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ON THE MAXIMUM MATCHINGS OF REGULAR MULTIGRAPHS

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The lower bounds on the cardinality of the maximum matchings of regular multigraphs are established in terms of the number of vertices, the degree of vertices and the edge-connectivity of a multigraph. The bounds are attained by infinitely many multigraphs, so are best possible.

1. Introduction

We consider *multigraphs* (no loops are allowed but more than one edge can join two vertices; these are called *multiple edges*), and briefly consider *pseudographs* (both loops and multiple edges are permitted). A multigraph with no multiple edges is called a *simple graph.* We simply use 'graph' if such a distinction is unnecessary. $G=(V, E)$ denotes a graph with vertex set V and edge set E. Throughout this note p denotes the number of vertices of G, i.e., $p=|V|$. A *matching* of a graph is a set of nonadjacent edges, and a *maximum matching,* denoted by $M(G)$, of G is one of maximum cardinality. $n(G)$ denotes the number of *unsaturated vertices* (i.e., vertices with which no edge of a matching is incident) in $M(G)$. Therefore

$$|M(G)|=\tfrac{1}{2}(p-n(G)). \tag{1}$$

For a subset S of V, we denote by $o(G, S)$ the number of odd connected components (i.e., components with an odd number of vertices) of the graph $G-S$ (which is obtained from G by the removal of all the vertices in S). Then it is known [2, 3] that

$$n(G)=\max\{o(G, S)-|S| \mid S\subset V\}. \tag{2}$$

The degree $d_G(v)$ of a vertex v in G is the number of edges with v as an endvertex each loop being counted twice. A graph G is said to be *regular of degree r* if $d_G(v)=r$ for every $v\in V$; such graphs are also called *r-regular.* G is *λ-edge-connected* if any removal of less than λ edges results neither in a disconnected graph nor in a trivial graph. G is *λ-odd-connected* if any removal of less than λ edges results neither in a disconnected graph with an odd connected

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component nor in a trivial graph. Note that λ-edge-connected implies λ-odd-connected, and that a λ-odd-connected r-regular graph is $(\lambda+1)$-odd-connected if $\lambda \neq r \pmod 2$.

Using a linear polynomial of p, we expressed lower bounds on the cardinality of the maximum matching for various classes of graphs: planar simple graphs, 4-connected graphs, trees, and arbitrary simple graphs [6, 7, 9]. On the other hand, Weinstein obtained other lower bounds for arbitrary simple graphs which are strong essentially for regular simple graphs [12]. (We independently found the same results only for regular simple graphs [6, 8].) In this note we give lower bounds for regular multigraphs which are best possible in the sense that infinitely many regular multigraphs attain the bounds. Note that the underlying simple graphs of a regular multigraph is no longer regular, although the cardinality of the maximum matchings of a multigraph is identical to that of the underlying simple graph. Therefore bounds for regular multigraphs obtained in this note are often sharper than Weinstein's bound applied to the underlying simple graphs.

2. Main results

$\lfloor x \rfloor$ means the greatest integer $\leq x$, and $\lceil x \rceil$ the least integer $\geq x$.

Theorem 1. *Let $G=(V,E)$ be an r-regular λ-odd-connected multigraph with p vertices, and let $r \geq 3$, $\lambda \geq 1$, and $r = \lambda \pmod 2$. Then*

$$|M(G)| \geq \begin{cases} \min\{\lfloor p/2 \rfloor, \lceil (r+\lambda)p/(3r+\lambda) \rceil\}, & \text{if } r \text{ is even;} \\ \lceil (r+\lambda)p/(3r+\lambda) \rceil, & \text{otherwise.} \end{cases}$$

Remark 1. When $r \neq \lambda \pmod 2$, G is $(\lambda+1)$-odd-connected. Therefore we can obtain a bound for such a case simply by replacing λ by $\lambda+1$ in the above equation.

Remark 2. From Theorem 1 and Remark 1 we immediately obtain the following fact [3, 10]: if G is r-regular and $(r-1)$-edge-connected, then $|M(G)| = \lfloor p/2 \rfloor$.

Remark 3. Vizing has shown that a multigraph G can be edge-coloured with at most $\Delta_G + m_G$ colours, where Δ_G is the maximum degree, and m_G is the multiplicity of G (i.e., the maximum number of edges joining two vertices) [11]. This fact yields a bound for r-regular multigraphs:

$$|M(G)| \geq \lceil rp/2(r+m_G) \rceil.$$

Note that the bound in Theorem 1 is sharper than this when

$$m_G > r(r-\lambda)/2(r+\lambda).$$

A part of Theorem 1 is slightly improved as follows.

