On Invariants for Difference Equations and Systems of Difference Equations of Rational Form

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C. J. Schinas (J. Math. Anal. Appl., 216 (1997), 164–179) has presented some invariants for difference equations and systems of difference equations of rational form with constant and periodic coefficients (of certain period). We report that the presented invariants as well as their difference equations can be generalized.

1. INTRODUCTION

The study of the integrability, exact solvability, or linearisability of discrete nonlinear systems governed by differential–difference and pure difference equations has become the focus of interest and an active domain of research in recent years [1–4]. In fact much effort has been invested during the past few years in taking analytical techniques, concepts, theorems, etc., we know and love in the theory of differential equations and finding the analogous techniques, etc., for differential–difference and difference equations [4–6]. One of the effective analytical techniques devised is the construction of the invariant or integral of motion of a given differential–difference or difference equation.

A nontrivial function $I_n$,

$$I_n = I_n(n, x_n, x_{n+1}, \ldots, x_{n+N-1}), x_n = x(n), \quad (1.1a)$$

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is said to be an invariant for an \(N\)th-order ordinary difference equation

\[
x_{n+N} = f(n, x_n, x_{n+1}, \ldots, x_{n+N-1})
\]

(1.1b)

if

\[
I(n+1) = I(n) \quad \text{or} \quad I_{n+1} = I_n.
\]

(1.1c)

Recently Schinas [7] has presented some invariants for difference equations and systems of difference equations of rational form with constant and periodic coefficients (of certain period). We report that the derived invariants as well its difference equations can be generalised.

2. SECOND-ORDER DIFFERENCE EQUATIONS

A. Autonomous Difference Equations

Consider an autonomous second-order difference equation defined by

\[
\begin{align*}
x(n+1) &= y(n), \\
y(n+1) &= \frac{P(x(n), y(n))}{Q(x(n), y(n))}
\end{align*}
\]

or

\[
\begin{align*}
x' &= y, \\
y' &= \frac{P(x, y)}{Q(x, y)},
\end{align*}
\]

(2.1)

where \(P(x, y)\) and \(Q(x, y)\) are polynomials in \(x\) and \(y\). Our aim is to derive an invariant or integral of motion \(I_n(x, y)\) for (2.1). Given a difference equation it is not clear what form of integral \(I_n(x, y)\) one has to choose. Thus in order to derive \(I_n(x, y)\) for (2.1) we make the assumption that it is a ratio of two biquadratics given by

\[
I_n = \frac{A_1 + A_2 y + A_3 y^2 + A_4 x + A_5 xy + A_6 x^2 + A_7 x^2 y + A_8 x^2 y^2}{B_1 + B_2 y + B_3 y^2 + B_4 x + B_5 xy + B_6 y^2 + B_7 x^2 + B_8 x^2 y + B_9 x^2 y^2},
\]

(2.2)

where \(A_i\) and \(B_i\), \(i = 1, 2, \ldots, 9\) are parameters. Equation (2.2) can be rewritten as

\[
I_n(x, y) = \frac{I_0(x) + I_1(x) y + I_2(x) y^2}{I_3(x) + I_4(x) y + I_5(x) y^2} = \frac{X(x, y)}{Y(x, y)},
\]

(2.3)
where
\[ I_0(x) = (A_1 + A_4 x + A_7 x^2), \quad I_1(x) = (A_2 + A_5 x + A_8 x^2), \]
\[ I_2(x) = (A_3 + A_6 x + A_9 x^2), \quad I_3(x) = (B_1 + B_4 x + B_7 x^2), \]
\[ I_4(x) = (B_2 + B_5 x + B_8 x^2), \]
and
\[ I_5(x) = (B_3 + B_6 x + B_9 x^2). \]

Requiring that
\[ I_{n+1}(x', y') = I_n(x, y), \tag{2.4} \]
we obtain a quadratic equation in \( y' \):
\[ \left[ I_2(y) Y(x, y) - I_5(y) X(x, y) \right] y'^2 + \left[ I_4(y) Y(x, y) - I_5(y) X(x, y) \right] y' + \left[ I_0(y) Y(x, y) - I_5(y) X(x, y) \right] = 0. \tag{2.5} \]

