Four-dimensional Yetter–Drinfeld module algebras over $H_4$

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Abstract

In this paper, we classify four-dimensional Yetter–Drinfeld module algebras over Sweedler’s 4-dimensional Hopf algebra $H_4$. All $H_4$-Azumaya algebras of dimension 4 have been given. In particular, all Yetter–Drinfeld $H_4$-module algebra structures on the matrix algebra $M_2(k)$ are classified.

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Introduction

Let $k$ be a commutative ring with unity and $H$ be a Hopf algebra over $k$ with a bijective antipode. In [3,4] we introduced $H$-Azumaya algebras and the Brauer group $BQ(k,H)$ classifying the Azumaya algebras in the Yetter–Drinfeld module category $_{H\mathcal{YD}}^H$. If $(H,R)$ is a quasitriangular (or coquasitriangular) Hopf algebra, then we consider the equivariant Brauer group $BM(k,H,R)$ (or $BC(k,H,R)$) which is a subgroup of $BQ(k,H)$. The Brauer group $BQ(k,H)$ of $H$ is isomorphic to the equivariant Brauer...
group $\text{BM}(k, D(H), R)$ of the quantum double of $H$ with the canonical $R$-matrix $R$ when $H$ is a finite Hopf algebra. In [15] the equivariant Brauer group $\text{BM}(k, H_4, R_0)$ of the triangular Hopf algebra $(H_4, R_0)$, Sweedler’s 4-dimensional Hopf algebra over a field $k$, has been computed. The group is isomorphic to the direct product of the Brauer–Wall group $\text{BW}(k)$ and the additive group $k^+$, whose elements are represented by the $2 \times 2$ matrix algebra $M_2(k)$ with Galois actions of $H_4$. On the other hand, the equivariant Brauer group $\text{BM}(k, H_4, R_0)$ can be computed via an exact sequence constructed in [19] through the computation of the group of the bi-Galois objects of the transmuted braided group of $(H_4, R_0)$. Those bi-Galois objects consist of generalized quaternion algebras and 4-dimensional commutative algebras with Yetter–Drinfeld $H_4$-module structures. It is therefore natural to consider Yetter–Drinfeld $H_4$-module structures on 4-dimensional algebras and particularly the Yetter–Drinfeld Azumaya structures on the matrix algebras $M_2(k)$. An interesting question raised by S. Montgomery and H.-J. Schneider is that how many non-isomorphic $H_4$-Azumaya structures exist on $M_2(k)$. We will answer this question and go further to classify all 4-dimensional $H_4$-Azumaya algebras. To this end, we will first classify all 4-dimensional Yetter–Drinfeld $H_4$-module algebras. There are 29 types of 4-dimensional Yetter–Drinfeld $H_4$-module algebras in terms of Yetter–Drinfeld module structures. For each type of $H_4$-module algebras, we give a further classification of non-isomorphic algebra structures.

In Section 3, we classify all 4-dimensional $H_4$-Azumaya algebras. There are five classes of 4-dimensional $H_4$-Azumaya algebras: the class of quaternion algebras with trivial $H_4$-action and trivial $H_4$-coactions; the class of quaternion algebras with non-trivial $\mathbb{Z}_2$-gradings; the class of generalized quaternion algebras with $H_4$-coactions induced by $H_4$-actions through the quasitriangular structures of $H_4$; and the class of generalized quaternion algebras with $H_4$-actions induced by $H_4$-coactions through one cotriangular structure of $H_4$; and the class of generalized quaternion algebras with $H_4$-actions and $H_4$-coactions not induced from each or another. The last class of $H_4$-Azumaya algebras are projective as $D(H_4)$-modules. The detailed classification up to isomorphism is given in Theorem 3.6.

In the final section, we will classify all non-isomorphic Yetter–Drinfeld $H_4$-module algebra structures on $M_2(k)$ of $2 \times 2$ matrices. The classification of $\mathbb{Z}_2$-graded algebra structures on $M_2(k)$ as a special case of our classification has been done in [2,10]. It turns out only four types of four-dimensional Yetter–Drinfeld $H_4$-modules admit the matrix algebra structure. The rich $H_4$-Azumaya algebra structures on $M_2(k)$ are highlighted in Theorem 4.10, where all Yetter–Drinfeld module algebra structures on $M_2(k)$ except
one type are $H_4$-Azumaya algebra structures. Moreover, there are infinitely many non-isomorphic elementary $H_4$-Azumaya algebra structures on $M_2(k)$ although they represent the same element 1 of the Brauer group $BQ(k, H_4)$.

1. Preliminaries

1.1. Yetter–Drinfeld module algebras

Throughout, we work over a fixed field $k$ with $\text{char}(k) \neq 2$. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over $k$; all maps are $k$-linear; $\dim$, $\otimes$ and $\text{Hom}$ stand for $\dim_k$, $\otimes_k$ and $\text{Hom}_k$, respectively. For the theory of Hopf algebras and quantum groups, we refer to [11,13,14,17]. Let $k^\times$ denote the multiplicative group of all non-zero elements in the field $k$, and $k^{\times 2} := \{\alpha^2 \mid \alpha \in k^\times\}$ is the subgroup of square elements.

