{R})$, the set of all fuzzy numbers on the real numbers. In Section 4 the nearest trapezoidal fuzzy number to an arbitrary generalized left right fuzzy number was introduced and a simple method for computing it, was presented. Section 5 contains some properties of the nearest approximation. In Section 6 any power of a trapezoidal fuzzy number is approximated with a trapezoidal one. In Section 7 we approximate multiplication of two trapezoidal fuzzy number with a trapezoidal one. Finally, we have come to conclusion in Section 8.

2. Basic concepts

Let $F(\mathbb{R})$ be the set of all fuzzy numbers on $\mathbb{R}$, i.e., the space of all fuzzy sets which are normal, fuzzy convex, upper semicontinuous with bounded supports [15].

Definition 2.1. A fuzzy set $\widetilde{A}$ is called a generalized left right fuzzy number (LR fuzzy number) if its membership function satisfy the following [5]

$$\mu_{\tilde{A}}(x) = \begin{cases} l_{\tilde{A}}(x), & l \leqslant x \leqslant m_l, \\ 1, & m_l \leqslant x \leqslant m_r, \\ r_{\tilde{A}}(x), & m_r \leqslant x \leqslant r, \\ 0, & \text{otherwise,} \end{cases}$$

where $l_{\tilde{A}}(x)$ is the left spread membership function that is an increasing continuous function on $[l, m_l]$ and $r_{\tilde{A}}(x)$ is the right spread membership function that is a decreasing continuous function on $[m_r, r]$ such that

$$l_{\tilde{A}}(l) = \begin{cases} 1, & l = m_l; \\ 0, & l \neq m_l, \end{cases} \qquad r_{\tilde{A}}(r) = \begin{cases} 1, & m_r = r; \\ 0, & m_r \neq r, \end{cases}$$

and $l_{\tilde{A}}(m_l) = r_{\tilde{A}}(m_r) = 1$. In addition, if $l_{\tilde{A}}(x)$ and $r_{\tilde{A}}(x)$ are linear, then $\widetilde{A}$ is a trapezoidal fuzzy number which is denoted by (l, m_l, m_r, r). In this case if $m_l = m_r (= m)$, we denote it by (l, m, r), which is a triangular fuzzy number. Let $TRF(\mathbb{R})$ be the set of trapezoidal fuzzy numbers on $\mathbb{R}$.

The α-cut representation of a fuzzy number $\widetilde{A}$ is the pair of functions $(L_{\tilde{A}}(\alpha), R_{\tilde{A}}(\alpha))$, defined by

$$L_{\tilde{A}}(\alpha) = \begin{cases} \inf\{x | \mu_{\tilde{A}}(x) \geqslant \alpha\} & \alpha > 0, \\ \inf\{x | x \in \operatorname{supp}(\mu)\} & \alpha = 0; \end{cases}$$

$$R_{\tilde{A}}(\alpha) = \begin{cases} \sup\{x | \mu_{\tilde{A}}(x) \geqslant \alpha\} & \alpha > 0, \\ \sup\{x | x \in \operatorname{supp}(\mu)\} & \alpha = 0. \end{cases}$$

Obviously, when $\widetilde{A} = (l, m_l, m_r, r)$, then $L_{\tilde{A}}(0) = l$ and $R_{\tilde{A}}(0) = r$.

Definition 2.2. A continuous function $s:[0,1]\to[0,1]$ with the following properties is called a *regular reducing* function [11]

1. $s(0)=0$,
2. $s(1)=1$,
3. $s(\alpha)$ is increasing,
4. $\int_0^1 s(\alpha)\mathrm{d}\alpha=\frac{1}{2}$.

Definition 2.3. The *value* and *ambiguity* of a fuzzy numbers $\widetilde{A}$ with respect to s, are defined by the following relations [14]

$$\begin{aligned}\mathrm{Val}(\widetilde{A})&=\int_0^1 s(\alpha)[R_{\tilde{A}}(\alpha)+L_{\tilde{A}}(\alpha)]\mathrm{d}\alpha,\\ \mathrm{Amb}(\widetilde{A})&=\int_0^1 s(\alpha)[R_{\tilde{A}}(\alpha)-L_{\tilde{A}}(\alpha)]\mathrm{d}\alpha.\end{aligned}$$

It is clear that if $\widetilde{A}=a$ be a crisp real number then $\mathrm{Val}(\widetilde{A})=a$ and $\mathrm{Amb}(\widetilde{A})=0$.

Definition 2.4. Let s be a regular reducing function. Then I_s, called a source number with respect to s, is defined by

$$I_s=\int_0^1 s(\alpha)\alpha\mathrm{d}\alpha. \tag{2.1}$$

3. Source metric

Definition 3.1. For $\widetilde{A},\widetilde{B}\in F(\mathbb{R})$, we define *source distance* of $\widetilde{A}$ and $\widetilde{B}$ by

$$D(\widetilde{A},\widetilde{B})=\frac{1}{2}\left\{|\mathrm{Val}(\widetilde{A})-\mathrm{Val}(\widetilde{B})|+|\mathrm{Amb}(\widetilde{A})-\mathrm{Amb}(\widetilde{B})|+\max\{|t_b-t_a|,|m_b-m_a|\}\right\},$$

where $[m_a,t_a]$ and $[m_b,t_b]$ are the cores of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, respectively.