Solving Eq. (2.5) for \( y' \) we can explicitly find the admissible form of \( P(x, y) \) and \( Q(x, y) \) in Eq. (2.1). We note that our aim is not to present the most general form of \( y' \) here. It is straightforward to show that for the parametric restrictions
\[ A_8 = A_6, \quad A_4 = A_2, \quad \text{and} \quad A_7 = A_3; \]
\[ B_8 = B_6, \quad B_4 = B_2, \quad \text{and} \quad B_7 = B_3, \tag{2.6} \]
Eq. (2.5) reduces to
\[ \left[ (I_2 I_3 - I_0 I_5) + x(I_2 I_4 - I_1 I_5) \right] y'^2 + \left[ (I_1 I_3 - I_0 I_4) + x^2(I_1 I_5 - I_2 I_4) \right] y' + x\left[ (I_0 I_4 - I_1 I_5) + x(I_0 I_5 - I_2 I_4) \right] = 0, \tag{2.7} \]
where \( I_i, i = 0, 1, 2, 3, 4, \) and 5, are functions of \( y \) only and therefore the autonomous second-order difference Eq. (2.1) becomes
\[ y' = \frac{(I_0 I_4 - I_1 I_5) - x(I_2 I_3 - I_0 I_5)}{(I_2 I_3 - I_0 I_5) - x(I_1 I_5 - I_2 I_4)} = \frac{P(x, y)}{Q(x, y)}. \tag{2.8} \]

The invariant \( I_n, \) Eq. (2.3), takes the form
\[ I_n(x, y) \]
\[ = \frac{A_1 + A_3(x + y) + A_5(x^2 + y^2) + A_7 x y + A_8(x + y) y} {B_1 + B_3(x + y) + B_5(x^2 + y^2) + B_7 x y + B_9(x + y) y + B_9 x^2 y^2}. \tag{2.9} \]
It is appropriate to mention here that the autonomous second-order difference equation (2.8) and its invariant (2.9) were first identified by Quispel et al. [8]. Also, they have shown that the invariant (2.9) can be parametrised in terms of Jacobi elliptic functions. See also [9] for invariants of higher order autonomous difference equations of type (2.8). It is clear that the second-order difference equation (3.1) and its invariant (3.2) given in [7] are a particular case of the above result.

B. Nonautonomous Difference Equations

Consider a nonautonomous second-order difference equation having the form

\[ x' = y, \quad y' = \frac{P(n, x, y)}{Q(n, x, y)}, \tag{2.10} \]

where \( P(n, x, y) \) and \( Q(n, x, y) \) are polynomials in \( x \) and \( y \). Here again we look for an invariant of Eq. (2.10) which is a ratio of two biquadratics having the form

\[ I_n = \frac{I_0(n, x) + I_1(n, x)y + I_2(n, x)y^2}{I_3(n, x) + I_4(n, x)y + I_5(n, x)y^2}, \tag{2.11} \]

where

\[
\begin{align*}
I_0 &= (I_{00} + I_{02}x + I_{03}x^2), \\
I_1 &= (I_{11} + I_{12}x + I_{13}x^2), \\
I_2 &= (I_{21} + I_{22}x + I_{23}x^2), \\
I_3 &= (I_{31} + I_{32}x + I_{33}x^2), \\
I_4 &= (I_{41} + I_{42}x + I_{43}x^2), \\
I_5 &= (I_{51} + I_{52}x + I_{53}x^2),
\end{align*}
\]

and

\[ I_n = (I_{ij}x + I_{jk}x^2), \]

where \( I_{ij}, i, j = 0, 1, 2, 3, 4, \) and 5, are unknown functions of the variable \( n \) to be determined.

Demanding that

\[ I_{n+1}(n + 1, x', y') = I_n(n, x, y), \tag{2.12} \]

we obtain a quadratic equation in \( y' \). Solving it yields the required forms of \( P(n, x, y) \) and \( Q(n, x, y) \). One of the interesting cases we identified is the
nonautonomous difference equation

\[ x' = y, \quad y' = \frac{f_1(n, y) - xf_2(n, y)}{f_2(n, y) - xf_3(n, y)}, \]

where

\[ f_1(n, y) = (\alpha + ay)(g + y' + \beta' y^2) - (d + \alpha' y)(y + by + \lambda y^2), \]
\[ f_2(n, y) = (d + \alpha' y)(\beta + \lambda y + ey^2), \]

and

\[ f_3(n, y) = -(\alpha + ay)(\beta + \lambda y + ey^2), \]

and its invariant

\[ I_n = \frac{(g + y' + \beta' y^2) + (y + by + \lambda y^2)x + (\beta + \lambda y + ey^2)x^2}{(d + \alpha' y) + x(\alpha + ay)} \]

\[ = \frac{g + y' + \beta' y^2 + y + by + \lambda y^2 + \beta + \lambda y + ey^2}{a + b + \lambda + e}, \]