Let $H$ be a Hopf algebra with a bijective antipode $S$. A Yetter–Drinfeld $H$-module (simply, YD $H$-module) $M$ is a crossed $H$-bimodule. That is, $M$ is at once a left $H$-module and a right $H$-comodule satisfying the following equivalent compatibility conditions:

(i) $\sum h(1) \cdot m(0) \otimes h(2)m(1) = \sum (h(2) \cdot m)(0) \otimes (h(2) \cdot m)(1)h(1)$,
(ii) $\sum (h \cdot m)(0) \otimes (h \cdot m)(1) = \sum h(2) \cdot m(0) \otimes h(3)m(1)S^{-1}(h(1))$,

where $h \in H$, $m \in M$, and the sigma notations for a comodule and for a comultiplication can be found in the reference book [17]. A Yetter–Drinfeld $H$-module algebra (simply, YD $H$-module algebra) is a YD $H$-module satisfying the following equivalent compatibility conditions:

(i) $\sum h(1) \cdot m(0) \otimes h(2)m(1) = \sum (h(2) \cdot m)(0) \otimes (h(2) \cdot m)(1)h(1)$,
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(i) $\sum h(1) \cdot m(0) \otimes h(2)m(1) = \sum (h(2) \cdot m)(0) \otimes (h(2) \cdot m)(1)h(1)$,
(ii) $\sum (h \cdot m)(0) \otimes (h \cdot m)(1) = \sum h(2) \cdot m(0) \otimes h(3)m(1)S^{-1}(h(1))$,
\[(h \cdot m^*)(m) = m^*(S(h) \cdot m),\]
\[\sum m^*_0(m)m^*_1 = \sum m^*(m(0))S^{-1}(m(1)),\]
where \(h \in H\), \(m^* \in M^*\) and \(m \in M\). The endomorphism algebra \(\text{End}_k(M)\) is a \(\text{YD } H\)-module algebra with the \(H\)-structures induced by those of \(M\), i.e., for \(h \in H\), \(f \in \text{End}_k(M)\) and \(m \in M\),
\[(h \cdot f)(m) = \sum h(1) \cdot f(S(h(2)) \cdot m),\]
\[\chi(f)(m) = \sum f(m(0))(1) \otimes \sum f(m(0))(1)S(m(1))\]
for \(m \in M\), \(h \in H\) and \(f \in \text{End}_k(M)\). A \(\text{YD } H\)-module algebra is called \textit{elementary} if it is isomorphic to \(\text{End}(M)\) for some finite \(\text{YD } H\)-module \(M\).

Now let \(H\) be a quasitriangular Hopf algebra, that is, \(H\) is a Hopf algebra with an invertible element \(R = \sum R^1 \otimes R^2\) in \(H \otimes H\) satisfying the following axioms \((r = R)\):

(QT1) \(\sum \Delta(R^1) \otimes R^2 = \sum R^1 \otimes r^1 \otimes R^2 r^2\),
(QT2) \(\sum \epsilon(R^1)R^2 = 1\),
(QT3) \(\sum R^1 \otimes \Delta(R^2) = \sum R^1 r^1 \otimes r^2 \otimes R^2\),
(QT4) \(\sum R^1 \epsilon(R^2) = 1\),
(QT5) \(\Delta^{\text{cop}}(h)R = R \Delta(h)\),

where \(\Delta^{\text{cop}} = \tau \Delta\) is the comultiplication of the Hopf algebra \(H^{\text{cop}}\) and \(\tau\) is the flip map.

If \(A\) is a left \(H\)-module algebra, then \(A\) is simultaneously a \(\text{YD } H\)-module algebra with the right \(H^{\text{op}}\)-comodule structure given by
\[A \rightarrow A \otimes H^{\text{op}}, \quad a \mapsto \sum (R(2) \cdot a) \otimes R(1), \quad a \in A.\]

1.2. \(H\)-Azumaya algebras

A \(\text{YD } H\)-module algebra \(A\) is called \(H\)-Azumaya if the following \(\text{YD } H\)-module algebra maps are isomorphisms:

\[F : A \# \bar{A} \rightarrow \text{End}(A), \quad F(a \# \bar{b})(c) = \sum ac(0)(c(1) \cdot b),\]
\[G : \bar{A} \# A \rightarrow \text{End}(A)^{\text{op}}, \quad G(\bar{a} \# b)(c) = \sum a(0)(a(1) \cdot c)b.\]
If a YD $H$-module algebra is $H$-Azumaya, then its left center

$$A^A := \{ x \in A \mid xy = \sum y(0)(y(1) \cdot x), \ \forall y \in A \}.$$  

and its right center

$$A^A := \{ x \in A \mid yx = \sum x(0)(x(1) \cdot y), \ \forall y \in A \}$$

are both trivial (see [4]).

