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# Centers of generic algebras with involution

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**Abstract**

Let $F$ be an infinite field of characteristic different from 2. Let $n$ be a positive integer, and let $V = M_n(F) \oplus M_n(F)$. The projective symplectic and orthogonal groups, $PSp_n$ and $PO_n$, act on $V$ by simultaneous conjugation. Results of Procesi and Rowen have shown that $F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are the centers of the generic division algebras with symplectic and orthogonal involutions, respectively. Saltman has shown that $F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are stably isomorphic over $F$ for all $n$ even, and that for all $n$ odd $F(V)^{PO_n}$ is stably rational over $F$. Saltman has also shown that for all $n$ for which the highest power of 2 dividing $n$ is less than 8, $F(V)^{PSp_n}$ and therefore $F(V)^{PO_n}$ are stably rational over $F$. We show that the result is also true for all $n$ for which the highest power of 2 dividing $n$ is 8.


*Keywords:* Rationality; Flasque classes; Generic algebra; Symplectic and orthogonal groups

## 1. Introduction

Let $F$ be an infinite field of characteristic different from 2. Let $V = M_n(F) \oplus M_n(F)$, then the general linear group $GL_n$ acts on $V$ by simultaneous conjugation, and since its center acts trivially, we obtain an action of the projective general linear group $PGL_n$ on $V$. Let $PO_n$ and $PSp_n$, the projective orthogonal and symplectic groups. Results of Procesi [7]

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and Rowen [8] show that $F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are the centers of the generic division algebras of degree $n$ with symplectic and orthogonal involutions, respectively.

Let $G$ be a finite group and let $F$ be a field. Given a $ZG$-lattice $M$, let $F[M]$ denote the group algebra of the abelian group $M$, and let $F(M)$ be its quotient field. There is an action of $G$ on $F(M)$ via the $G$-action on $M$. Questions of rationality of $F(M)^G$, the fixed subfield of $F(M)$ under the action of $G$, are referred to as lattice invariant problems. The special case where $M = ZG$ is referred to as the Noether setting of $G$, and denoted by $F(G)$.

In [10] Saltman shows that $F(V)^{PSp_n}$ is stably isomorphic to the invariants of a certain lattice over the Weyl group, $W$, of $PSp_n$. This Weyl group is the wreath product of $Z/2Z$ by $S_m$, the symmetric group on $m$ letters with $m = n/2$. Saltman also shows that $F(V)^{PSp_n}$ and $F(V)^{PO_n}$ are stably isomorphic over $F$ for all $n$ even, and that for all $n$ odd $F(V)^{PO_n}$ is stably rational over $F$. Saltman further shows that $F(V)^{PSp_n}$, and thus $F(V)^{PO_n}$, is stably rational over $F$ for all $n$ for which the highest power of 2 dividing $n$ is less than 8. The main result of this article is that $F(V)^{PSp_n}$, and thus $F(V)^{PO_n}$, are also stably rational over $F$ for $n = 8s$ with $s$ odd. The proof goes as follows. Let $n = 2m$, and let $U_m$ be the standard rank $m$ integral representation of $S_m$, and let $I_m$ be defined by the exact sequence

$$0 \to I_m \to U_m \to Z \to 0.$$

Let $D_2 = \mathrm{Hom}(I_m/2I_m, F^*)$. In Theorems 3.2 and 3.3 we show that $F(V)^{PSp_n}$ is stably isomorphic to the invariants of the Noether setting of the group $D_2 \rtimes W$. In Theorem 3.4 we show that the invariants of the Noether settings of $D_2 \rtimes W$ and of $D_2 \rtimes S_m$ are stably isomorphic over $F$, and consequently $F(V)^{PSp_n}$ is stably isomorphic to the invariant of $F(D_2 \rtimes S_m)$. The main result, Theorem 3.5, now follows from Corollary 2.8, in which we show that the invariants of the Noether setting $D_2 \rtimes S_4$ are stably rational over $F$, from results of Katsylo and Schofield [5,9] on matrix invariants of composite size, and from Saltman's result on the rationality of $F(V)^{PO_s}$ for $s$ odd.

