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Total domination in inflated graphs

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ABSTRACT

The inflation G_I of a graph G is obtained from G by replacing every vertex x of degree $d(x)$ by a clique $X = K_{d(x)}$ and each edge xy by an edge between two vertices of the corresponding cliques X and Y of G_I in such a way that the edges of G_I which come from the edges of G form a matching of G_I. A set S of vertices in a graph G is a total dominating set, abbreviated TDS, of G if every vertex of G is adjacent to a vertex in S. The minimum cardinality of a TDS of G is the total domination number $\gamma_t(G)$ of G. In this paper, we investigate total domination in inflated graphs. We provide an upper bound on the total domination number of an inflated graph in terms of its order and matching number. We show that if G is a connected graph of order $n \geq 2$, then $\gamma_t(G_I) \geq 2n/3$, and we characterize the graphs achieving equality in this bound. Further, if we restrict the minimum degree of G to be at least 2, then we show that $\gamma_t(G_I) \geq n$, with equality if and only if G has a perfect matching. If we increase the minimum degree requirement of G to be at least 3, then we show $\gamma_t(G_I) \geq n$, with equality if and only if every minimum TDS of G_I is a perfect total dominating set of G_I, where a perfect total dominating set is a TDS with the property that every vertex is adjacent to precisely one vertex of the set.

1. Introduction

In this paper, we continue the study of domination in inflated graphs first introduced by Dunbar and Haynes [2] and studied, for example, in [3,4,8,9]. A *total dominating set*, abbreviated TDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $V(G)$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS of G. A TDS of G of cardinality $\gamma_t(G)$ is called a *$\gamma_t(G)$-set*. Total domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [5,6]. A recent survey of total domination in graphs can be found in [7].

For notation and graph theory terminology, we in general follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n(G)$ and edge set E of size $m(G)$. The minimum and maximum degrees among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A cycle on n vertices is denoted by C_n, and a path on n vertices by P_n. A vertex of degree 1 in G is called a *leaf* of G. We denote the set of leaves in G by $L(G)$ and we let $\ell(G) = |L(G)|$. A *support vertex* is a vertex that is adjacent to a leaf, while a *strong support vertex* is adjacent to at least two leaves.

The *open neighborhood* of v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. For subsets $S, T \subseteq V$, the set S *totally dominates* the set T if $T \subseteq N(S)$. If S and T are disjoint subsets of V, then by $G[S, T]$ we denote the set of all edges in G that join a vertex of S and a vertex of T. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$.

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A TDS S in a graph $G = (V, E)$ is a *perfect total dominating set*, abbreviated PTDS, if every vertex is adjacent to precisely one vertex of S, that is, if $|N(v) \cap S| = 1$ for each vertex $v \in V$.

Two edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G which we denote by $\alpha'(G)$. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M. If M is a matching in G, an *M-matched vertex* is a vertex incident with an edge in M while an *M-unmatched vertex* is a vertex not incident with an edge in M.

A set $P \subseteq V$ is a *paired-dominating set*, abbreviated PDS, if P is a total dominating set, with the added requirement that the subgraph induced by P contains a perfect matching (not necessarily induced). The *paired-domination number* of G, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a PDS of G.

The *corona* $H \circ \overline{K}_2$ of a graph H and the empty graph $\overline{K}_2$, as defined in [5], is the graph constructed from a copy of H by adding for each vertex $v \in V(H)$, two new vertices v' and v'' and the two pendant edges vv' and vv''. Hence, $H \circ \overline{K}_2$ has order $3|V(H)|$.

1.1. Inflated graph

For the notation for inflated graphs, we follow [3]. The *inflation* or *inflated graph* G_I of a graph G without isolated vertices is obtained as follows: each vertex x_i of degree $d(x_i)$ of G is replaced by a clique $X_i \cong K_{d(x_i)}$ and each edge x_ix_j of G is replaced by an edge uv in such a way that $u \in X_i$, $v \in X_j$, and two different edges of G are replaced by non-adjacent edges of G_I. There are two different kinds of edges in G_I. The edges of the clique X_i are colored red and the X_i's are called the *red cliques* (a red clique X_i is reduced to a vertex if x_i is a leaf of G). The other ones, which correspond to the edges of G, are colored *blue* and they form a perfect matching of G_I. Every vertex of G_I belongs to exactly one red clique and is incident with exactly one blue edge. For notational simplicity, we denote the vertex set of a red clique X_i by X_i.