Let $H$ be a quasitriangular Hopf algebra with a quasitriangular structure $R = \sum R^1 \otimes R^2$. Then the category $\mathcal{M}_H$ of the left $H$-modules is a braided category, denoted by $\mathcal{M}_H^R$. Explicitly, let $M$ and $N$ be left $H$-modules. Then there is an $H$-module isomorphism

$$\Phi : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto \sum R^2 \cdot n \otimes R^1 \cdot m.$$  

In this case a left $H$-module algebra $A$ is called $H$-Azumaya with respect to $R$ if $A$ is $H$-Azumaya as a YD $H$-module algebra with the right $H^{\text{op}}$-comodule structure (1).

Let $H$ be a finite-dimensional Hopf algebra over $k$ and let $D(H) = H^{\text{cop}} \bowtie H$ be the Drinfeld quantum double of $H$. Then $D(H)$ is a quasitriangular Hopf algebra with the canonical quasitriangular structure

$$\mathcal{R} = \sum_{i=1}^n (\varepsilon \bowtie h_i) \otimes (\overline{h_i} \bowtie 1),$$

where $\{ h_1, h_2, \ldots, h_n \}$ is a $k$-basis of $H$ and $\{ \overline{h_1}, \overline{h_2}, \ldots, \overline{h_n} \}$ is the dual basis of $H^*$, the dual Hopf algebra of $H$. It is well known that an algebra $A$ is a YD $H$-module algebra if and only if $A$ is a left $D(H)$-module algebra (see [13]). Moreover, a YD $H$-module algebra $A$ is $H$-Azumaya if and only if the left $D(H)$-module algebra $A$ is $D(H)$-Azumaya with respect to the canonical quasitriangular structure $\mathcal{R}$. Thus in order to study YD $H$-module algebras, we suffice to study the left $D(H)$-module algebras.

1.3. Drinfeld double of $H_4$

Sweedler’s 4-dimensional Hopf algebra is a special case of Taft’s Hopf algebras. The Drinfeld doubles of Taft’s Hopf algebras and their finite representations were investigated in [5–8]. Let us recall some results which we need throughout the paper.

Sweedler’s 4-dimensional Hopf algebra $H_4$ is generated by two elements $g$ and $h$ subject to the relations:

$$g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.$$  

The coalgebra structure and the antipode are determined by
$$\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes g + 1 \otimes h, \quad \varepsilon(x) = 0,$$
$$\varepsilon(h) = 0, \quad S(g) = g^{-1} = g, \quad S(h) = gh.$$  

Moreover, $H_4$ has a canonical basis $\{1, g, h, gh\}$.

Let $D_4$ be the algebra generated by $a, b, c$ and $d$ subject to the relations:

$$ba = -ab, \quad db = -bd, \quad ca = -ac, \quad dc = -cd, \quad bc = cb,$$
$$a^2 = 0, \quad b^2 = 1, \quad c^2 = 1, \quad d^2 = 0, \quad da + ad = 1 - bc.$$  

Then $D_4$ is a Hopf algebra with the coalgebra structure and the antipode given by

$$\Delta(a) = a \otimes b + 1 \otimes a, \quad \varepsilon(a) = 0, \quad S(a) = -ab = ba,$$
$$\Delta(b) = b \otimes b, \quad \varepsilon(b) = 1, \quad S(b) = b^{-1} = b,$$
$$\Delta(c) = c \otimes c, \quad \varepsilon(c) = 1, \quad S(c) = c^{-1} = c,$$
$$\Delta(d) = d \otimes c + 1 \otimes d, \quad \varepsilon(d) = 0, \quad S(d) = -dc = cd.$$  

$D_4$ is a $2^4$-dimensional Hopf algebra. $D_4$ has a canonical basis $\{a^i b^j c^l d^k \mid 0 \leq i, j, l, k \leq 1\}$, and is not semisimple and is isomorphic to $D(H_4)$ as a Hopf algebra. The Hopf algebra isomorphism is given by

$$D(H_4) = H_4^{*\text{cop}} \triangleright H_4 \to D_4, \quad h^i g^j \mapsto \sum_{0 \leq m < 2} \frac{1}{2} (-1)^m c^m d^i a^i b^j$$  

for all $0 \leq s, t, i, j \leq 1$, where $\{h^i g^j \mid 0 \leq i, j \leq 1\}$ is the basis of $H_4$, and $\{h^i g^j \mid 0 \leq i, j \leq 1\}$ is the dual basis of $H_4^{*\text{cop}}$. The canonical quasitriangular structure on $D_4$ reads as follows:

$$\mathcal{R} = \frac{1}{2} (1 \otimes 1 + b \otimes 1 + 1 \otimes c - b \otimes c + a \otimes d + ab \otimes d + a \otimes cd - ab \otimes cd).$$  

### 1.4. Indecomposable representations of $D_4$

Let $J(D_4)$ stand for the Jacobson radical of $D_4$. Then $J(D_4)^3 = 0$ by [8, Corollary 2.4]. This means that the Loewy length of $D_4$ is 3. In order to study the left $D_4$-module algebras of dimension 4, we need first to give the structures of all indecomposable $D_4$-modules with dimension not exceeding 4. We will follow the notations of [8].