Theorem 3.1. *For $\widetilde{A},\widetilde{B},\widetilde{C}\in F(\mathbb{R})$ the source distance, D, satisfies the following properties*:

1. $D(\widetilde{A},\widetilde{A})=0$,
2. $D(\widetilde{A},\widetilde{B})=D(\widetilde{B},\widetilde{A})$,
3. $D(\widetilde{A},\widetilde{C})\leqslant D(\widetilde{A},\widetilde{B})+D(\widetilde{B},\widetilde{C})$.

Example 3.1. Let $\mu_{\tilde{A}}(x)=\chi_{\{a\}}(x)$ and $\mu_{\tilde{B}}(x)=\chi_{\{b\}}(x)$, then

$$D(\widetilde{A},\widetilde{B})=\frac{1}{2}(|a-b|+|0-0|+|a-b|)=|a-b|.$$

Theorem 3.2. *For $\widetilde{A},\widetilde{B},\widetilde{A}',\widetilde{B}'\in TFR(\mathbb{R})$ and nonnegative real number k, the source distance D satisfies the following properties*:

1. $D(k\widetilde{A},k\widetilde{B})=kD(\widetilde{A},\widetilde{B})$,
2. $D(\widetilde{A}+\widetilde{B},\widetilde{A}'+\widetilde{B}')\leqslant D(\widetilde{A},\widetilde{A}')+D(\widetilde{B},\widetilde{B}')$.

Theorem 3.3. *Let* $\widetilde{A}, \widetilde{B} \in TRF(\mathbb{R})$, *then* $D(\widetilde{A}, \widetilde{B}) = 0$ *if and only if* $\widetilde{A} = \widetilde{B}$.

Proof. If $\widetilde{A} = \widetilde{B}$, from Theorem (3.1) we have $D(\widetilde{A}, \widetilde{B}) = 0$. Let $\widetilde{A} = (l_a, m_a, t_a, r_a)$ and $\widetilde{B} = (l_b, m_b, t_b, r_b)$ are two trapezoidal fuzzy numbers. If $D(\widetilde{A}, \widetilde{B}) = 0$ then

$$\begin{cases} \text{(a)}\max\{|t_b - t_a|, |m_b - m_a|\} = 0, \\ \text{(b)}\,\mathrm{Val}(\widetilde{A}) = \mathrm{Val}(\widetilde{B}), \\ \text{(c)}\,\mathrm{Amb}(\widetilde{A}) = \mathrm{Amb}(\widetilde{B}). \end{cases}$$

From (a), we have max $\{|t_b - t_a|, |m_b - m_a|\} = 0$ and hence $m_a = m_b$ and $t_a = t_b$. From (b) and (c)

$$\mathrm{Val}(\widetilde{A}) + \mathrm{Amb}(\widetilde{A}) = 2\int_0^1 s(\alpha)R_{\tilde{A}}(\alpha)\mathrm{d}\alpha = 2\int_0^1 s(\alpha)R_{\tilde{B}}(\alpha)\mathrm{d}\alpha = \mathrm{Val}(\widetilde{B}) + \mathrm{Amb}(\widetilde{B}),$$
$$\mathrm{Val}(\widetilde{A}) - \mathrm{Amb}(\widetilde{A}) = 2\int_0^1 s(\alpha)L_{\tilde{A}}(\alpha)\mathrm{d}\alpha = 2\int_0^1 s(\alpha)L_{\tilde{B}}(\alpha)\mathrm{d}\alpha = \mathrm{Val}(\widetilde{B}) - \mathrm{Amb}(\widetilde{B}),$$

and hence

$$\frac{1}{2}(1 - 2I_s)r_a + t_aI_s = \frac{1}{2}(1 - 2I_s)r_b + t_bI_s,$$
$$m_aI_s + \frac{1}{2}(1 - 2I_s)l_a = m_bI_s + \frac{1}{2}(1 - 2I_s)l_b,$$

which imply $l_a = l_b$, $r_a = r_b$ and $\widetilde{A} = \widetilde{B}$. □

Corollary 3.4. *The source distance*, D, *is a metric on* $TRF(\mathbb{R})$.

Proof. By Theorems (3.3) and (3.1) the proof is clear. □

By an example we show that the source distance D, is a pseudo-metric on $F(\mathbb{R})$. Let $\widetilde{A} = \left(-3 + \frac{3\pi}{2}, 3, 9 - \frac{3\pi}{2}\right)$ and

$$\mu_{\tilde{B}}(x) = \begin{cases} \dfrac{2}{(x-3)^2 + 1} - 1, & 2 \leqslant x \leqslant 4, \\ 0, & \text{otherwise}, \end{cases}$$

thus $D(\widetilde{A}, \widetilde{B}) = 0$, but $\widetilde{A} \neq \widetilde{B}$.