## 2. Preliminary results and definitions

Let $G$ be a finite group and let $F$ be a field. A $ZG$-lattice $M$ is a finitely generated $Z$-free $ZG$-module and as in the introduction, $F(M)$ denotes the quotient field of the group algebra of the abelian group $M$. We let $\mathcal{L}_G$ denote the category of $ZG$-lattices.

**Definition 2.1.** Let $G$ be a finite group and let $M$ be a $ZG$-lattice. $M$ is said to be a permutation module if it has a $Z$-basis permuted by $G$. $M$ is said to be stably permutation if there exist permutation modules $P$ and $P'$ such that $M \oplus P \cong P'$. $M$ is said to be invertible or permutation projective if it is a direct summand of a permutation module. $M$ is said to be quasi-permutation if there exists a $ZG$-exact sequence $0 \to M \to P \to R \to 0$ with $P$ and $R$ permutation.

Let $G$ be a finite group. An equivalence relation is defined on $\mathcal{L}_G$ as follows. $M$ and $M'$ in $\mathcal{L}_G$ are equivalent if there exist permutation modules $P$ and $P'$ such that $M \oplus P \cong$

$M' \oplus P'$. The set of equivalence classes forms an abelian monoid under the direct sum. The zero element is the class of all stably permutation lattices. The equivalence class of $M$ will be denoted by $[M]$. For any integer $n$, $H^n(G, M)$ will denote the $n$th Tate cohomology group of $G$ with coefficients in $M$. A $ZG$-lattice $M$ is flasque if $H^{-1}(H, M) = 0$ for all subgroups $H$ of $G$. A flasque resolution of $M$ is an exact sequence

$$0 \to M \to P \to E \to 0$$

with $P$ permutation and $E$ flasque. It follows directly from [4, Lemma 1.1], that any $ZG$-lattice has a flasque resolution. The flasque class of $M$ is $[E]$, and will be denoted by $\phi(M)$. By [2, Lemma 5, Section 1] $\phi(M)$ is independent of the flasque resolution of $M$. The lattices whose flasque class is 0 are the quasi-permutation lattices.

**Definition 2.2.** Let $L$ and $K$ be extension fields of a field $F$, and let $G$ be a finite subgroup of their groups of $F$-automorphisms. Then $L$ and $K$ are stably isomorphic as $G$-fields if there exist $G$-trivial indeterminates $x_1 \ldots x_n$ and $y_1 \ldots y_r$ such that $L(x_1 \ldots x_n) \cong K(y_1 \ldots y_r)$ as $F$-algebras, and the isomorphism respects their $G$-actions. If $F = K$ we also say that $K$ is stably rational over $F$.

**Notation 2.3.** For any positive integer $k$, $Z_k$ will denote $Z/kZ$. Henceforth $G = S_m$ the symmetric group on $m$ letters unless otherwise specified, and $F$ will be an infinite field of characteristic different from 2. We will denote by $H$ the subgroup of $G$ generated by $S_{m-2}$ and the transposition $(m-1, m)$.

**Definition 2.4.** We define the $ZG$-lattice $U_m$ to be the standard rank $m$ permutation representation of $G$, more precisely $U_m$ has the set $\{u_1, \ldots, u_m\}$ as a $Z$-basis and for $g \in G$, $g(u_i) = u_{g(i)}$. We define $B$ to be the sublattice of $U_m$ with $Z$-basis $\{u_i + u_m\colon 1 \leqslant i \leqslant m\}$. Finally we define $I_m$ by the following exact sequence $0 \to I_m \to U_m \to Z \to 0$.

**Remark 2.5.** There exists an exact sequence $0 \to B \to U_m \to Z_2 \to 0$ where the map $U_m \to Z_2$ sends $u_i$ to 1.