Let $M$ be an indecomposable $D_4$-module with $\dim M \leq 4$. We list all indecomposable $D_4$-modules according to their Loewy length. There are four simple $D_4$-modules (up to isomorphism); two are of dimension one and two are of dimension two. In the following, denote $J(D_4)$ by $J$ for short.

One-dimensional simple modules: $V(1, r), r \in \mathbb{Z}_2, \quad a \cdot v = d \cdot v = 0, \quad b \cdot v = c \cdot v = (-1)^r v, \quad v \in V(1, r).$  

(2)
Two-dimensional simple modules: \( V(2, r), r \in \mathbb{Z}_2 \). \( V(2, r) \) has a standard \( k \)-basis \( \{v_1, v_2\} \) such that
\[
\begin{align*}
    a \cdot v_1 &= v_2, & d \cdot v_1 &= 0, & b \cdot v_1 &= (-1)^r v_1, & c \cdot v_1 &= (-1)^{r+1} v_1, \\
    a \cdot v_2 &= 0, & d \cdot v_2 &= 2 v_1, & b \cdot v_2 &= (-1)^{r+1} v_2, & c \cdot v_2 &= (-1)^r v_2.
\end{align*}
\] (3)

The simple module \( V(2, r), r = 0, 1 \), are both projective and injective.

Four-dimensional projective modules of Loewy length 3: Let \( P(1, r) \) be the projective cover of \( V(1, r), r = 0, 1 \). Then \( P(1, r) \) is the injective envelope of \( V(1, r) \) as well, \( r \in \mathbb{Z}_2 \). \( P(1, r) \) has a standard \( k \)-basis \( \{v_1, v_2, v_3, v_4\} \) such that
\[
\begin{align*}
    a \cdot v_1 &= v_2, & d \cdot v_1 &= v_3, & b \cdot v_1 &= (-1)^r v_1, & c \cdot v_1 &= (-1)^r v_1, \\
    a \cdot v_2 &= 0, & d \cdot v_2 &= -v_4, & b \cdot v_2 &= (-1)^{r+1} v_2, & c \cdot v_2 &= (-1)^{r+1} v_2, \\
    a \cdot v_3 &= v_4, & d \cdot v_3 &= 0, & b \cdot v_3 &= (-1)^{r+1} v_3, & c \cdot v_3 &= (-1)^{r+1} v_3, \\
    a \cdot v_4 &= 0, & d \cdot v_4 &= 0, & b \cdot v_4 &= (-1)^r v_4, & c \cdot v_4 &= (-1)^r v_4.
\end{align*}
\] (4)

Note that
\[
\begin{align*}
    \text{soc}(P(1, r)) &= J^2 \cdot P(1, r) \cong V(1, r), \\
    \text{soc}^2(P(1, r))/\text{soc}(P(1, r)) &= (J \cdot P(1, r))/(J^2 \cdot P(1, r)) \cong 2V(1, r + 1) \quad \text{and} \\
    P(1, r)/\text{soc}^2(P(1, r)) &= P(1, r)/(J \cdot P(1, r)) \cong V(1, r).
\end{align*}
\]

There are infinitely many non-isomorphic indecomposable \( D_4 \)-modules with Loewy length 2. We list them according to the lengths and the co-lengths of their socles. We say that an indecomposable module \( M \) with Loewy length 2 is of \((s, t)\)-type if \( l(\text{soc}(M)) = t \) and \( l(M/\text{soc}(M)) = s \).

If \( M \) is of \((1, 1)\)-type then \( M \cong M_1(1, r, \eta) \), where \( r \in \mathbb{Z}_2 \), and \( \eta = \infty \) or \( \eta \in k \). The indecomposable module \( M_1(1, r, \infty) \) is of dimension 2 and has a standard basis \( \{v_1, v_2\} \) with the \( D_4 \)-action given by
\[
\begin{align*}
    a \cdot v_1 &= 0, & d \cdot v_1 &= v_2, & b \cdot v_1 &= (-1)^{r+1} v_1, & c \cdot v_1 &= (-1)^{r+1} v_1, \\
    a \cdot v_2 &= 0, & d \cdot v_2 &= 0, & b \cdot v_2 &= (-1)^r v_2, & c \cdot v_2 &= (-1)^r v_2.
\end{align*}
\] (5)

It is easy to see that \( M_1(1, r, \infty) \) is a submodule of \( P(1, r) \) and a quotient module of \( P(1, r + 1) \). Moreover, we have \( \text{soc}(M_1(1, r, \infty)) = J \cdot M_1(1, r, \infty) \cong V(1, r) \), and \( M_1(1, r, \infty)/\text{soc}(M_1(1, r, \infty)) \cong V(1, r + 1) \).