(2.13)

where \( \alpha(n), \beta(n), \gamma(n), \) and \( \lambda(n) \) are periodic functions of period 2; that is, \( \alpha(n + 2) = \alpha'' = \alpha, \) and \( a, b, d, e, \) and \( g \) are parameters. Thus the second-order difference equation, Theorem 3.1, given in [7], is a particular case of our result.

3. THIRD-ORDER DIFFERENCE EQUATIONS

A. Autonomous Case

Consider an autonomous third-order difference equation having the form

\[ x' = y, \quad y' = z, \quad z' = \frac{P(x, y, z)}{Q(x, y, z)}, \]

(3.1)

where \( P(x, y, z) \) and \( Q(x, y, z) \) are polynomials in \( x, y, \) and \( z. \) Here also we make the assumption that the invariant \( I_n(x, y, z) \) of (3.1) is a ratio of two triquadratics given by

\[ I_n(x, y, z) = \frac{I_0(y, z) + xI_1(y, z) + x^2I_2(y, z)}{I_3(y, z) + xI_4(y, z) + x^2I_5(y, z)} = \frac{X(x, y, z)}{Y(x, y, z)}, \]

(3.2)
where

\begin{align*}
I_0 &= (A_1 z^2 + A_2 z + A_3) y^2 + (A_4 z^2 + A_5 z + A_6) y \\
&
+ (A_7 z^2 + A_8 z + A_9),
I_1 &= (A_{10} z^2 + A_{11} z + A_{12}) y^2 + (A_{13} z^2 + A_{14} z + A_{15}) y \\
&
+ (A_{16} z^2 + A_{17} z + A_{18}),
I_2 &= (A_{19} z^2 + A_{20} z + A_{21}) y^2 + (A_{22} z^2 + A_{23} z + A_{24}) y \\
&
+ (A_{25} z^2 + A_{26} z + A_{27}),
I_3 &= (B_1 z^2 + B_2 z + B_3) y^2 + (B_4 z^2 + B_5 z + B_6) y \\
&
+ (B_7 z^2 + B_8 z + B_9),
I_4 &= (B_{10} z^2 + B_{11} z + B_{12}) y^2 + (B_{13} z^2 + B_{14} z + B_{15}) y \\
&
+ (B_{16} z^2 + B_{17} z + B_{18}),
I_5 &= (B_{19} z^2 + B_{20} z + B_{21}) y^2 + (B_{22} z^2 + B_{23} z + B_{24}) y \\
&
+ (B_{25} z^2 + B_{26} z + B_{27}),
\end{align*}

and

\begin{align*}
I_{n+1}(x', y', z') &= I_n(x, y, z),
\end{align*}

where \(A_i\) and \(B_i\), \(i = 1, 2, \ldots, 27\), are parameters.

As before, the condition for invariance on the integral \(I_n(x, y, z)\), Eq. (3.2), that is,

\begin{align*}
I_{n+1}(x', y', z') &= I_n(x, y, z),
\end{align*}

yields a quadratic equation in \(z'\) and the solution of it determines the admissible form of \(P(x, y, z)\) and \(Q(x, y, z)\) in Eq. (3.1). We mention here also that our aim is not to present the most general form of \(z'\). It is straightforward to check that for the parametric restrictions