**Lemma 2.6.** *Let $D_2 = \mathrm{Hom}(I_m/2I_m, F^*)$. The field $F(B \otimes I_m)^G$ is stably isomorphic to the invariants of the Noether setting of the group $G' = D_2 \rtimes G$.*

**Proof.** We tensor the exact sequence of Remark 2.5 by $I_m$ to obtain

$$0 \to B \otimes I_m \to U_m \otimes I_m \to I_m/2I_m \to 0.$$

This sequence gives an embedding of $F(B \otimes I_m)$ into $F(U_m \otimes I_m)$ which respects the $G$-actions. By Galois theory $F(U_m \otimes I_m)$ is a Galois extension of $F(B \otimes I_m)$ with group $D_2$. Now we have Galois extensions

$$F(B \otimes I_m)^G \subset F(B \otimes I_m) \subset F(U_m \otimes I_m)$$

and it is easy to see that the Galois group of $F(U_m \otimes I_m)$ over $F(B \otimes I_m)^G$ is $G'$. Since the action of $G'$ on $F(U_m \otimes I_m)$ is faithful and $F$-linear, $F(U_m \otimes I_m)$ is stably isomorphic to $F(G')$ as $G'$-fields by [1, Lemma 1.3] and the result follows. □

**Proposition 2.7.** *Let $G$ be the symmetric group on* 4 *letters. The flasque class of the $ZG$-lattice $B \otimes I_4$ is equal to* 0*.*

**Proof.** There is a map from $U_4 \otimes U_4$ to $B$ sending $u_i \otimes u_j$ to $u_i - u_j$ if $i \neq j$ and $u_i \otimes u_i$ to $2u_i$. Now form the pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B \otimes I_4 & \longrightarrow & B \otimes U_4 & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & B \otimes I_4 & \longrightarrow & S & \longrightarrow & U_4 \otimes U_4 & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & M & \longrightarrow & M & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathrm{Res}^G_{S_3} B \cong \mathrm{Res}^G_{S_3} U_4$ and since $U_4 \cong ZG/S_3$ the middle horizontal sequence splits by [2, Lemma 2, Section 1]. Furthermore, we have $B \otimes U_4 \cong U_4 \otimes U_4$ by Frobenius reciprocity. Therefore we have an exact sequence

$$0 \to B \otimes I_4 \to M \oplus B \otimes U_4 \to U_4 \otimes U_4 \to 0.$$

By [2, Lemma 7, Section 1], $\phi(M) = \phi(B \otimes I_4)$, where as above $\phi(M)$ denotes the flasque class of $M$. Now changing to multiplicative notation, we let the set $\{y_{ij}\colon 1 \leqslant i, j \leqslant 4\}$ be a $Z$-basis for $U_4 \otimes U_4$, with $y_{ij}$ corresponding to $u_i \otimes u_j$. We define the sets

$$\begin{aligned}
A_1 &= \{t_{ij} = y_{ij} y_{ji}\colon 1 \leqslant i < j \leqslant 4\},\\
A_2 &= \left\{x_{1j} = y_{11} y_{1j}^{-2} y_{jj}^{-1}\colon 2 \leqslant j \leqslant 4\right\},\\
A_3 &= \{w_{ij} = y_{1i} y_{ij} y_{j1}\colon 1 < i < j \leqslant 4\}.
\end{aligned}$$

We also defined $x_{ij} = y_{ii} y_{ij}^{-2} y_{jj}^{-1}$ for all $i < j$. It is not difficult to check that $A_1 \cup A_2 \cup A_3$ is a basis for $M$ over $Z$. As above $H$ is defined to be the subgroup of $G$ generated by $S_2$ and the transposition $(3,4)$. The set $A_1$ is a $Z$-basis for $ZG/H$, since $A_1$ is a transitive $G$-set and the stabilizer of $t_{34}$ is $H$. The idea for the remainder of this proof comes from the proof of [6, Lemma 4, Section 3]. Let $V_4$ be the Klein four-group, then $G/V_4 \cong S_3$ and we let $T = ZS_3/S_2$. Consider the exact sequence

$$0 \to K \to M \oplus Z \to U_4 \oplus T \to 0,$$

where the map $M \oplus Z \to U_4 \oplus T$ is defined as follows. First let $\{t_1, t_2, t_3\}$ be a multiplicative Z-basis for $T$. Then