The indecomposable module \( M_1(1, r, \eta), r \in \mathbb{Z}_2, \eta \in k \), has a standard basis \( \{v_1, v_2\} \) with the \( D_4 \)-action given by
\[
\begin{align*}
    a \cdot v_1 &= v_2, & d \cdot v_1 &= -\eta v_2, & b \cdot v_1 &= (-1)^{r+1} v_1, & c \cdot v_1 &= (-1)^{r+1} v_1, \\
    a \cdot v_2 &= 0, & d \cdot v_2 &= 0, & b \cdot v_2 &= (-1)^r v_2, & c \cdot v_2 &= (-1)^r v_2.
\end{align*}
\] (6)
$M_1(1, r, \eta)$ is a submodule of $P(1, r)$ and a quotient module of $P(1, r + 1)$. Moreover,

$$
soc(M_1(1, r, \eta)) = J \cdot M_1(1, r, \eta) \cong V(1, r) \quad \text{and} \quad M_1(1, r, \eta)/soc(M_1(1, r, \eta)) \cong V(1, r + 1).
$$

The indecomposable module of $(2, 1)$-type is given by the syzygy functor $\Omega$. Let $V(1, r), r = 0, 1$, be the one-dimensional simple modules. Then the syzygy $\Omega^1 V(1, r) \cong \text{Ker}(P(1, r) \twoheadrightarrow V(1, r)) = J \cdot P(1, r)$ is of $(2, 1)$-type. $\Omega V(1, r)$ has a standard $k$-basis \{v_1, v_2, v_3\} with the $D_4$-action given by

$$
a \cdot v_1 = 0, \quad d \cdot v_1 = -v_3, \quad b \cdot v_1 = (-1)^{r+1}v_1, \quad c \cdot v_1 = (-1)^r v_1,
$$

$$
a \cdot v_2 = v_3, \quad d \cdot v_2 = 0, \quad b \cdot v_2 = (-1)^{r+1}v_2, \quad c \cdot v_2 = (-1)^r v_2,
$$

$$
a \cdot v_3 = 0, \quad d \cdot v_3 = 0, \quad b \cdot v_3 = (-1)^r v_3, \quad c \cdot v_3 = (-1)^r v_3. \quad (7)
$$

If $M$ is of $(1, 2)$-type, then $M \cong \Omega^{-1} V(1, r)$ for some $r \in \mathbb{Z}_2$, where $\Omega^{-1}$ is the cosyzygy functor. That is,

$$
\Omega^{-1} V(1, r) \cong Coker(V(1, r) \twoheadrightarrow P(1, r)) = P(1, r)/soc(P(1, r)).
$$

The module $\Omega^{-1} V(1, r)$ has a standard $k$-basis \{v_1, v_2, v_3\} with the $D_4$-action given by

$$
a \cdot v_1 = v_2, \quad d \cdot v_1 = v_3, \quad b \cdot v_1 = (-1)^r v_1, \quad c \cdot v_1 = (-1)^r v_1,
$$

$$
a \cdot v_2 = 0, \quad d \cdot v_2 = 0, \quad b \cdot v_2 = (-1)^{r+1} v_2, \quad c \cdot v_2 = (-1)^{r+1} v_2,
$$

$$
a \cdot v_3 = 0, \quad d \cdot v_3 = 0, \quad b \cdot v_3 = (-1)^{r+1} v_3, \quad c \cdot v_3 = (-1)^{r+1} v_3. \quad (8)
$$

There are three classes of indecomposable modules of $(2, 2)$-type. If $M$ is of $(2, 2)$-type and contains a submodule of $(1, 1)$-type, then $M$ is isomorphic to the module $M_2(1, r, \eta)$, where $r \in \mathbb{Z}_2$, $\eta = \infty$ or $\eta \in k$. Using the pullback diagrams given in [8, p. 15], one can describe the structures of the modules $M_2(1, r, \eta), r \in \mathbb{Z}_2, \eta = \infty$ or $\eta \in k$. If $M$ is of $(2, 2)$-type and does not contain any submodule of $(1, 1)$-type, then $M$ is isomorphic to a submodule of $2P(1, r)$ for some $r \in \mathbb{Z}_2$ (see [8]). Using the structure of $P(1, r)$, one can prove that $M$ has the structure described in (11) below (see Lemma 2.30). Now we list these indecomposable modules of $(2, 2)$-type as follows.

The module $M_2(1, r, \infty), r \in \mathbb{Z}_2$, has a standard basis \{v_1, v_2, v_3, v_4\} with the $D_4$-action given by

$$
a \cdot v_1 = 0, \quad d \cdot v_1 = v_3, \quad b \cdot v_1 = (-1)^{r+1} v_1, \quad c \cdot v_1 = (-1)^r v_1,
$$

$$
a \cdot v_2 = v_3, \quad d \cdot v_2 = v_4, \quad b \cdot v_2 = (-1)^{r+1} v_2, \quad c \cdot v_2 = (-1)^{r+1} v_2,
$$

$$
a \cdot v_3 = 0, \quad d \cdot v_3 = 0, \quad b \cdot v_3 = (-1)^r v_3, \quad c \cdot v_3 = (-1)^r v_3,
$$