Following the notation of Kang et al. [8], if x_i and x_j are two adjacent vertices of G, the vertex of X_i (respectively, X_j) incident with the blue edge of G_I replacing the edge x_ix_j of G is called x_ix_j (respectively, x_jx_i) in G_I. By definition, every leaf in G corresponds to a leaf in G_I and every support vertex in G corresponds to a support vertex in G_I. Further, every support vertex x_jx_i in G_I is adjacent only to other vertices in the clique X_j and to the (unique) leaf x_ix_j adjacent to it in G_I (where x_i is a leaf in G adjacent to the vertex x_j).

As remarked by Favaron [3], G_I is the line-graph of the subdivision $S(G)$ of G which is obtained by replacing each edge of G by a path of length 2. In particular, we note that G_I is claw-free. Further, $n(G_I) = \sum_{x_i \in V(G)} d_G(x_i) = 2m(G)$, $\delta(G_I) = \delta(G)$ and $\Delta(G_I) = \Delta(G)$.

2. Total domination in inflated graphs

Our aim in the paper is to study total domination in inflated graphs. First, we provide an upper bound on the total domination number of an inflated graph in terms of its order and matching number. For this purpose, we define $\phi_L(G)$ as the maximum possible number of leaves of G that are M-unmatched taken over all maximum matchings M in G. For example, for $k \geq 3$ if G is the graph of order $3k$ obtained from a cycle C_k on k vertices by adding a pendant edge to each vertex and then subdividing exactly once every edge on the cycle, then $\alpha'(G) = k$ and there is a maximum matching M in G consisting entirely of cycle edges, whence $\phi_L(G) = k$.

Lemma 1. *If G is a graph with no isolated vertex, then $\gamma_t(G_I) \leq 2n(G) - 2\alpha'(G) - \phi_L(G)$.*

Proof. Let $n = n(G)$, $\alpha' = \alpha'(G)$ and let $\phi_L = \phi_L(G)$. Among all maximum matching in G, let M be one that maximizes the number of leaves that are M-unmatched. Let $\Phi_L(G)$ denote the set of M-unmatched leaves in G and let $\Phi_{\geq 2}(G)$ denote the set of M-unmatched vertices in G of degree at least two. Then, $\phi_L = |\Phi_L(G)|$ and $|\Phi_{\geq 2}(G)| = n - 2\alpha' - \phi_L$. Let $V(G) = \{x_1, x_2, \ldots, x_n\}$. Renaming vertices, if necessary, we may assume that $M = \{x_{2i-1}x_{2i} \mid 1 \leq i \leq \alpha'\}$. Let S_M denote the vertices in G_I corresponding to the M-matched vertices of G, that is,

$$S_M = \bigcup_{i=1}^{\alpha'} \{x_{2i-1}x_{2i}, x_{2i}x_{2i-1}\}.$$

For $i \in \{1, 2, \ldots, 2\alpha'\}$, the vertex x_i is M-matched in G. If i is odd, then $x_ix_{i+1} \in M$, implying that $\{x_ix_{i+1}, x_{i+1}x_i\} (\subseteq S_M)$ totally dominates the vertices in the red cliques X_i and X_{i+1}. If i is even, then $x_ix_{i-1} \in M$, implying that $\{x_ix_{i-1}, x_{i-1}x_i\} (\subseteq S_M)$ totally dominates the vertices in the red cliques X_{i-1} and X_i. Hence for $i \in \{1, 2, \ldots, 2\alpha'\}$, the set S_M totally dominates the vertices in the red clique X_i.