4. The nearest trapezoidal fuzzy number

Definition 4.1. The $\widetilde{A}^* \in TRF(\mathbb{R})$, is the nearest trapezoidal fuzzy number to an arbitrary LR fuzzy number $\widetilde{B}$ if and only if

$$D(\widetilde{A}^*, \widetilde{B}) = \min_{\tilde{A} \in TRF(\mathbb{R})} D(\widetilde{A}, \widetilde{B}).$$

Theorem 4.1. *The* $\widetilde{A}^* \in TRF(\mathbb{R})$, *is the nearest trapezoidal fuzzy number to an arbitrary LR fuzzy number* $\widetilde{C}$ *if and only if*

$$D(\widetilde{A}^*, \widetilde{C}) = 0.$$

Proof. If $D(\widetilde{A}^*, \widetilde{C}) = 0$ then for all $\widetilde{A} \in TRF(\mathbb{R})$ we have $D(\widetilde{A}, \widetilde{C}) \geqslant D(\widetilde{A}^*, \widetilde{C})$, i.e., $\widetilde{A}^*$ is the nearest trapezoidal fuzzy number to $\widetilde{C}$. Conversely, If $D(\widetilde{A}^*, \widetilde{C}) = \beta$ then we show that $\beta = 0$. Suppose $\beta > 0$. We will show that there is a $\widetilde{B} \in TRF(\mathbb{R})$ such that $D(\widetilde{B}, \widetilde{C}) < D(\widetilde{A}^*, \widetilde{C})$. Let for generalized LR fuzzy number $\widetilde{C}$, the quantities $\mathrm{Val}(\widetilde{C})$ and $\mathrm{Amb}(\widetilde{C})$ with respect to regular reducing function s, are specified. We want to find a trapezoidal fuzzy number which has the same value and ambiguity as $\widetilde{C}$, also modal values of both fuzzy numbers are the same, with these properties we will have $D(\widetilde{B}, \widetilde{C}) = 0$. Let $\widetilde{B}$ be a trapezoidal fuzzy number (l_b, m_b, t_b, r_b). We want to find l_b, m_b, t_b and r_b. It is clear that

$$\begin{cases} L_{\tilde{B}}(\alpha) = (m_b - l_b)\alpha + l_b, \\ R_{\tilde{B}}(\alpha) = r_b - (r_b - t_b)\alpha. \end{cases}$$

Thus we should have

$$\begin{cases} \int_0^1 s(\alpha)[R_{\tilde{B}}(\alpha) + L_{\tilde{B}}(\alpha)]\mathrm{d}\alpha = \mathrm{Val}(\widetilde{C}), \\ \int_0^1 s(\alpha)[R_{\tilde{B}}(\alpha) - L_{\tilde{B}}(\alpha)]\mathrm{d}\alpha = \mathrm{Amb}(\widetilde{C}), \end{cases}$$

and

$$\begin{cases} r_b \int_0^1 s(\alpha)\mathrm{d}\alpha - (r_b - t_b)I_s = \frac{1}{2}(\mathrm{Val}(\widetilde{C}) + \mathrm{Amb}(\widetilde{C})), \\ (m_b - l_b)I_s + l_b \int_0^1 s(\alpha)\mathrm{d}\alpha = \frac{1}{2}(\mathrm{Val}(\widetilde{C}) - \mathrm{Amb}(\widetilde{C})), \end{cases}$$

and hence

$$\begin{cases} \frac{1}{2}r_b - (r_b - t_b)I_s = \frac{1}{2}(\mathrm{Val}(\widetilde{C}) + \mathrm{Amb}(\widetilde{C})), \\ (m_b - l_b)I_s + \frac{1}{2}l_b = \frac{1}{2}(\mathrm{Val}(\widetilde{C}) - \mathrm{Amb}(\widetilde{C})). \end{cases}$$

Thus

$$\begin{cases} m_b = m_c, \\ l_b = \dfrac{\mathrm{Val}(\widetilde{C}) - \mathrm{Amb}(\widetilde{C}) - 2m_b I_s}{1 - 2I_s}, \\ t_b = t_c, \\ r_b = \dfrac{\mathrm{Val}(\widetilde{C}) + \mathrm{Amb}(\widetilde{C}) - 2t_b I_s}{1 - 2I_s}. \end{cases} \tag{4.1}$$