$$
a \cdot v_4 = 0, \quad d \cdot v_4 = 0, \quad b \cdot v_4 = (-1)^r v_4, \quad c \cdot v_4 = (-1)^r v_4. \quad (9)
\(M_2(1, r, \infty)\) is a submodule of \(2P(1, r)\) and a quotient module of \(2P(1, r + 1)\). Moreover, \(\text{soc}(M_2(1, r, \infty)) = J \cdot M_2(1, r, \infty) \cong 2V(1, r)\) and \(M_2(1, r, \infty)/\text{soc}(M_2(1, r, \infty)) \cong 2V(1, r + 1)\). Furthermore, \(M_2(1, r, \infty)\) contains a unique submodule of \((1, 1)\)-type, which is equal to \(kv_1 + kv_3\) and isomorphic to \(M_1(1, r, \infty)\).

The second class of \((2, 2)\)-type indecomposable modules \(M_2(1, r, \eta), r \in \mathbb{Z}_2, \eta \in k\), has a standard basis \(\{v_1, v_2, v_3, v_4\}\) with the \(D_4\)-action given by

\[
\begin{align*}
a \cdot v_1 &= v_3, & d \cdot v_1 &= -\eta v_3, & b \cdot v_1 &= (-1)^{r+1} v_1, & c \cdot v_1 &= (-1)^{r+1} v_1, \\
a \cdot v_2 &= v_4, & d \cdot v_2 &= -v_3 - \eta v_4, & b \cdot v_2 &= (-1)^{r+1} v_2, & c \cdot v_2 &= (-1)^{r+1} v_2, \\
a \cdot v_3 &= 0, & d \cdot v_3 &= 0, & b \cdot v_3 &= (-1)^r v_3, & c \cdot v_3 &= (-1)^r v_3, \\
a \cdot v_4 &= 0, & d \cdot v_4 &= 0, & b \cdot v_4 &= (-1)^r v_4, & c \cdot v_4 &= (-1)^r v_4. \\
\end{align*}
\]

\(M_2(1, r, \eta)\) is a submodule of \(2P(1, r)\) and a quotient module of \(2P(1, r + 1)\). Moreover, \(\text{soc}(M_2(1, r, \eta)) = J \cdot M_2(1, r, \eta) \cong 2V(1, r)\) and \(M_2(1, r, \eta)/\text{soc}(M_2(1, r, \eta)) \cong 2V(1, r + 1)\). Furthermore, \(M_2(1, r, \eta)\) contains a unique submodule of \((1, 1)\)-type, which is equal to \(kv_1 + kv_3\) and isomorphic to \(M_1(1, r, \eta)\).

The last class of \((2, 2)\)-type indecomposable modules \(M_2(1, r, \delta, \eta), r \in \mathbb{Z}_2, \delta, \eta \in k\), are determined by an irreducible quadratic polynomial \(X^2 + \eta X + \delta\) in \(k[X]\). The module \(M_2(1, r, \delta, \eta)\) does not contain submodules of \((1, 1)\)-type. It has a \(k\)-basis \(\{v_1, v_2, v_3, v_4\}\) with the \(D_4\)-action given by

\[
\begin{align*}
a \cdot v_1 &= v_3, & d \cdot v_1 &= v_4, & b \cdot v_1 &= (-1)^{r+1} v_1, & c \cdot v_1 &= (-1)^{r+1} v_1, \\
a \cdot v_2 &= v_4, & d \cdot v_2 &= \eta v_4 - \delta v_3, & b \cdot v_2 &= (-1)^{r+1} v_2, & c \cdot v_2 &= (-1)^{r+1} v_2, \\
a \cdot v_3 &= 0, & d \cdot v_3 &= 0, & b \cdot v_3 &= (-1)^r v_3, & c \cdot v_3 &= (-1)^r v_3, \\
a \cdot v_4 &= 0, & d \cdot v_4 &= 0, & b \cdot v_4 &= (-1)^r v_4, & c \cdot v_4 &= (-1)^r v_4. \\
\end{align*}
\]

\(M_2(1, r, \delta, \eta)\) is a submodule of \(2P(1, r)\) and a quotient module of \(2P(1, r + 1)\). Moreover,

\[
\text{soc}(M_2(1, r, \delta, \eta)) = J \cdot M_2(1, r, \delta, \eta) \cong 2V(1, r) \quad \text{and} \quad M_2(1, r, \delta, \eta)/\text{soc}(M_2(1, r, \delta, \eta)) \cong 2V(1, r + 1).
\]

Note that the indecomposable module \(M_2(1, r, \delta, \eta)\) may exist only when \(k\) is not algebraically closed. The following lemma shows that any indecomposable module of \((2, 2)\)-type containing no submodules of \((1, 1)\)-type is isomorphic to some \(M_2(1, r, \delta, \eta)\) defined by (11), and that \(M_2(1, r, \delta, \eta)\) is uniquely determined by \(\delta\) and \(\eta\), which we will use in Section 2.