For $i \in \{2\alpha' + 1, \ldots, n\}$, suppose $x_i \in \Phi_L(G)$ (and so, x_i is a leaf in G that is M-unmatched). Let x_j be the neighbor of x_i in G, and let $S_i = \{x_jx_i\}$. The maximality of the matching M implies that x_j is M-matched, and so the red clique X_j contains a vertex of S_M, namely x_jx_{j+1} if j is odd or x_jx_{j-1} if j is even, and this vertex of S_M is different from the vertex x_jx_i. Thus the vertex in S_i is totally dominated by S_M. Moreover, the vertex in X_i is totally dominated by S_i.

For $i \in \{2\alpha'+1, \ldots, n\}$, suppose $x_i \in \Phi_{\geq 2}(G)$ (and so, $d(x_i) \geq 2$ and x_i is M-unmatched). Let S_i be an arbitrary 2-element subset of X_i. Then, S_i totally dominates the vertices in X_i. Hence the set

$$D = S_M \cup \left(\bigcup_{i=2\alpha'+1}^{n} S_i \right)$$

is a TDS of G_I, whence

$$\begin{aligned} \gamma_t(G_I) &\leq |S_M| + \sum_{i=2\alpha'+1}^{n} |S_i| \\ &= 2\alpha' + 2|\Phi_{\geq 2}(G)| + |\Phi(G)| \\ &= 2\alpha' + 2(n - 2\alpha' - \phi_L) + \phi_L \\ &= 2n - 2\alpha' - \phi_L. \quad \square \end{aligned}$$

We remark that if we restrict the graph G in the statement of Lemma 1 to have minimum degree at least two, then the TDS D constructed in the proof of Lemma 1 for the inflated graph G_I is a paired-dominating set of G_I. Hence as a consequence of the proof of Lemma 1, we have the following result due to Kang et al. [8].

Corollary 2 ([8]). *If G is a graph with $\delta(G) \geq 2$, then $\gamma_{\mathrm{pr}}(G_I) \leq 2n(G) - 2\alpha'(G)$.*

Next we establish lower bounds on the total domination number of an inflated graph. Recall that $L(G)$ denotes the set of leaves in a graph G and that $\ell(G) = |L(G)|$. We begin by establishing a lower bound on the total domination number of an inflated graph with minimum degree one in terms of the number of leaves in the graph.

Theorem 3. *Let G be a graph with $\delta(G) = 1$. Then, $\gamma_t(G_I) \geq \ell(G)$, with equality if and only if every vertex of G is a leaf or a strong support vertex.*

Proof. Let S be a $\gamma_t(G_I)$-set. If x_i is a leaf of G that is adjacent to a vertex x_j, then the vertex x_jx_i in X_j belongs to S in order to totally dominate the vertex x_ix_j in X_i. Hence for every leaf of G, there corresponds a unique vertex in G_I that belongs to S. Thus, $\gamma_t(G_I) = |S| \geq \ell(G)$.

Suppose that $\gamma_t(G_I) = \ell(G)$. Then, S consists precisely of these $\ell(G)$ support vertices in G_I that totally dominate the set $L(G_I)$ of leaves in G_I. Suppose that x_k is a vertex of G that is neither a leaf nor a strong support vertex of G. If x_k is a support vertex of G that is adjacent to a leaf x_r, then the red clique X_k contains exactly one support vertex, namely x_kx_r, and this support vertex is adjacent to the leaf x_rx_k in X_r. Thus the set S, which consists of the support vertices of G_I, contains no neighbor of x_kx_r. Hence the vertex x_kx_r is not totally dominated by S, contradicting the fact that S is a TDS in G_I. Therefore, x_k is not a support vertex of G. Thus no vertex in X_k is a support vertex in G_I. Since every support vertex in G_I is adjacent only to other vertices in the red clique that contains it and to the (unique) leaf adjacent to it, no vertex in the red clique X_k is totally dominated by S, a contradiction. Therefore, every vertex of G is either a leaf or a strong support vertex of G.

Conversely, if every vertex of G is either a leaf or a strong support vertex of G, then either $G = G_I = K_2$, in which case $\gamma_t(G_I) = \ell(G) = 2$, or $n(G) \geq 3$, in which case the set of support vertices in G_I totally dominate G_I and, once again, $\gamma_t(G_I) = \ell(G)$ since the number of support vertices in G_I is equal to the number of leaves in G. $\square$

We next provide a lower bound on the total domination number of an inflated graph G_I in terms of the number of vertices of the graph G. By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty.