Theorem 2. *If G is a connected r-regular multigraph, and $r \geq 3$ and odd, then*

$$|M(G)| \geq \left\lceil \frac{(r^2-r-1)p-(r-1)}{r(3r-5)} \right\rceil.$$

Both the bound in Theorem 1 for the case of $\lambda \geq 2$ and the bound in Theorem 2 are best possible in the sense that neither the coefficient of p nor the constant term can be improved as shown in the following section.

3. Constructions

In this section we construct infinitely many r-regular λ-edge-connected graphs $G(r, \lambda, h)$ ($h = 1, 2, \ldots$) which attain the bound in Theorem 1 for the case of $\lambda \geq 2$ or the bound in Theorem 2.

3.1. *Construction of $G(r, \lambda, h)$ for Theorem* 1

Let $r \geq 3$, $2 \leq \lambda \leq r-2$ and $r \equiv \lambda \pmod 2$. Resembling Weinstein's construction [12, p. 1504], we first define a bipartite graph $B(r, \lambda, h) = (X \cup Y, E)$ with $|X| = rh$ and $|Y| = \lambda h$, where each edge in E has one endvertex in X and the other in Y. Take $X \cup Y$ the set of nonnegative integers $<(r+\lambda)h$ and take Y the set of the even nonnegative integers $<2\lambda h$. Let $B(r, \lambda, h)$ contain edges of the following two kinds:

(i) $(y, y \oplus 1 \oplus 2k)$, $y \in Y$ and $0 \leq k < \lambda$, where $\oplus$ means addition (mod $2\lambda h$);

(ii) (x, y) where $x \geq 2\lambda h$, $y \in Y$ and $\frac{1}{2}y \equiv x \pmod h$.

Clearly $B(r, \lambda, h)$ is λ-edge-connected, and $d_B(v) = \lambda$ if $v \in X$, or $d_B(v) = r$ if $v \in Y$.

$H(r, \lambda)$ denotes a multigraph containing exactly three vertices of respective degree r, r and $r-\lambda$.

$G(r, \lambda, h)$ is constructed through a $B(r, \lambda, h) = (X \cup Y, E)$ and $H(r, \lambda)$'s as follows: for each $x \in X$, let $H_x(r, \lambda)$ be a copy of $H(r, \lambda)$ such that $H_x(r, \lambda)$ has no vertex common with $B(r, \lambda, h)$ or with any $H_{x'}$, $x' \neq x$; let $G(r, \lambda, h)$ be the graph obtained from a $B(r, \lambda, h)$ and $H_x(r, \lambda)$'s ($x \in X$) by identifying each $x \in X$ with the vertex of degree $r-\lambda$ in $H_x(r, \lambda)$. (We depict $G(4, 2, 2)$ in Fig. 1.)

Clearly $G(r, \lambda, h)$ constructed as above is an r-regular λ-edge-connected graph with $p = (3r+\lambda)h$ vertices, and $n(G(r, \lambda, h)) = (r-\lambda)h$. (Take $S = Y$ in (2).) Therefore $n(G(r, \lambda, h)) = (r-\lambda)p/(3r+\lambda)$. Hence $G(r, \lambda, h)$ attains the bound in Theorem 1.

3.2. *Construction of $G(r, 1, h)$ for Theorem* 2

Ler r be odd. We first give the definition of the terminology on a tree. A connected graph with no cycle is called a *tree*. A vertex of degree 1 in a tree is

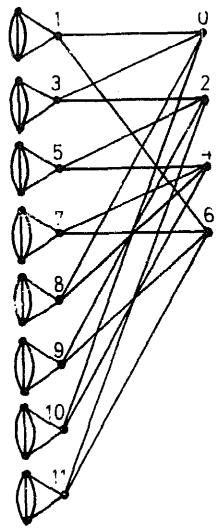

Fig. 1. $G(4, 2, 2)$.

called a *leaf*. For $b \geqslant 1$ we define a *b-ary tree* as a tree such that

(a) There is one specially designated vertex called the *root* of the tree; and

(b) The degree of each vertex other than the leaves is $b+1$.

Note that the definition of a b-ary tree is different from those in [1, 4]. A vertex of degree $b+1$ in a b-ary tree is called an *internal vertex*. The *depth* of a vertex v in a b-ary tree is the length of the path from the root to v. (The length of a path is the number of edges on the path.) The *height of a tree* is the length of a longest path from the root to a leaf. A b-ary tree is said to be *complete* if for some integer κ, every vertex of depth less than k is an internal vertex and every vertex of depth k is a leaf. We denote by $T(b, h)$ the complete b-ary tree of height h ($\geqslant 1$).