- $t_{12} \to u_1 u_2 t_1 t_2$ and for all $g \in G$ $t_{g(1),g(2)} = u_{g(1)} u_{g(2)} t_{\bar{g}(1)} t_{\bar{g}(2)}$ where $\bar{g} = gV_4$.
- $x_{12} \to (u_1 u_2 t_1 t_2)^{-1}$ and for all $g \in G$ $x_{g(1),g(2)} = (u_{g(1)} u_{g(2)} t_{\bar{g}(1)} t_{\bar{g}(2)})^{-1}$ where $\bar{g} = gV_4$.
- $w_{ij} \to u_1 u_i u_j t_1 t_2 t_3$.
- $z \to u_1 u_2 u_3 u_4 t_1 t_2 t_3$.

A direct calculation shows that this map is a group epimorphism whose kernel $K$ has as a $Z$-basis

$$\{t_{ij} x_{ij}\colon 1 \leqslant i < j \leqslant m\}.$$

In terms of the $y_{ij}$'s, $t_{ij} x_{ij} = y_{ii} y_{ij}^{-1} y_{jj}^{-1} y_{ji}$, and so $K \cong ZG/H^-$, that is $ZG/H$ tensored with the sign representation of $G$. Therefore, we have

$$0 \to ZG/H^- \to M \oplus Z \to U \oplus T \to 0.$$

By [2, Lemma 7, Section 1] $\phi(M) = \phi(ZG/H^-)$ and the latter is equal to 0, since the following sequence is exact

$$0 \to ZG/H^- \to ZG/S_2 \to ZG/H \to 0. \qquad \square$$

**Corollary 2.8.** *The invariants of the Noether setting of the group $G' = T_2 \rtimes S_4$ are stably rational over the base field $F$.*

**Proof.** By Lemma 2.6, $F(B \otimes I_4)^{S_4}$ and $F(G')^{G'}$ are stably isomorphic, and by [1, Lemma 1.4], $F(B \otimes I_4)^{S_4}$ is stably rational over $F$ since $\phi(B \otimes I_4) = 0$. $\square$

## 3. The center

In [10], Saltman gives a description of $Z_{2m}$, the center of the generic division algebra of degree $2m$ with symplectic involution over a base field $F$, as a lattice invariant problem over the Weyl group $W$ of $PSp_n$. This description will formulated in Lemma 3.1. Let $T$ be the direct sum of $m$ copies of $Z_2$, let $G = S_m$ and let $W$ be the semidirect product of $T$ by $G$, where $G$ acts on $T$ by permutating the summands; equivalently $W$ is the wreath product of $T$ by $Z_2$. We use the following notation some of which is the same as in Section 2.

- $G' = S_{2m}$, and $G = S_m$.
- $W = T \rtimes G$, where we let $\{\sigma_1, \ldots, \sigma_m\}$ generate $T$ and $T$ embeds into $G'$ by sending $\sigma_i$ to the transposition $(i, m+i)$.
- $U_{2m} \cong ZG'/S_{2m-1}$ and we let the set $\{u_i\colon i = 1, \ldots, 2m\}$ be its $Z$-basis.
- $U_m \cong ZG/S_{m-1}$.
- $I_{2m}$ will be defined by the exact sequence $0 \to I_{2m} \to U_{2m} \to Z \to 0$.
- $I_m$ will be defined by the exact sequence $0 \to I_m \to U_m \to Z \to 0$.
- $Y'$ will be the $ZW$-lattice with $Z$-basis $\{y_{ij}\colon i, j = 1, \ldots, 2m\}$ and with the following $W$-action.

We let $G$ act on the set $\{1, \ldots, 2m\}$ with the usual action on $\{1, \ldots, m\}$ and for $k \in \{1, \ldots, m\}$ $g(m+k) = g(k)$. Moreover,

$$\sigma_i(k) = \begin{cases} k & \text{if } i \neq k \bmod m, \\ m+i & \text{if } i = k \bmod m \text{ and } i \leqslant m, \\ i-m & \text{if } i = k \bmod m \text{ and } i > m. \end{cases}$$

Now for $w \in W$, $wy_{ij} = y_{w(i),w(j)}$.

The following lemma is in [10], we include the constructive part of the proof as it will be needed later.