It remains to show that $\widetilde{B} = (l_b, m_b, t_b, r_b)$ defined by (4.1), is well defined. It is clear that

$$1 - 2I_s = 1 - 2\psi \int_0^1 s(\alpha)\mathrm{d}\alpha = 1 - \psi, \quad \psi \in (0, 1),$$

$$\mathrm{Val}(\widetilde{C}) + \mathrm{Amb}(\widetilde{C}) = 2\int_0^1 s(\alpha)R_{\tilde{B}}(\alpha)\mathrm{d}\alpha = 2R_{\tilde{B}}(\phi)\int_0^1 s(\alpha)\mathrm{d}\alpha = R_{\tilde{B}}(\phi) \geqslant t_b,$$

$$\mathrm{Val}(\widetilde{C}) - \mathrm{Amb}(\widetilde{C}) = 2\int_0^1 s(\alpha)L_{\tilde{B}}(\alpha)\mathrm{d}\alpha = 2L_{\tilde{B}}(\tau)\int_0^1 s(\alpha)\mathrm{d}\alpha = L_{\tilde{B}}(\tau) \leqslant m_b.$$

Thus

$$l_b = \frac{\mathrm{Val}(\widetilde{C}) - \mathrm{Amb}(\widetilde{C}) - 2m_b I_s}{1 - 2I_s} \leqslant \frac{m_b(1 - 2I_s)}{1 - 2I_s} = m_b,$$
$$r_b = \frac{\mathrm{Val}(\widetilde{C}) + \mathrm{Amb}(\widetilde{C}) - 2t_b I_s}{1 - 2I_s} \geqslant \frac{t_b(1 - 2I_s)}{1 - 2I_s} = t_b,$$

and it completed the proof. □

Theorem 4.2. *The nearest trapezoidal fuzzy number related to an arbitrary LR fuzzy number is unique.*

Proof. Let $\widetilde{A}_1^*, \widetilde{A}_2^*$ be two nearest trapezoidal fuzzy number to an arbitrary LR fuzzy number $\widetilde{C}$, i.e., $D(\widetilde{A}_1^*, \widetilde{C}) = 0$ and $D(\widetilde{A}_2^*, \widetilde{C}) = 0$. By Theorem (3.1)

$$0 \leqslant D(\widetilde{A}_1^*, \widetilde{A}_2^*) \leqslant D(\widetilde{A}_1^*, \widetilde{C}) + D(\widetilde{C}, \widetilde{A}_2^*) = 0$$

thus $D(\widetilde{A}_1^*, \widetilde{A}_2^*) = 0$ and from Theorem (3.3) it is clear that $\widetilde{A}_1^* = \widetilde{A}_2^*$. □

Corollary 4.3. *The trapezoidal fuzzy number $\widetilde{A}$, is the nearest fuzzy number to an arbitrary LR fuzzy number $\widetilde{B}$ if and only if*

$$\begin{cases} \max\{|t_b - t_a|, |m_b - m_a|\} = 0, \\ \mathrm{Val}(\widetilde{A}) = \mathrm{Val}(\widetilde{B}), \\ \mathrm{Amb}(\widetilde{A}) = \mathrm{Amb}(\widetilde{B}). \end{cases}$$

By the last corollary $\widetilde{A}$, is the nearest fuzzy number to an arbitrary LR fuzzy number $\widetilde{B}$ if and only if

$$\begin{cases} m_a = m_b, \\ l_a = \dfrac{\mathrm{Val}(\widetilde{B}) - \mathrm{Amb}(\widetilde{B}) - 2m_b I_s}{1 - 2I_s}, \\ t_a = t_b, \\ r_a = \dfrac{\mathrm{Val}(\widetilde{B}) + \mathrm{Amb}(\widetilde{B}) - 2t_b I_s}{1 - 2I_s}. \end{cases} \tag{4.2}$$

Example 4.1. Let $s(\alpha) = \alpha$ and

$$\mu_{\widetilde{A}}(x) = \begin{cases} \log_3(x), & 1 \leqslant x \leqslant 3, \\ 1, & 3 \leqslant x \leqslant 4, \\ \log_3(7 - x), & 4 \leqslant x \leqslant 6, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore $\widetilde{B} = (0.44188, 3, 4, 6.55812)$, see Fig. 1.

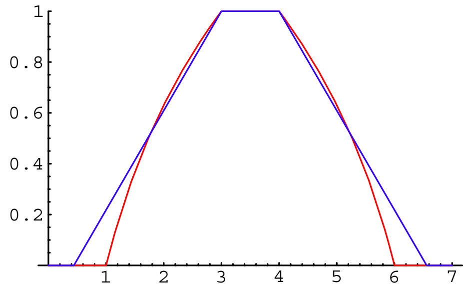

Fig. 1. $\widetilde{B}$ is the nearest trapezoidal fuzzy number to $\widetilde{A}$.

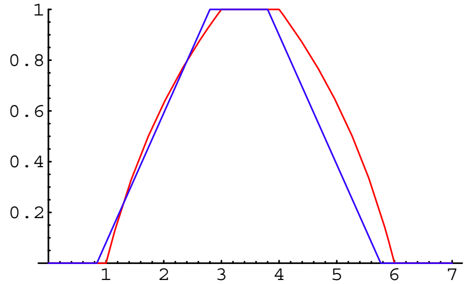

Fig. 2. The nearest trapezoidal fuzzy number to $\widetilde{A}$ using method of [2].