$G(r, 1, h)$ is constructed through a $T(r, h)$ and $H(r, 1)$'s as follows: for each leaf x of $T(r, h)$, let $H_x(r, 1)$ be a copy of $H(r, 1)$ such that $H_x(r, 1)$ has no vertex common with $T(r, h)$ or with any $H_{x'}$, $x' \neq x$; let $G(r, 1, h)$ be the graph obtained from $T(r, h)$ and $H_x(r, 1)$'s by identifying each leaf x of $T(r, h)$ with the vertex of $H_x(r, 1)$ which has degree $r-1$. (We depict $G(3, 1, 2)$ in Fig. 2.)

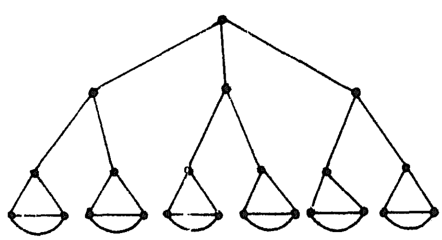

Fig. 2. $G(3, 1, 2)$.

Clearly $G(r, 1, h)$ is a connected r-regular graph with $p = \{r(3r-5)(r-1)^{h-1}-2\}/(r-2)$ vertices, and $n(G(r, 1, h)) = (r-1)^h$. Note that the maximum of (2) is attained for $G(r, 1, h)$ when S is the set of internal vertices of even depth in $T(r, h)$ if h is odd, or of odd depth otherwise. Therefore

$$n(G(r,1,h)) = \frac{(r-1)(r-2)}{r(3r-5)}\left(p+\frac{2}{r-2}\right).$$

Hence $G(r, 1, h)$ attains the bound in Theorem 2.

4. Proof of Theorems 1 and 2

For a graph $G = (V, E)$ and a subset S of V, $u(S)$ denotes the number of edges with one endvertex in S and the other in $V - S$. Let I be the set of all positive integers, and I_o the set of all positive odd integers.

Proof of Theorem 1. Suppose that $n(G) \geqslant 2$. Then it follows from (2) that there exists a nonempty proper subset S of V such that $o(G, S) - s = n(G)$, where $s = |S|$. Let $G_m = (V_m, E_m)$ $(m = 1, 2, \ldots)$ be the mth connected component of $G - S$. Define $J_i = \{m \mid |V_m| = i\}$ and $t_i = |J_i|$ for $i \in I$. Clearly

$$o(G,S) = \sum_{i \in I_o} t_i, \tag{3}$$

and

$$p = s + \sum_{i \in I} i t_i. \tag{4}$$

Since G is r-regular, we have

$$rs \geqslant u(S) \geqslant \sum_{i \in I_o} \left(\sum_{j \in J_i} u(V_j)\right). \tag{5}$$

Since the multigraph G is λ-odd-connected and V_j is a nonempty proper subset of V, for $j \in J_i$

$$u(V_j) \geqslant \begin{cases} \lambda, & \text{if } i \in I_o; \quad (6a) \\ r, & \text{if } i = 1. \quad (6b) \end{cases}$$

Combining (5) and (6), we obtain

$$rs \geqslant rt_1 + \lambda \sum_{i \in I_o'} t_i, \tag{7}$$

where $I_o' = I_o - \{1\}$. Using (2), (3) and (7), we have

$$r \cdot n(G) \leqslant (r-\lambda) \sum_{i \in I_o'} t_i. \tag{8}$$

On the other hand, combining (4) and (7) yields

$$rp \geq (3r+\lambda) \sum_{i \in I_o'} t_i. \tag{9}$$

Combining (8) and (9) yields

$$n(G) \leq (r-\lambda)p/(3r+\lambda). \tag{10}$$

Thus we have shown that $n(G) \leq 1$ or (10) holds. Especially (10) holds if p is even, since $n(G)$ would not be 1. Our claim immediately follows from (1) and these facts.

Next we give the following lemma, which characterizes the configuration of graphs attaining the lower bound in Theorem 2.

Lemma 1. *Let $r \geq 3$ and odd, and let a connected r-regular multigraph $G=(V, E)$ satisfy the following condition: $n(G)/\{p+2/(r-2)\}$ is maximum among all connected r-regular multigraphs. Let S be a subset of V such that $n(G)=o(G, S)-s$, where $s=|S| \geq 1$. V_i, t_i and J_i $(i \in I)$ are defined as in the proof of Theorem* 1. *Then*

(a) $t_i=0$ *unless* $i=1$ *or* 3;

(b) $G-V_j$ *contains exactly r connected components if* $j \in J_1$, *i.e.*, $|V_j|=1$;

(c) $u(V_j)=1$ *if* $j \in J_3$;

(d) *G contains no edges with both endvertices in S: and*

(e) *G is obtained from an* $(r-1)$*-ary tree and copies of the graph* $H(r, 1)$ *by identifying each leaf of the tree with the vertex of degree* $r-1$ *in a distinct copy of* $H(r, 1)$. ($H(r, 1)$ *was defined in Section* 3.)