\begin{align*}
A_{25} &= A_{21} = A_1, A_{26} = A_{24} = A_{16} = A_{12} = A_4 = A_2, \\
A_{27} &= A_7 = A_3, A_{17} = A_{15} = A_5, \\
A_{18} &= A_9 = A_6, A_{22} = A_{20} = A_{10}, \\
A_{23} &= A_{13} = A_{11}, \\
B_{25} &= B_{21} = B_1, B_{26} = B_{24} = B_{16} = B_{12} = B_4 = B_2, \\
B_{27} &= B_5 = B_3, B_{17} = B_{15} = B_9, \\
B_{18} &= B_9 = B_6, B_{22} = B_{20} = B_{10}, \\
B_{23} &= B_{13} = B_{11},
\end{align*}
the quadratic equation in \( z' \) reduces to the simple equation
\[
[(I_2 I_3 - I_0 I_4) + x(I_2 I_4 - I_1 I_5)] z'^2 \\
+ [(I_1 I_3 - I_0 I_4) + x^2(I_1 I_5 - I_2 I_4)] z' \\
+ x[I(I_0 I_4 - I_1 I_3) + x(I_0 I_5 - I_2 I_3)] = 0
\] (3.5)
and therefore the autonomous third-order difference equation (3.1) becomes
\[
z' = \frac{(I_0 I_4 - I_1 I_3) - x(I_2 I_3 - I_0 I_5)}{(I_2 I_3 - I_0 I_4) - x(I_1 I_5 - I_2 I_4)},
\] (3.6)
where
\[
I_0(y, z) = (A_1 z^2 + A_2 z + A_3) y^2 + (A_2 z^2 + A_5 z + A_6) y \\
+ (A_2 z^2 + A_6 z + A_9),
\]
\[
I_1(y, z) = (A_{10} z^2 + A_{11} z + A_2) y^2 + (A_{11} z^2 \\
+ A_{14} z + A_3) y + (A_2 z^2 + A_5 z + A_6),
\]
\[
I_2(y, z) = (A_{19} z^2 + A_{10} z + A_1) y^2 + (A_{10} z^2 \\
+ A_{11} z + A_2) y + (A_2 z^2 + A_5 z + A_3),
\]
\[
I_3(y, z) = (B_1 z^2 + B_2 z + B_3) y^2 + (B_2 z^2 + B_5 z + B_6) y \\
+ (B_3 z^2 + B_6 z + B_9),
\]
\[
I_4(y, z) = (B_{10} z^2 + B_{11} z + B_2) y^2 + (B_{11} z^2 + B_{14} z + B_3) y \\
+ (B_2 z^2 + B_5 z + B_6),
\]
and
\[
I_5(y, z) = (B_{19} z^2 + B_{10} z + B_1) y^2 + (B_{10} z^2 + B_{11} z + B_2) y \\
+ (B_2 z^2 + B_5 z + B_3).
\]

The invariant \( I_n(x, y, z) \) of the above difference equation (3.6) takes the form
\[
I_n(x, y, z) = \frac{I_0(y, z) + xI_1(y, z) + x^2I_2(y, z)}{I_5(y, z) + xI_4(y, z) + x^2I_5(y, z)}.
\] (3.7)
Thus we have generalised the third difference equation (3.11) and its invariant, (3.12), given in Ref. [7].

B. Nonautonomous Case

Consider a nonautonomous third-order difference equation defined by

\[ x' = y, \quad y' = z, \quad z' = \frac{P(n, x, y, z)}{Q(n, x, y, z)}, \tag{3.8} \]

where \( P(n, x, y, z) \) and \( Q(n, x, y, z) \) are polynomials in \( x, y, \) and \( z \). Proceeding in a way similar to that described in the previous section for the second-order difference equation we find the following:

**Theorem.** The nonautonomous third-order difference equation defined by

\[ x' = y, \quad y' = z, \quad z' = \frac{f_1(n, y, z) - x f_2(n, y, z)}{f_2(n, y, z) - x f_3(n, y, z)}, \tag{3.9} \]

where

\[
\begin{align*}
    f_1 &= (\beta'' + \alpha'z + \alpha''y + dyz) I_0(n, y, z) \\
    &\quad - (e + \beta y + \beta'z + \alpha yz) I_1(n, y, z), \\
    f_2 &= (e + \beta y + \beta'z + \alpha yz) I_2(n, y, z), \\
    f_3 &= - (\beta'' + \alpha''y + \alpha'z + dyz) I_2(n, y, z), \\
    I_0 &= (a + \lambda'z + \epsilon''z^2) + (\lambda' + \omega''z + \rho''z^2)y + (\epsilon' + \delta''z + \gamma''z^2)y^2, \\
    I_1 &= (\lambda + \omega z + \delta z^2) + (\omega' + cz + \theta'z^2)y + (\rho' + \theta z + \kappa z^2)y^2,
\end{align*}
\]