Theorem 4. *Let G be a connected graph of order $n \geq 2$. Then, $\gamma_t(G_I) \geq 2n/3$, with equality if and only if G is the corona $H \circ \overline{K}_2$ of some connected graph H.*

Proof. Let $G = (V, E)$ and let $V(G) = \{x_1, x_2, \ldots, x_n\}$. Let S be a $\gamma_t(G_I)$-set. Let S_1 be the set of vertices in S that belong to a red clique that contains exactly one vertex of S, and let S_2 be the set of vertices in S that belong to a red clique that contains at least two vertices of S. Let (V_0, V_1, V_2) be a weak partition of V, where $V_0 = \{x_i\colon |S \cap X_i| = 0\}$, $V_1 = \{x_i\colon |S \cap X_i| = 1\}$ and $V_2 = \{x_i\colon |S \cap X_i| \geq 2\}$. For $i = 0, 1, 2$, let $n_i = |V_i|$, and so $n = n_0 + n_1 + n_2$. Further, $|S_1| = n_1$ and $|S_2| \geq 2n_2$, while $|S| = |S_1| + |S_2|$.

If $n_0 = 0$, then $|S| \geq n_1 + 2n_2 = n + n_2 \geq n > 2n/3$, which establishes the desired lower bound. Hence we may assume that $n_0 \geq 1$.

Let x be a vertex in a red clique that contains no vertex of S. Then, $x = x_ix_j$ for some integers i and j. We note that $S \cap X_i = \emptyset$. In order to totally dominate x, the vertex $x_jx_i \in X_j$ belongs to S. In order to totally dominate x_jx_i, the set S must contain a vertex of X_j different from x_jx_i. Hence, $|S \cap X_j| \geq 2$, and so $x_j \in V_2$ and $x_jx_i \in S_2$. Therefore, each vertex in a red clique that contains no vertex of S is totally dominated by a unique vertex in S_2. In particular, we note that the set V_0 is an independent set in G.

Let $A = V_0$ and let $B = N(A)$. Then, $|A| = n_0$. Let $|B| = b$. From our earlier observations, we note that A is an independent set and that $B \subseteq V_2$. We now construct a bipartite graph F with partite sets (A, B), where the edge set of F is the set of edges

$G[A, B]$ that join a vertex of A and a vertex of B in the graph G. Let F have order n_F and size m_F. Let Δ denote the maximum degree of a vertex of B in the graph F. Let $(B_1, B_2, \ldots, B_\Delta)$ be a weak partition of B, where $d_F(v) = i$ for each vertex $v \in B_i$ in the graph F for $i = 1, 2, \ldots, \Delta$. Thus for $i = 1, 2, \ldots, \Delta$, each vertex in B_i is adjacent to exactly i vertices of A in F. For $i = 1, 2, \ldots, \Delta$, let $|B_i| = b_i$, and so

$$b = |B| = \sum_{i=1}^{\Delta} b_i. \tag{1}$$

By definition of the set V_2, if $x_j \in V_2$, then $|S \cap X_j| \geq 2$. Since each vertex in a red clique that contains no vertex of S is totally dominated by a unique vertex in S_2, we observe that if $x_i \in B_i$ for some $i \geq 2$, then $|S \cap X_i| \geq i$. For $i = 1, 2, \ldots, \Delta$, we define a function $f : B \to \{1, 2, \ldots, \Delta\}$ as follows: for $v \in B_1$, define $f(v) = 2$, while for $v \in B_i$ for some i with $2 \leq i \leq \Delta$, define $f(v) = i$. Thus if $v \in B$, say $v = x_j$, then $|S \cap X_j| \geq f(v)$. We define $f(B) = \sum_{v \in B} f(v)$. By definition of the function f, we have that