Proof. Let $j \in J_i$, and $u=u(V_j)$. Let $H_m(r, 1)$ $(m=1, 2, \ldots, u)$ be a copy of $H(r, 1)$. We now construct a new r-regular multigraph G' from $G-V_j$ as follows: (i) make the disjoint union of $G-V_j$ and $H_1(r, 1), \ldots, H_u(r, 1)$; and (ii) for each m $(m=1, \ldots, u)$, add an edge joining the vertex of degree $r-1$ in H_m and one of the vertices of $G-V_j$ which were adjacent to a vertex in V_j, so that the resulting multigraph G' becomes r-regular. Suppose that G' contains exactly k connected components $D_1, \ldots, D_k$. We denote by p_m' the number of vertices of D_m, and by u_m the number of edges of G with one endvertex in V_j and the other in the vertex set of D_m for $m=1, \ldots, k$. Clearly

$$1 \leq k \leq u, \tag{11}$$

and

$$u=\sum_{m=1}^{k} u_m. \tag{12}$$

Let

$$\pi=\sum_{m=1}^{k} [n(D_m)\{(r-2)p+2\}-n(G)\{(r-2)p_m'+2\}],$$

then $\pi \leq 0$. For, otherwise, we have

$$\frac{n(G)}{p+2/(r-2)} < \max\left\{\frac{n(D_m)}{p'_m+2/(r-2)} \,\middle|\, m=1,\ldots,k\right\},$$

which contradicts the assumption on the maximality of G. $F_m = (X_m, Y_m)$ $(m = 1, \ldots, k)$ denotes a connected component of $G - V_j$ which corresponds to D_m, and p_m denotes the number of vertices in F_m. We define n_m for $m = 1, \ldots, k$ as follows:

$$n_m = o(F_m, S \cap X_m) - |S \cap X_m|.$$

Then we have

$$p = i + \sum_{m=1}^{k} p_m, \tag{13}$$

$$p'_m = p_m + 3u_m, \tag{14}$$

$$n(D_m) \geq n_m + u_m, \tag{15}$$

and

$$n(G) = \begin{cases} \sum_{m=1}^{k} n_m + 1, & \text{if } i \text{ is odd}; \quad (16a) \\ \sum_{m=1}^{k} n_m, & \text{otherwise}. \quad (16b) \end{cases}$$

We separate two cases.

Case 1: *i is even.* Using (13)–(16), we obtain

$$\begin{aligned} n(D_m)&\{(r-2)p+2\} - n(G)\{(r-2)p'_m+2\} \\ &\geq (r-2)\left\{n_m\left(\sum_{\substack{z=1\\(z\neq m)}}^{k} p_z\right) - p_m\left(\sum_{\substack{z=1\\(z\neq m)}}^{k} n_z\right)\right\} \\ &\quad + n_m\{(r-2)i+2\} + u_m\{p(r-2)+2-3(r-2)n(G)\} - 2n(G). \end{aligned} \tag{17}$$

Noting that

$$\sum_{m=1}^{k}\left\{n_m\left(\sum_{\substack{z=1\\(z\neq m)}}^{k} p_z\right) - p_m\left(\sum_{\substack{z=1\\(z\neq m)}}^{k} n_z\right)\right\} = 0, \tag{18}$$

and using (12) and (16)–(17), we have

$$\pi \geq n(G)\{(r-2)i + 2 - 3(r-2)u - 2k\} + p(r-2)u + 2u. \tag{19}$$

From Theorem 1 (by taking $\lambda = 1$ in (10)), we have

$$p \geq (3r+1)n(G)/(r-1). \tag{20}$$

Substituting (11) and (20) into (19), we obtain

$$\pi \geq n(G)\{(r-1)(r-2)i + 2(r-1) + 2(r-3)u\}/(r-1) + 2u > 0. \tag{21}$$

This is a contradiction. Thus we have shown that if i is even, then

$$t_i = 0. \tag{22}$$

Case 2: *i is odd.* Using (13)–(16), we obtain

$$\begin{aligned} n(D_m)\{(r-2)p+2\} - n(G)\{(r-2)p'_m+2\} \\ \geq (r-2)\left\{n_m\left(\sum_{\substack{z=1\\(z\neq m)}}^{k} p_z\right) - p_m\left(\sum_{\substack{z=1\\(z\neq m)}}^{k} n_z\right)\right\} \\ + n_m\{(r-2)i+2\} - p_m(r-2) + u_m\{p(r-2)+2-3(r-2)n(G)\} - 2n(G) \end{aligned} \tag{23}$$