The nearest trapezoidal fuzzy number to $\widetilde{A}$ in this example using the introduced method in [2] is (0.84003, 2.80092, 3.80092, 5.76181), see Fig. 2.

The nearest trapezoidal fuzzy number to $\widetilde{A}$ in this example using the introduced method in [4] is (0.48633, 2.97779, 4.02221, 6.51367).

We compare five methods: (1)—used in this paper, (2)—used in [2], (3)—used in [9,1,10], (4)—used in [4] and (5)—used in [7], by some examples, Table 1. In this table, $a = \frac{3}{2}\sqrt{\frac{\pi}{2}}$, $b = (8 - 3\sqrt{2})\frac{\sqrt{\pi}}{4}$, $c = (3\sqrt{2} - 4)\frac{\sqrt{\pi}}{4}$, $d = (4 - \sqrt{2})\frac{3\sqrt{\pi}}{4}$, $p = \frac{1}{4}(3\sqrt{2\pi} - 8\sqrt{\ln 2})$ and $q = -\frac{3}{2}\sqrt{\frac{\pi}{2}} + 4\sqrt{\ln 2}$.

As we see, for all fuzzy numbers there is no trapezoidal fuzzy number computed by method (3), also by this method a fuzzy number can only be approximated with a symmetric triangular fuzzy number. As it seems, there is no guaranty to have an approximating fuzzy number with the same core as original fuzzy number for methods (3) and (4) even the original fuzzy number is a trapezoidal one. By method (3) a real interval is approximated by a triangular fuzzy number. However by our method a crisp real interval is the nearest one to itself. Moreover to find the approximating fuzzy number one must check four conditions for method (2) and five conditions for method (4), however by our method it'll be known by explicit relations (4.2). By method (5) we can not approximate an unbounded fuzzy number (a fuzzy number is bounded if its support is compact [12,13]). In Table 1, the first and the last examples are unbounded and we computed the approximations for compact supports then we took limit and used error function $\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt\right)$. Also the results for methods (2) and (5), are exactly the same.

Table 1
Numerical results of examples

$L(\alpha)/R(\alpha)$	(1)	(2)	(3)	(4)	(5)
	0.43601	0.50052	0.06790	0.52195	0.50052
$1-0.3\sqrt{-\ln\alpha}$	1	0.96775	1.67725	0.89256	0.96775
$2+0.7\sqrt{-\ln\alpha}$	2	2.07526	1.67725	2.10743	2.07526
	3.31598	3.16546	3.2866	2.9722	3.16546
	0.44188	0.84003	0.4905	0.48633	0.84003
3^{α}	3	2.80092	3.5	2.97779	2.80092
$7-3^{\alpha}$	4	4.19908	3.5	4.02221	4.19908
	6.55812	6.15997	6.5095	6.51367	6.15997
	1	1	0.75	1	1
1	1	1	1.5	1	1
2	2	2	1.5	1	2
	2	2	2.25	2	2
	1	1	–0.5	1	1
$\alpha+1$	2	2	2	2	2
$5-3^{\alpha}$	2	2	2	2	2
	5	5	4.5	2.5	5
	1	1	0.5	–0.33333	1
1	1	1	1.75	0.66667	1
$3-\alpha$	2	2	1.75	2.33333	2
	3	3	3	2.33333	3
	$\mu-a\sigma$	$\mu-b\sigma$	$\mu-d\sigma$	$\mu-p\sigma$	$\mu-b\sigma$
$\mu-\sigma\sqrt{-\ln\alpha}$	μ	$\mu-c\sigma$	μ	$\mu-q\sigma$	$\mu-c\sigma$
$\mu+\sigma\sqrt{-\ln\alpha}$	μ	$\mu+c\sigma$	μ	$\mu+q\sigma$	$\mu+c\sigma$
	$\mu+a\sigma$	$\mu+b\sigma$	$\mu+d\sigma$	$\mu+p\sigma$	$\mu+b\sigma$

5. Properties

Some properties of the approximation operators are presented by Grzegorzewski and Mrówka [7]. In this section we consider some properties of the approximation operator suggested in Section 4.

Let $T : F(\mathbb{R}) \to TRF(\mathbb{R})$ be the approximation operator which produce the nearest trapezoidal fuzzy number to a given original fuzzy number using (4.2).