**Lemma 1.1.**

(a) Let \(M\) be a \((2, 2)\)-type indecomposable \(D_4\)-module such that \(M\) does not contain any submodules of \((1, 1)\)-type. Then \(M\) is isomorphic to \(M_2(1, r, \delta, \eta)\) for some \(r \in \mathbb{Z}_2\) and \(\delta, \eta \in k\) such that \(X^2 + \eta X + \delta\) is irreducible over \(k\).
(b) Let \( r \in \mathbb{Z}_2 \). Assume that \( \delta, \eta, \delta' \) and \( \eta' \) are elements in \( k \) such that \( X^2 + \eta X + \delta \) and \( X^2 + \eta' X + \delta' \) are irreducible in \( k[X] \). Then \( M_2(1, r, \delta, \eta) \cong M_2(1, r, \delta', \eta') \) if and only if \( \delta = \delta' \) and \( \eta = \eta' \).

**Proof.** (a) Assume that \( M \) is a \((2, 2)\)-type indecomposable \( D_4 \)-module such that \( M \) does not contain any submodules of \((1,1)\)-type. Then it follows from [8, Theorem 2.3(4), Lemma 3.6] that \( M \) is isomorphic to a submodule of \( 2P(1, r) \) for some \( r \in \mathbb{Z}_2 \). Hence we have \( M = M_0 \oplus M_1 = M_r \oplus M_{r+1} \), where \( M_r = \text{soc}(M) = \{ x \in M \mid b \cdot x = (-1)^r x \} \) and \( M_{r+1} = \{ x \in A \mid b \cdot x = (-1)^{r+1} x \} \). We also have that \( \dim M_0 = \dim M_1 = 2 \). It follows from [6, Lemma 2.2] that \( a \cdot M_{r+1} \subseteq M_r \) and \( d \cdot M_{r+1} \subseteq M_r \). We claim that \( a \cdot M_{r+1} = M_r \) and \( d \cdot M_{r+1} = M_r \). In fact, if \( a \cdot M_{r+1} \neq M_r \), then there exists a non-zero element \( x \in M_{r+1} \) such that \( a \cdot x = 0 \). Clearly, \( d \cdot x \neq 0 \). Otherwise, \( kx \) is a simple submodule of \( M \) which implies \( x \in \text{soc}(M) = M_r \), a contradiction. Hence span\( \{x, d \cdot x\} \) is a submodule of \((1,1)\)-type, which contradicts to the hypothesis that \( M \) does not contain any submodule of \((1,1)\)-type. Thus, we have proved \( a \cdot M_{r+1} = M_r \). Similarly, one can show \( d \cdot M_{r+1} = M_r \). Next, we claim that for any \( 0 \neq x \in M_{r+1}, a \cdot x \) and \( d \cdot x \) are linearly independent over \( k \). In fact, if not, then span\( \{x, a \cdot x\} \) is a submodule of \((1,1)\)-type, a contradiction. Now let \( 0 \neq v_4 \in M_r \). Then from the equations \( a \cdot M_{r+1} = M_r \) and \( d \cdot M_{r+1} = M_r \), one can choose elements \( v_1 \) and \( v_2 \) in \( M_{r+1} \) such that \( d \cdot v_1 = v_4 \) and \( a \cdot v_2 = v_4 \). Let \( v_3 = a \cdot v_1 \). Then \( \{v_1, v_2\} \) is a linear basis in \( M_{r+1} \) and \( \{v_3, v_4\} \) is a linear basis in \( M_r \). We also have \( d \cdot v_2 = -\delta v_3 + \eta v_4 \) for some \( \delta, \eta \in k \). If \( X^2 + \eta X + \delta \) is not reducible in \( k[X] \), then there is a \( \gamma \) in \( k \) such that \( \gamma^2 + \eta \gamma + \delta = 0 \). Let \( x = \gamma v_1 + v_2 \). Then \( 0 \neq x \in M_{r+1}, a \cdot x = \gamma v_3 + v_4 \) and \( d \cdot x = -\delta v_3 + (\eta + \gamma)v_4 \). Hence \( d \cdot x - (\eta + \gamma)(a \cdot x) = 0 \), a contradiction. Thus we have proved that \( X^2 + \eta X + \delta \) is irreducible in \( k[X] \). It is clear that \( M \cong M_2(1, r, \delta, \eta) \) as \( D_4 \)-modules and that \( \{v_1, v_2, v_3, v_4\} \) is a canonical basis for \( M \cong M_2(1, r, \delta, \eta) \) with the \( D_4 \)-action given by \((11)\).