$$f(B) = \left(\sum_{i=1}^{\Delta} i b_i \right) + b_1. \tag{2}$$

Since every vertex in A is adjacent to at least one vertex of B, we have that

$$n_0 = |A| \leq m_F = \sum_{i=1}^{\Delta} i b_i. \tag{3}$$

By Eqs. (1)–(3), we have that

$$\begin{aligned} n_F &= n_0 + b \\ &\leq \sum_{i=1}^{\Delta} (i+1) b_i \\ &= \frac{3}{2} \left[\left(\sum_{i=1}^{\Delta} i b_i \right) + b_1 \right] - b_1 - \left[\sum_{i=3}^{\Delta} \left(\frac{i}{2} - 1 \right) b_i \right] \\ &= \frac{3}{2} f(B) - b_1 - \left[\sum_{i=3}^{\Delta} \left(\frac{i}{2} - 1 \right) b_i \right] \\ &\leq \frac{3}{2} f(B), \end{aligned}$$

and so $f(B) \geq 2n_F/3$. Let

$$S_F = \bigcup_{x_i \in V(F)} (S \cap X_i) \quad \text{and} \quad \overline{S}_F = S \setminus S_F.$$

Then,

$$|S_F| = \sum_{x_i \in V(F)} |S \cap X_i| = \sum_{x_i \in B} |S \cap X_i| \geq \sum_{x_i \in B} f(x_i) = f(B).$$

For each vertex $x_i \in V(G) \setminus V(F)$, we note that $x_i \in V_1 \cup V_2$, and so $|S \cap X_i| \geq 1$, implying that $|\overline{S}_F| \geq n - n_F$. Since $n_F \leq n$, we have that

$$\begin{aligned} |S| &= |S_F| + |\overline{S}_F| \\ &\geq f(B) + (n - n_F) \\ &\geq 2n_F/3 + (n - n_F) \\ &= n - n_F/3 \\ &\geq n - n/3 \\ &= 2n/3. \end{aligned}$$

Hence, $\gamma_t(G_I) = |S| \geq 2n/3$. Suppose that equality holds. Then we must have equality throughout the above inequality chains. In particular, this implies that $n_F = n$, and so $F = G$. Further, $b = b_2$ and $n_0 = m_F$. Hence every vertex that belongs to A is a leaf in G, while every vertex of B is a strong support vertex in G that is adjacent to exactly two (leaves) of A. Since G is connected, we note that $G[B]$ is a connected graph. Hence, G is the corona $H \circ \overline{K}_2$ of some connected graph H (where $H = G[B]$). $\square$

In the introductory paper on total domination, Cockayne et al. [1] showed that if G is a connected graph on $n \geq 3$ vertices, then $\gamma_t(G) \leq 2n/3$. Hence as an immediate consequence of Theorem 4, we have the following result which shows that the total domination number of a graph is at most the total domination number of its inflation.

Corollary 5. *For every connected graph G with no isolated vertex, $\gamma_t(G) \leq \gamma_t(G_I)$, with equality if and only if $G = K_2$ or $G = P_3$.*

Proof. Let G have order $n \geq 2$. If $n = 2$, then $G = K_2$, and so $G_I = K_2$ and $\gamma_t(G) = \gamma_t(G_I) = 2$. Suppose $n \geq 3$. Then by the Cockayne–Dawes–Hedetniemi result and by Theorem 4, we have that $\gamma_t(G) \leq 2n/3 \leq \gamma_t(G_I)$. Suppose equality occurs. Then, $\gamma_t(G_I) = 2n/3$ and, by Theorem 4, G is the corona $H \circ \overline{K}_2$ of some connected graph H of order $n/3$. If $|V(H)| \geq 2$, then $\gamma_t(G) = |V(H)| = n/3 < \gamma_t(G_I)$, a contradiction. Hence, H is the trivial graph K_1, whence $G = P_3$. □

We show next that if we restrict our attention to graphs with minimum degree at least two, then the lower bound in Theorem 4 can be improved significantly.