Proposition 5.1. *The nearest trapezoidal approximation operator is* 1*-cut invariance.*

Proof. It is a necessary condition for this approximation that

$$[T(\widetilde{A})]^1 = [\widetilde{A}]^1. \qquad \square$$

Proposition 5.2. *The nearest trapezoidal approximation operator is invariant to translations.*

Proof. Let z be a real number. Let $\widetilde{A}$ denote a fuzzy number with α-cut $[L_{\tilde{A}}(\alpha), R_{\tilde{A}}(\alpha)]$. Then the α-cut of $\widetilde{A}+z$ translated by z is $[L_{\tilde{A}}(\alpha)+z, R_{\tilde{A}}(\alpha)+z]$. Therefore

$$\mathrm{Val}(\widetilde{A}+z) = \mathrm{Val}(\widetilde{A})+z, \quad \mathrm{Amb}(\widetilde{A}+z) = \mathrm{Amb}(\widetilde{A}).$$

Thus

$$\begin{cases} m_{az} = m_a + z, \\ l_{az} = \dfrac{\mathrm{Val}(\widetilde{A}+z) - \mathrm{Amb}(\widetilde{A}+z) - 2(m_a+z)I_s}{1-2I_s} = \dfrac{\mathrm{Val}(\widetilde{A}) - \mathrm{Amb}(\widetilde{A}) - 2m_aI_s}{1-2I_s} + z, \\ t_{az} = t_a + z, \\ r_{az} = \dfrac{\mathrm{Val}(\widetilde{A}+z) + \mathrm{Amb}(\widetilde{A}+z) - 2(t_a+z)I_s}{1-2I_s} = \dfrac{\mathrm{Val}(\widetilde{A}) + \mathrm{Amb}(\widetilde{A}) - 2t_aI_s}{1-2I_s} + z. \end{cases}$$

therefore $T(\widetilde{A}+z) = T(\widetilde{A}) + z$. □

Proposition 5.3. *The nearest trapezoidal approximation operator is scale invariant.*

Proof. Let $\lambda \neq 0$ be a real number. Thus

$$\mathrm{Val}(\lambda\widetilde{A}) = \lambda\mathrm{Val}(\widetilde{A}), \quad \mathrm{Amb}(\lambda\widetilde{A}) = \lambda\mathrm{Amb}(\widetilde{A}).$$

and

$$\begin{cases} m_{\lambda a} = \lambda m_a, \\ l_{\lambda a} = \dfrac{\mathrm{Val}(\lambda\widetilde{A}) - \mathrm{Amb}(\lambda\widetilde{A}) - 2(\lambda m_a)I_s}{1-2I_s} = \lambda \times \dfrac{\mathrm{Val}(\widetilde{A}) - \mathrm{Amb}(\widetilde{A}) - 2m_aI_s}{1-2I_s}, \\ t_{\lambda a} = \lambda t_a, \\ r_{\lambda a} = \dfrac{\mathrm{Val}(\lambda\widetilde{A}) + \mathrm{Amb}(\lambda\widetilde{A}) - 2(\lambda t_a)I_s}{1-2I_s} = \lambda \times \dfrac{\mathrm{Val}(\widetilde{A}) + \mathrm{Amb}(\widetilde{A}) - 2t_aI_s}{1-2I_s}. \end{cases}$$

Therefore $T(\lambda\widetilde{A}) = \lambda T(\widetilde{A})$. □

Proposition 5.4. *The nearest trapezoidal approximation operator fulfills the nearness criterion with respect to metric D defined in Definition 3.1 in the set of all trapezoidal fuzzy numbers.*

Proof. By Definition 4.1, we have

$$D(\widetilde{A}, T(\widetilde{A})) = \min_{\widetilde{B}\in TRF(\mathbb{R})} D(\widetilde{A}, \widetilde{B}),$$

therefore

$$D(\widetilde{A}, T(\widetilde{A})) \leqslant D(\widetilde{A}, \widetilde{B}), \quad \forall \widetilde{B} \in TRF(\mathbb{R}). \qquad \square$$

Proposition 5.5. *The nearest trapezoidal approximation operator is continuous.*

Proof. An approximation operator T is continuous if for any $\widetilde{A}, \widetilde{B} \in F(\mathbb{R})$ we have

$$\forall \epsilon > 0, \exists \delta > 0, D(\widetilde{A}, \widetilde{B}) < \delta \Rightarrow D(T(\widetilde{A}), T(\widetilde{B})) < \epsilon.$$

Let $D(\widetilde{A}, \widetilde{B}) < \delta$. By Theorem 3.1 we have

$$D(T(\widetilde{A}), T(\widetilde{B})) \leqslant D(T(\widetilde{A}), \widetilde{A}) + D(\widetilde{A}, \widetilde{B}) + D(\widetilde{B}, T(\widetilde{B}))$$

and by Theorem 4.1 we have $D(T(\widetilde{A}), \widetilde{A}) = D(\widetilde{B}, T(\widetilde{B})) = 0$. Thus

$$D(T(\widetilde{A}), T(\widetilde{B})) \leqslant D(\widetilde{A}, \widetilde{B}) < \delta.$$

Therefore its suffices to take $\delta \leqslant \epsilon$. □

Proposition 5.6. *The nearest trapezoidal approximation operator is monotonic on any set of fuzzy numbers with equal cores.*