Using (12), (14), (16), (18) and (23), we obtain

$$\pi \geq n(G)\{(r-2)i+2-3(r-2)u-2k\} + p(r-2)(u-1) + 2(u-1). \tag{24}$$

Substituting (20) into (24), we obtain

$$\pi \geq n(G)\{(r-2)(r-1)i+4(r-2)u-2k(r-1)-r(3r-7)\}/(r-1) + 2(u-1). \tag{25}$$

We consider three subcases.

Subcase (i): $i=1$. In this case, $k \leq u = r$. Substituting $u=r$ and $i=1$ into (25), we have that if $k \leq r-1$,

$$\pi \geq 2n(G)(r-1-k) + 2(r-1) > 0.$$

Therefore $k=r$. Thus we have established (b).

Subcase (ii): $i=3$. Substituting (11) and $i=3$ into (25), we have that if $u \geq 2$,

$$\pi \geq (u-1)\{2(r-3)n(G)/(r-1)+2\} > 0.$$

Therefore $u=1$. Thus we have proved (c).

Subcase (iii): $i>3$. Substituing (11) and $i>3$ into (25), we have

$$\pi > (u-1)\{2(r-3)n(G)/(r-1)+2\} \geq 0.$$

Thus we have shown that if $i>3$ and odd, then

$$t_i = 0. \tag{26}$$

Combining (22) and (26), we obtain (a).

Suppose that G contain an edge e with both endvertices in S. We now define G', D'_m, p'_m and π in the same fashion as above except that $G - V_i$ is replaced by $G - e$. Then we have $\pi > 0$. (Take $i=0$ and $u=2$ in (21).) Thus we have proved (d).

(e) immediately follows from (a)–(d).

Using Lemma 1, we can exactly decide $n(G)$ for a graph G which attains the lower bound in Theorem 2.

Lemma 2. *Let G satisfy the requirement of Lemma* 1, *then*

$$n(G)=\frac{(r-1)(r-2)}{r(3r-5)}\left(p+\frac{2}{r-2}\right).$$

Proof. S, s and t_m $(m \in I)$ are defined as in Lemma 1. By Lemma 1 we have

$$n(G)=t_1+t_3-s, \tag{27}$$

$$p=t_1+3t_3+s, \tag{28}$$

and

$$t_1+t_3+s-1=rt_1+t_3=rs. \tag{29}$$

From (29) we obtain

$$s=(r-1)t_1+1. \tag{30}$$

Combining (29) and (30), we have

$$r(r-2)t_1=t_3-r. \tag{31}$$

Combining (27)–(31), we obtain

$$\frac{n(G)}{(r-2)+2}=\frac{t_1+t_3-(rt_1+t_3)/r}{(r-2)t_1+3(r-2)t_3+(r-2)(r-1)t_1+r}$$
$$=\frac{(r-1)t_3}{r\{r(r-2)t_1+3(r-2)t_3+r\}}$$
$$=\frac{r-1}{r(3r-5)}.$$

Thus we have proved the claim.

Now Theorem 2 immediately follows from Lemma 2 and (1).

5. Lower bounds for regular pseudographs

Employing a proof-technique similar to those of Theorems 1 or 2, we can easily obtain the lower bounds on the cardinality of the maximum matchings of regular pseudographs. We show the results without proof.

Theorem 3. *Let G be an r-regular* λ*-odd-connected pseudograph with p vertices, and let* $r \geq 3, \lambda \geq 1$, *and* $r=\lambda \pmod 2$. *Then*

$$|M(G)| \geq \begin{cases} \min\{\lfloor p/2 \rfloor, \lceil \lambda p/(r+\lambda) \rceil\}, & \text{if } r \text{ is even;} \\ \lceil \lambda p/(r+\lambda) \rceil, & \text{otherwise.} \end{cases}$$

Theorem 4. *If G is a connected r-regular pseudograph, and $r \geq 3$ and odd, then* $|M(G)| \geq \lceil (p-1)/r \rceil$.

It is easy to construct infinitely many regular pseudographs which attain the bound in Theorem 3 for the case of $\lambda \geq 2$ or bound in Theorem 4.

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