Theorem 6. *Let G be a graph of order n with $\delta(G) \geq 2$. Then, $\gamma_t(G_I) \geq n$, with equality if and only if G has a perfect matching.*

Proof. Let $G = (V, E)$ and let $V(G) = \{x_1, x_2, \ldots, x_n\}$. We follow exactly the notation and terminology introduced in the first paragraph of the proof of Theorem 4. If $n_0 = 0$, then $|S| \geq n_1 + 2n_2 \geq n$, which establishes the desired lower bound. Hence we may assume that $n_0 \geq 1$.

Let x be a vertex in a red clique that contains no vertex of S. Then, $x = x_i x_j$ for some integers i and j. We note that $S \cap X_i = \emptyset$. In order to totally dominate x, the vertex $x_j x_i \in X_j$ belongs to S. In order to totally dominate $x_j x_i$, the set S must contain a vertex of X_j different from $x_j x_i$. Hence, $|S \cap X_j| \geq 2$, and so $x_j \in V_2$ and $x_j x_i \in S_2$. Therefore, each vertex in a red clique that contains no vertex of S is totally dominated by a unique vertex in S_2. Since there are n_0 red cliques that contain no vertex of S, and since each red clique contains at least δ vertices, we deduce that $|S_2| \geq n_0\delta$. As observed earlier, $|S_2| \geq 2n_2$. Since $\delta \geq 2$, we therefore have that

$$|S_2| = \frac{1}{\delta}|S_2| + \left(\frac{\delta - 1}{\delta}\right)|S_2| \geq \frac{1}{\delta}|S_2| + \frac{1}{2}|S_2| \geq n_0 + n_2. \tag{4}$$

Thus, $\gamma_t(G_I) = |S| = |S_1| + |S_2| \geq n_1 + (n_0 + n_2) = n$, as desired. Suppose next that $\gamma_t(G_I) = n$. We show that G has a perfect matching.

Suppose $n_0 = 0$. Then, $n = |S| \geq n_1 + 2n_2 \geq n$. Consequently, we must have equality throughout this inequality chain, implying that $n = n_1$. Hence, $S = S_1$ and $V = V_1$. Therefore, every red clique contains exactly one vertex of S. Let $x \in V$. Then, $x = x_i$ for some i, $1 \leq i \leq n$. Let $x_i x_j$ denote the vertex of the red clique X_i that belongs to S. Then, $x_j x_i$ is the vertex of the red clique X_j that belongs to S, and we set $x' = x_j$. Then the set $\cup_{x \in V}\{xx'\}$ is a perfect matching in G.

Hence we may assume that $n_0 \geq 1$, for otherwise G has a perfect matching as claimed. We must then have equality throughout the Inequality Chain (4), implying that $\delta = 2$ and that every red clique that contains no vertex of S has size 2. Further, $|S_2| = 2n_2$, and so every red clique that contains at least two vertices of S contains exactly two vertices of S. Further, every vertex of S_2 is adjacent to a vertex that belongs to a red clique containing no vertex of S. We now consider the bipartite graph F with partite sets V_0 and V_2, and with edge set consisting of all edges of G that join V_0 and V_2, that is, $E(F) = [V_0, V_2]$. Then, F is a 2-regular bipartite subgraph of G. Let M_F be a perfect matching in F. If $n_1 = 0$, then $G = F$ and M_F is a perfect matching in G. Hence we may assume that $n_1 \geq 1$, for otherwise G has a perfect matching as claimed. Let $H = G[V_1]$ be the subgraph of G induced by the set V_1. Let $x \in V(H)$. Then, $x = x_i$ for some i, $1 \leq i \leq n$. Let $x_i x_j$ denote the vertex of the red clique X_i that belongs to S. Then, $x_j x_i \in X_j$ belongs to S. If $x_j \in V_2$, then $x_j x_i \in S_2$, contradicting our earlier observation that $x_i x_j$ would then belong to a red clique X_i containing no vertex of S. Hence, $x_j \in V_1$ and we set $x' = x_j$. Let $M_H = \cup_{x \in V_1}\{xx'\}$. Then, $M_F \cup M_H$ is a perfect matching in G.