Proof. Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers with equal cores and $\widetilde{A} \subseteq \widetilde{B}$. Thus $m_a = m_b$ and $t_a = t_b$ and there exist two nonnegative functions $k_1(\alpha)$ and $k_2(\alpha)$ such that for $0 \leqslant \alpha \leqslant 1$,

$$L_{\tilde{A}}(\alpha) = L_{\tilde{B}}(\alpha) + k_1(\alpha),$$
$$R_{\tilde{A}}(\alpha) = R_{\tilde{B}}(\alpha) - k_2(\alpha).$$

It is clear that

$$\int_0^1 s(\alpha)k_1(\alpha)\mathrm{d}\alpha \geqslant 0,$$

thus

$$\int_0^1 s(\alpha)\{k_1(\alpha) - (m_a - m_b)\alpha\}\mathrm{d}\alpha \geqslant 0,$$

i.e.,

$$\mathrm{Val}(\widetilde{A}) - \mathrm{Amb}(\widetilde{A}) - 2m_a I_s \geqslant \mathrm{Val}(\widetilde{B}) - \mathrm{Amb}(\widetilde{B}) - 2m_b I_s,$$

therefore $l_{T(\widetilde{A})} \geqslant l_{T(\widetilde{B})}$. In a similar way it can be shown that $r_{T(\widetilde{A})} \leqslant r_{T(\widetilde{B})}$. Thus

$$T(\widetilde{A}) \subseteq T(\widetilde{B}). \qquad \square$$

Value of fuzzy numbers can be used as a ranking function, [4], i.e.,

$$\widetilde{A} \succ \widetilde{B} \Longleftrightarrow T(\widetilde{A}) \succ T(\widetilde{B}).$$

Proposition 5.7. *The nearest trapezoidal approximation operator is order invariant with respect to value function.*

Proof. The proof is trivial, because $\mathrm{Val}(T(\widetilde{A})) = \mathrm{Val}(\widetilde{A})$ and $\mathrm{Val}(T(\widetilde{B})) = \mathrm{Val}(\widetilde{B})$. □

6. Powers of a trapezoidal fuzzy number

As an application of nearest trapezoidal fuzzy number, we can find the nearest trapezoidal fuzzy number related to any power of a trapezoidal fuzzy number. Let $\widetilde{A}$ be a nonnegative trapezoidal fuzzy number (l_a, m_a, t_a, r_a). It is clear that if $\widetilde{A}^n$ be the nth power of $\widetilde{A}$, then $\widetilde{A}^n$ is a generalized LR fuzzy number where

$$\begin{cases} L_{\tilde{A}^n}(\alpha) = [(m_a - l_a)\alpha + l_a]^n, \\ R_{\tilde{A}^n}(\alpha) = [r_a - (r_a - t_a)\alpha]^n. \end{cases}$$

We can easily compute $\mathrm{Amb}(\widetilde{A}^n)$ and $\mathrm{Val}(\widetilde{A}^n)$, and also the modal value of $\widetilde{A}^n$ is the interval $[m_a^n, t_a^n]$.

Now let $\widetilde{B}^{(n)} = \left(l_b^{(n)}, m_b^{(n)}, t_b^{(n)}, r_b^{(n)}\right)$ be the nearest trapezoidal fuzzy number to $\widetilde{A}^n$. Therefore

$$m_b^{(n)} = m_a^n, \quad t_b^{(n)} = t_a^n, \tag{6.1}$$

and

$$l_b^{(n)} = \frac{\mathrm{Val}(\widetilde{A}^n) - \mathrm{Amb}(\widetilde{A}^n) - 2m_b I_s}{1 - 2I_s}, \quad r_b^{(n)} = \frac{\mathrm{Val}(\widetilde{A}^n) + \mathrm{Amb}(\widetilde{A}^n) - t_b I_s}{1 - 2I_s}. \tag{6.2}$$

Let $\widetilde{A} = (l_a, m_a, t_a, r_a)$ be an arbitrary nonnegative trapezoidal fuzzy number, also let $s(\alpha) = \alpha$. With a simple computation it can be shown that $\widetilde{B}^{(n)} = \left(l_b^{(n)}, m_a^n, t_a^n, r_b^{(n)}\right)$ where

$$l_b^{(n)} = \begin{cases} 2\dfrac{3l_a^{n+2} - m_a^{n+2}(n^2-1) - l_a^2 m_a^n(n^2+3n+2) + l_a m_a^{n+1}(2n^2+3n-2)}{(m_a - l_a)^2(n+1)(n+2)}, & l_a \neq m_a, \\ m_a^n, & l_a = m_a, \end{cases}$$

$$r_b^{(n)} = \begin{cases} -2\dfrac{-3r_a^{n+2} + t_a^{n+2}(n^2-1) + t_a^n r_a^2(n^2+3n+2) - t_a^{n+1} r_a(2n^2+3n-2)}{(r_a - t_a)^2(n+1)(n+2)}, & t_a \neq r_a, \\ t_a^n, & t_a = r_a. \end{cases}$$

Example 6.1. Let $s(\alpha) = \alpha$ and $\widetilde{A} = (2, 3, 4, 5)$.