and

\[
I_2 = (e + \rho z + \gamma z^2) + (\delta' + \theta''z + \kappa'z^2)y + (\gamma' + \kappa''z + bz^2)y^2,
\]

where \( a(n), \beta(n), \gamma(n), \delta(n), \omega(n), \epsilon(n), \rho(n), \kappa(n), \) and \( \theta(n) \) are periodic functions of period 3 and \( a, b, c, d, \) and \( e \) are parameters, possesses an invariant

\[
I_n = \frac{I_0(n, y, z) + x f_3(n, y, z) + x^2 I_2(n, y, z)}{[e + \beta y + \beta'z + \alpha yz] + x(\beta'' + \alpha''y + \alpha'z + dyz)]. \tag{3.9}
\]

**Proof.** The requirement of invariant of \( I_n(n, x, y, z) \), Eq. (3.9), that is,

\[
I_{n+1}(n + 1, x', y', z') = I_n(n, x, y, z), \tag{3.10}
\]
yields the following quadratic equation for $z'$:

\[
\left[ \left( e + \beta y + \beta' z + \alpha y z \right) + x \left( \beta'' + \alpha'' y + \alpha' z + d y z \right) I_1(n, y, z) \right] z^2
\]

\[
\left[ \left( e + \beta y + \beta' z + \alpha y z \right) I_4(n, y, z) \right]
\]

\[
-\left[ \left( \beta'' + \alpha'' y + \alpha' z + d y z \right) \left[ I_0(n, y, z) + x^2 I_2(n, y, z) \right] \right] z'
\]

\[
x \left[ \left( \beta'' + \alpha'' y + \alpha' z + d y z \right) I_0(n, y, z) - \left( e + \beta y + \beta' z + \alpha y z \right) \right]
\]

\[
\times \left[ I_1(n, y, z) + x I_2(n, y, z) \right] = 0. \tag{3.11}
\]

Solving Eq. (3.11) for $z'$, we obtain the nonautonomous difference equation (3.9).

Here also, we mention that the difference equation and its invariant given in Theorem 3.2 of [7] are a special case of the above result.

4. TWO-COUPLED DIFFERENCE EQUATIONS

In this section we report only the results.

**Theorem.** A system of two-coupled difference equations defined by

\[
X_{n+1} = \frac{A_{12} + A_1 x_n + A_7 x_n^2}{(A_7 + A_3 x_n + A_9 x_n^2 + A_5 x_n y_n) x_{n-1}},
\]

\[
y_{n+1} = \frac{A_{11} + A_2 y_n + A_8 y_n^2}{(A_8 + A_4 y_n + A_{10} y_n^2 + A_5 x_n y_n) y_{n-1}}, \tag{4.1}
\]

where $A_i$, $i = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, \text{ and } 12$, are parameters possessing an invariant

\[
I_n = \frac{X(x_n, y_n, x_{n-1}, y_{n-1})}{x_n x_{n-1} y_n y_{n-1}}, \tag{4.2}
\]

where

\[
X = \left( A_1 y_n + A_2 x_n + A_3 x_n^2 y_n + A_4 x_n y_n^2 + A_6 x_n y_n \right) x_{n-1} y_{n-1}
\]

\[
+ \left( A_3 x_n + A_5 x_n y_n + A_7 + A_9 x_n^2 \right) y_n x_{n-1}^2 y_{n-1}
\]

\[
+ \left( A_4 y_n + A_5 x_n y_n + A_8 + A_{10} y_n^2 \right) x_n x_{n-1}^2 y_{n-1}^2
\]

\[
+ \left( A_2 y_n + A_8 y_n^2 + A_{11} \right) x_n x_{n-1} + \left( A_1 x_n + A_7 x_n^2 + A_{12} \right) y_n y_{n-1}.
\]
Since the proof of the theorem is a straightforward one, we have omitted its details here. Clearly the two-coupled difference equation and its invariant (4.1) and (4.2) generalize the Theorem 2.10 of Ref. [7].

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