**Lemma 3.1.** *We have an epimorphism from $Y'$ to $I_{2m}/I_m$ with kernel $Y_1$, such that $F(Y_1)^W$ is stably isomorphic to the center of the generic division algebra of degree $2m$ with symplectic involution.*

**Proof.** A $Z$-basis for $I_{2m}$ is the set $\{u_i - u_m, u_{m+i} - u_{2m}\colon i = 1, \ldots, m-1, u_m - u_{2m}\}$, and the sublattice with $Z$-basis $\{u_i - u_m + u_{m+i} - u_{2m}\colon i = 1, \ldots, m-1\}$ is isomorphic to $I_m$. Therefore, we have an exact sequence of $ZW$-lattices

$$0 \to I_m \to I_{2m} \to S \to 0.$$

We also have the exact sequence

$$0 \to Y \to Y' \to I_{2m} \to 0$$

where the map $Y' \to I_{2m}$ sends $y_{ij}$ to $u_i - u_j$. We form the pullback of the maps $Y' \to I_{2m}$ and $I_m \to I_{2m}$ to obtain the diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & S & \longrightarrow & S & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Y' & \longrightarrow & I_{2m} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & Y_1 & \longrightarrow & I_m & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By [10, Proposition 1.5], $F(Y_1)^W$ is stably isomorphic $F(V)^{PSp_n}$ and the latter is the center of the generic division algebra of degree $2m$ with symplectic involution by [7, pp. 377–378] and [8, p. 184]. $\square$

Recall that $D_2 = \mathrm{Hom}(I_m/2I_m, F^*)$. Let $T_1$ be the subgroup of $T$ generated by $\{\sigma_1, \dots, \sigma_{m-1}\}$. We define

$$\tilde{U} \cong Z[W/T_1 S_{m-1}] \otimes I_{T/T_1},$$

where $I_{T/T_1}$ is the kernel of the augmentation map from $Z[T/T_1]$ to $Z$. We have $\mathrm{Res}_W^{G'} U_{2m} = \mathrm{Res}_W^{G'} ZG'/S_{2m-1} = ZW/T_1 S_{m-1}$ by Mackey's subgroup theorem [3, Theorem 10.13].

**Theorem 3.2.** *Keeping the above notation, there is a $ZW$-lattice $Y''$ defined by the exact sequence*

$$0 \to Y_1 \to Y'' \to \tilde{U} \to 0$$

*such that $F(Y'')^W$ is equivalent to the invariants of the Noether setting of the group $D_2 \rtimes W$.*

**Proof.** As in Lemma 3.1, $S \cong I_{2m}/I_m$. Let $s_i = u_i - u_m + I_m$ for $i = 1, \dots, m-1$, and let $z = u_m - u_{2m} + I_m$. Then it is immediate that the set $\{s_1, \dots, s_{m-1}, z\}$ is a $Z$-basis for $S$. Now consider the sublattice $K$ of $S$ with $Z$-basis $\{t_i = 2s_i + z\colon i = 1, \dots, m-1,\ t_m = z\}$. For $i = 1, \dots, m-1$ we have

$$
\begin{aligned}
t_i &= 2(u_i - u_m) + u_m - u_{2m} + I_m \\
&= u_i - u_m - (u_{m+i} - u_{2m}) + u_m - u_{2m} + I_m = u_i - u_{m+i} + I_m.
\end{aligned}
$$

So $K$ is isomorphic to $U_m$ as a $G$-module, and $\sigma_i(t_k) = -t_k$ if $k = i$, $\sigma_i(t_k) = t_k$ otherwise. We have the $W$-exact sequence

$$0 \to K \to U_{2m} \to U_m \to 0$$

where the map $U_{2m} \to U_m$ sends $u_i$ to $u_{i \bmod m}$. Therefore $K = ZW \otimes_{TS_{m-1}} I_{T/T_1}$ and hence $K \cong \tilde{U}$.