Therefore $\widetilde{B}^{(2)} = \left(\frac{7}{2}, 9, 16, \frac{49}{2}\right)$, see Fig. 3.

Example 6.2. Let $s(\alpha) = \alpha$ and $a \geqslant 0$.

If $\mu_{\tilde{A}}(x) = \chi_{\{a\}}(x)$ then $\mu_{\widetilde{B}^{(n)}}(x) = \chi_{\{a^n\}}(x)$.

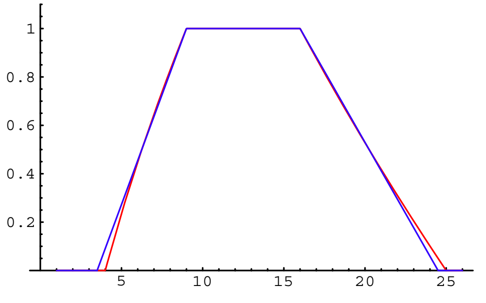

Fig. 3. The nearest trapezoidal fuzzy number to $\widetilde{A}^2$.

7. Nearest trapezoidal fuzzy number to multiplication of two trapezoidal fuzzy numbers

Let $\widetilde{A} = (l_a, m_a, t_a, r_a), \widetilde{B} = (l_b, m_b, t_b, r_b)$ be two nonnegative trapezoidal fuzzy numbers and $\widetilde{C} = \widetilde{A} \otimes \widetilde{B}$, be the multiplication of the two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ based on extension principle. We know that $\widetilde{C}$ is a generalized LR fuzzy number but not a trapezoidal one. Now using the nearest method we can approximate it by a trapezoidal fuzzy number $\widetilde{F} = (l_f, m_f, t_f, r_f)$, where

$$\begin{cases} l_f = \frac{3}{2}(m_a - l_a)(m_b - l_b) + 2[(m_a - l_a)l_b + (m_b - l_b)l_a] + 3l_a l_b - 2m_a m_b, \\ m_f = m_a m_b, \\ t_f = t_a t_b, \\ r_f = \frac{3}{2}(r_a - t_a)(r_b - t_b) - 2[(r_a - t_a)r_b + (r_b - t_b)r_a] + 3r_a r_b - 2t_a t_b. \end{cases}$$

Lemma 7.1. *The nearest fuzzy number $\widetilde{F}$ to $\widetilde{C} = \widetilde{A} \otimes \widetilde{B}$, is well defined, i.e., $l_f \leqslant m_f$ and $r_f \geqslant t_f$.*

Proof. It is trivial from proof of Theorem 4.1. □

Example 7.1. Let $\widetilde{A} = (1, 2, 3)$ and $\widetilde{B} = (2, 4, 5)$.

Hence $\widetilde{C} = \widetilde{A} \otimes \widetilde{B}$ and $\widetilde{F} = (1, 8, 14.5)$, the nearest triangular fuzzy number, are shown in the Fig. 4.

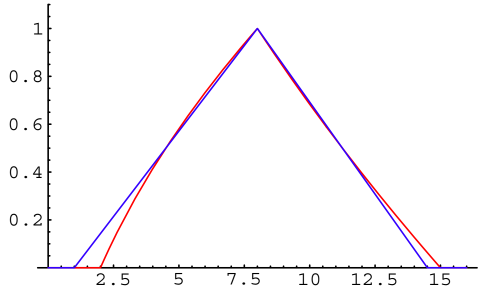

Fig. 4. The nearest triangular fuzzy number.

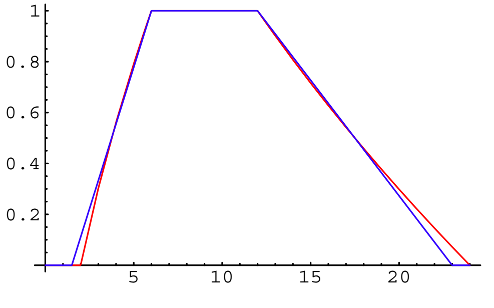

Fig. 5. The nearest trapezoidal fuzzy number.

Example 7.2. Let $\widetilde{A} = (1, 2, 3, 4)$ and $\widetilde{B} = (2, 3, 4, 6)$.

Hence $\widetilde{C} = \widetilde{A} \otimes \widetilde{B}$ and $\widetilde{F} = (1.5, 6, 12, 23)$, the nearest trapezoidal fuzzy number, are shown in Fig. 5.

8. Conclusions

In this work a metric was defined on $TRF(\mathbb{R})$, also a method was presented to find the unique nearest trapezoidal fuzzy number related to an arbitrary LR fuzzy number using this metric.

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