Conversely, suppose G has a perfect matching. Then, by Corollary 2, $\gamma_{\rm pr}(G_I) \leq n$. Since the paired-domination number of a graph is at least its total domination, this implies that $\gamma_t(G_I) \leq n$. As shown earlier, $\gamma_t(G_I) \geq n$. Consequently, $\gamma_t(G_I) = n$. □

Since the paired-domination number of a graph is at least its total domination, we have the following consequence of Theorem 6 and its proof.

Corollary 7 ([8]). *Let G be a graph of order n with $\delta(G) \geq 2$. Then, $\gamma_{\rm pr}(G_I) \geq n$, with equality if and only if G has a perfect matching.*

As a consequence of the proof of Theorem 6, we have the following result.

Corollary 8. *Let G be a graph of order n with $\delta(G) \geq 3$. Then, $\gamma_t(G_I) \geq n$, with equality if and only if every $\gamma_t(G_I)$-set is a perfect total dominating set of G_I.*

Proof. We shall follow the notation introduced in the proof of Theorem 6. In particular, let $G = (V, E)$ have order $n = n(G)$. Let $\delta(G) = \delta \geq 3$ and let S be a $\gamma_t(G_I)$-set. By Theorem 6, $\gamma_t(G_I) \geq n$. Suppose that $\gamma_t(G_I) = n$. We show that S is a perfect total dominating set of G_I. If $n_0 = 1$, then by Inequality Chain (4) and since $\delta \geq 3$, we have that $|S_2| \geq \frac{1}{\delta}|S_2| + \frac{2}{3}|S_2| > n_0 + n_2$, implying that $n = \gamma_t(G_I) = |S| = |S_1| + |S_2| > n_1 + (n_0 + n_2) = n$, a contradiction. Hence, $n_0 = 0$, and so $S = S_1$ and $V = V_1$.

Proceeding as in the proof of Theorem 6, we note that n is even and, renaming the vertices of G if necessary, that the set S corresponds to a perfect matching $M = \{x_{2i-1}x_{2i} \mid 1 \le i \le n/2\}$. For odd j, $1 \le j \le (n-1)/2$, every vertex in X_j different from x_jx_{j+1} is uniquely totally dominated by x_jx_{j+1}, while the vertex x_jx_{j+1} is uniquely totally dominated by $x_{j+1}x_j$. For even j, $2 \le j \le n/2$, every vertex in X_j different from x_jx_{j-1} is uniquely totally dominated by x_jx_{j-1}, while the vertex x_jx_{j-1} is uniquely totally dominated by $x_{j-1}x_j$. In both cases, every vertex of X_j is adjacent to precisely one vertex of S. Thus, S is a PTDS of G_I. Conversely, suppose that every $\gamma_t(G_I)$-set is a PTDS of G_I. Let S be a $\gamma_t(G_I)$-set. Since $\delta \ge 3$, the PTDS S contains at most one vertex from every red clique, and so $S_2 = \emptyset$ and $n_2 = 0$. Hence, $n = n_0 + n_1$ and $n \le \gamma_t(G_I) = |S| = |S_1| = n_1 \le n$. Consequently, we must have equality throughout this inequality chain, implying that $\gamma_t(G_I) = n$. □

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References

[1] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, Total domination in graphs, Networks 10 (1980) 211–219.
[2] J.E. Dunbar, T.W. Haynes, Domination in infated graphs, Congr. Numer. 118 (1996) 143–154.
[3] O. Favaron, Irredundance in inflated graphs, J. Graph Theory 28 (1998) 97–104.
[4] O. Favaron, Inflated graphs with equal independent number and upper irredundance number, Discrete Math. 236 (2001) 81–94.
[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.
[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: advanced topics, Marcel Dekker, Inc., New York, 1998.
[7] M.A. Henning, Recent results on total domination in graphs: a survey, Discrete Math. 309 (2009) 32–63.
[8] L. Kang, M.Y. Sohn, T.C.E. Cheng, Paired-domination in inflated graphs, Theoret. Comput. Sci. 320 (2004) 485–494.
[9] J. Puech, The lower irredundance and domination parameters are equal for inflated trees, J. Combin. Math. Combin. Comput. 33 (2000) 117–127.