A simple computation now shows that $S/\tilde{U} \cong D_2' = \mathrm{Hom}(D_2, F^*)$. Now we form the pullback of the maps $Y' \to S$ and $\tilde{U} \to S$ to obtain

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & D_2' & \longrightarrow & D_2' & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y' & \longrightarrow & S & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & Y'' & \longrightarrow & \tilde{U} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle vertical sequence gives a $W$-embedding of $F(Y'')$ into $F(Y')$ and by Galois theory $F(Y')^{D_2} \cong F(Y')$ as $W$-fields. Furthermore, $F(Y')^{D_2 \rtimes W} \cong F(Y'')^W$. Since the action of $D_2 \rtimes W$ on $F(Y')$ is faithful and $F$-linear, $F(Y')^{D_2 \rtimes W}$ is stably isomorphic to $F(D_2 \rtimes W)^{D_2 \rtimes W}$ by [1, Lemma 1.3]. □

**Theorem 3.3.** *The fields $F(Y'')$ and $F(Y_1)$ are stably isomorphic as $W$-fields. Consequently the center of the generic division algebra of degree 2m with symplectic involution is equivalent to the invariants of the Noether setting $F(D_2 \rtimes W)$.*

**Proof.** We have the $ZW$-exact sequence

$$0 \to Y_1 \to Y'' \to \tilde{U} \to 0$$

and since $U_m \cong ZW/TS_{m-1}$ we also have

$$0 \to \tilde{U} \to U_{2m} \to U_m \to 0.$$

Let $L = F(Y_1)$. Then $F(Y'') \cong L_\alpha(\tilde{U})$ for some $\alpha \in \mathrm{Ext}^1_W(\tilde{U}, L^*)$. We show that $\mathrm{Ext}^1_W(\tilde{U}, L^*) = 0$. We have

$$\mathrm{Ext}^1_W\bigl(\tilde{U}, L^*\bigr) \cong \mathrm{Ext}^1_W\bigl(ZW \otimes_{ZTS_{m-1}} I_{T/T_1}, L^*\bigr) \cong \mathrm{Ext}^1_{TS_{m-1}}\bigl(I_{T/T_1}, L^*\bigr)$$

by Shapiro's Lemma. Set $\hat{I} = \mathrm{Hom}(I_{T/T_1}, L^*)$, then the inflation-restriction sequence gives

$$0 \to H^1\bigl(TS_{m-1}/T_1S_{m-1}, \hat{I}^{T_1S_{m-1}}\bigr) \to H^1\bigl(TS_{m-1}, \hat{I}\bigr) \to H^1\bigl(T_1S_{m-1}, \hat{I}\bigr).$$

Now $\hat{I} \cong L^*$ as a $T_1S_{m-1}$-module, so $H^1(T_1S_{m-1}, \hat{I}) = 0$ by Hilbert's Theorem 90, and $\hat{I}^{T_1S_{m-1}} = \mathrm{Hom}_{T_1S_{m-1}}(I, L^*) \cong (L^*)^{T_1S_{m-1}} \cong (L^{T_1S_{m-1}})^*$. Then

$$H^1\bigl(TS_{m-1}/T_1S_{m-1}, \hat{I}^{T_1S_{m-1}}\bigr) \cong H^1\bigl(TS_{m-1}/T_1S_{m-1}, \bigl(L^{T_1S_{m-1}}\bigr)^*\bigr) = 0$$

again by Hilbert's Theorem 90, since the action of $TS_{m-1}/T_1S_{m-1}$ on $L^{T_1S_{m-1}}$ is faithful. Therefore $\mathrm{Ext}^1_W(\tilde{U}, L^*) = 0$ and hence $F(Y'') \cong L(\tilde{U}) \cong F(Y_1 \oplus \tilde{U})$ as $W$-fields.

Since $\mathrm{Res}^W_G \tilde{U} \cong U_m$ we choose the same $Z$-basis for both, namely the set $\{u_1, \ldots, u_m\}$ with the natural action of $W$, and we view this basis as multiplicative. So $L(\tilde{U}) = L(u_1, \ldots, u_m)$. Set

$$z_i = \frac{1 + u_i^{-1}}{1 - u_i^{-1}},$$

then $L(u_1, \ldots, u_m) = L(z_1, \ldots, z_m)$ and the action of $W$ is $L$-linear, since $\sigma_i(z_k) = (-1)^{\delta_{ik}} z_k$ where $\delta$ is the Kronecker delta. Therefore $F(Y_1 \oplus \tilde{U}) = L(z_1, \ldots, z_m)$ and $L = F(Y'')$ are stably isomorphic as $W$-fields by [1, Lemma 1.3]. By Theorem 3.2 the last statement follows. □

**Theorem 3.4.** *The invariants of the Noether settings of the groups $D_2 \rtimes W$ and $D_2 \rtimes G$ are stably isomorphic over $F$. Consequently the center of the generic division algebra of degree $2m$ with symplectic involution is equivalent to the invariants of the Noether setting $F(D_2 \rtimes G)$.*

**Proof.** By [1, Lemma 1.3] $F(M)$ and $F(D_2 \rtimes W)$ are stably isomorphic over $F$ for any $F$-vector space $M$ on which $D_2 \rtimes W$ acts linearly and faithfully. Note that $D_2 \rtimes W = (D_2 \times T) \rtimes G$ by definition. So let $M = F \otimes_Z (U_m \otimes I_m \oplus U_m)$ with the usual $G$-action and with the following actions of $T$ and $D_2$. We let $T$ act trivially on $U_m \otimes I_m$. Recall that $D_2 = \mathrm{Hom}(I_m/2I_m, F^*)$. We obtain a faithful $D_2 \rtimes G$-faithful action on $F(U_m \otimes I_m)$ via the exact sequence of Remark 2.5, namely

$$0 \to B \to U_m \to Z_2 \to 0,$$

which we now tensor by $I_m$ over $Z$

$$0 \to B \otimes I_m \to U_m \otimes I_m \to I_m/2I_m \to 0.$$

Now by Kummer theory $D_2 \rtimes G$ acts $F$-linearly and faithfully on $F(U_m \otimes I_m)$. We take $\{y_{ij}\colon i, j = 1, \ldots, m,\ i \neq j\}$ as a multiplicative basis for $U_m \otimes I_m$. We let $D_2$ act trivially on $U_m$, and we let $\sigma_k(u_i) = -u_i$ if $k = i$ and $\sigma_k(u_i) = u_i$, otherwise. Then

$$F(M)^T = F(y_{ij})\big(u_1^2, \ldots, u_m^2\big)$$

and by [1, Lemma 1.3] $F(M)^T$ is stably isomorphic to $F(U_m \otimes I_m)$ as a $D_2 \rtimes G$-field. Since the action of $D_2 \rtimes G$ on $F(U_m \otimes I_m)$ is linear and faithful, $F(U_m \otimes I_m)$ is stably isomorphic to the Noether setting $F(D_2 \rtimes G)$. The last statement follows from Theorem 3.3. □

**Theorem 3.5.** *For $s$ odd, the center of the generic division algebra of degree $8s$ with involution over an infinite field $F$ of characteristic different from 2 is stably rational over $F$.*

**Proof.** We let $G$ denote the symmetric group on 4 letters, and hence $W = T \rtimes G$ where now $T$ is direct sum of 4 copies of $Z_2$. Let $n = 8s$. By [10, Corollary 0.6 and Theorem 1.1] the centers of the generic division algebras of even degree with symplectic and orthogonal involutions are stably isomorphic. Let $Z_s$, $Z_8$ and $Z_n$ be the centers of the generic division algebras of degrees $s$, 8, and $n$ with the appropriate involutions. The proof of [10, Lemma 5.2] shows that stable rationality of $Z_s$ and $Z_8$ implies stable rationality of $Z_n$. An earlier proof from which this result follows can be found in [5,9]. By [10, Theorem 1.2] $Z_s$ is rational over $F$, thus it remains to prove that $Z_8$ is stably rational over $F$. We keep all the above notation. By Theorem 3.3, $Z_8$ is stably isomorphic to the invariants of the Noether setting $F(D_2 \rtimes W)$. By Theorem 3.4, these invariants are stably isomorphic to those of $D_2 \rtimes G$ and the result now follows by Corollary 2.8. □

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