(b) It is obvious that \( M_2(1, r, \delta, \eta) \cong M_2(1, r, \delta', \eta') \) if \( \delta = \delta' \) and \( \eta = \eta' \). Conversely, assume that there is a \( D_4 \)-module isomorphism \( f \) from \( M_2(1, r, \delta, \eta) \) to \( M_2(1, r, \delta', \eta') \). Let \( \{v_1, v_2, v_3, v_4\} \) and \( \{v'_1, v'_2, v'_3, v'_4\} \) be the canonical base of \( M_2(1, r, \delta, \eta) \) and \( M_2(1, r, \delta', \eta') \) given by \((11)\) respectively. Then by the action of \( b \) on the two bases, we know that \( f(kv_1 + kv_2) = kv'_1 + kv'_2 \). Hence there exists an invertible matrix \((a_{ij}) \in M_2(k)\) such that

\[
 f(v_1) = a_{11}v'_1 + a_{12}v'_2 \quad \text{and} \quad f(v_2) = a_{21}v'_1 + a_{22}v'_2.
\]

Since \( f \) is a \( D_4 \)-module homomorphism and \( v_3 = a \cdot v_1 \) and \( v_4 = a \cdot v_2 \), we have

\[
 f(v_3) = a_{11}v'_3 + a_{12}v'_4 \quad \text{and} \quad f(v_4) = a_{21}v'_3 + a_{22}v'_4.
\]

On the other hand, since \( d \cdot v_1 = v_4 \), we also have \( f(v_4) = f(d \cdot v_1) = d \cdot f(v_1) = d \cdot (a_{11}v'_1 + a_{12}v'_2) = a_{11}d \cdot v'_1 + a_{12}d \cdot v'_2 \). Now applying \( d \cdot v'_1 = v'_4 \) and \( d \cdot v'_2 = \eta'v'_4 - \delta'v'_3 \) in the last equation, one obtains \( f(v_4) = -\delta'\alpha_{12}v'_3 + (\alpha_{11} + \eta'\alpha_{12})v'_4 \). Hence we have \( f(v_4) = a_{21}v'_3 + a_{22}v'_4 = -\delta'\alpha_{12}v'_3 + (\alpha_{11} + \eta'\alpha_{12})v'_4 \). This shows that \( \alpha_{21} = -\delta'\alpha_{12} \) and \( \alpha_{22} = \alpha_{11} + \eta'\alpha_{12} \).
Similarly, from $f(\eta v_4 - \delta v_3) = f(d \cdot v_2) = d \cdot f(v_2)$, it follows that $f(v_3) = \delta^{-1}(\eta f(v_4) - d \cdot f(v_2)) = \delta^{-1}(\eta f(v_4) - d \cdot (\alpha_{21}v'_3 + \alpha_{22}v'_2)) = \delta^{-1}(\eta(\alpha_{21}v'_3 + \alpha_{22}v'_2) - \alpha_{11}v'_4 - \alpha_{22}(\eta'v_4 - \delta'v'_3)) = \delta^{-1}((\eta\alpha_{21} + \delta'\alpha_{22})v'_3 + ((\eta - \eta')\alpha_{22} - \alpha_{21})v'_4)$. Thus we have $f(v_3) = \alpha_{11}v'_3 + \alpha_{12}v'_4 = \delta^{-1}((\eta\alpha_{21} + \delta'\alpha_{22})v'_3 + ((\eta - \eta')\alpha_{22} - \alpha_{21})v'_4)$, from which one gets

$$\alpha_{11} = \delta^{-1}(\eta\alpha_{21} + \delta'\alpha_{22})$$ and $$\alpha_{12} = \delta^{-1}((\eta - \eta')\alpha_{22} - \alpha_{21}).$$

Now replacing $\alpha_{21}$ with $-\delta'\alpha_{12}$ and $\alpha_{22}$ with $\alpha_{11} + \eta'\alpha_{12}$ respectively in the above two equations and rearranging the order, one obtains that

$$\begin{cases} (\eta - \eta')\alpha_{11} + (\eta'(\eta - \eta') - (\delta - \delta'))\alpha_{12} = 0, \\ (\delta - \delta')\alpha_{11} + \delta'(\eta - \eta')\alpha_{12} = 0. \end{cases}$$

Since $(\alpha_{ij})$ is an invertible matrix, $(\alpha_{11}, \alpha_{12}) \neq (0, 0)$ and so

$$\begin{vmatrix} \eta - \eta' & \eta'(\eta - \eta') - (\delta - \delta') \\ \delta - \delta' & \delta'(\eta - \eta') \end{vmatrix} = 0,$$

which is equivalent to the equation: $(\delta - \delta')^2 + \eta'(\delta - \delta')(\eta' - \eta) + \delta'(\eta' - \eta)^2 = 0$. If $\eta' - \eta \neq 0$, then $(\delta - \delta')/(\eta' - \eta)$ would be a solution to the equation $x^2 + \eta'x + \delta' = 0$, which contradicts the fact that $X^2 + \eta'X + \delta'$ is an irreducible polynomial in $k[X]$. Thus one must have that $\eta = \eta'$ and consequently $\delta = \delta'$. $\square$

For later use, we list some basic properties of the described indecomposable modules.

**Lemma 1.2.** Let $r \in \mathbb{Z}_2$. Then

(a) $\text{End}_{D_4}(V(l, r)) \cong k$, $1 \leq l \leq 2$.

(b) $\text{End}_{D_4}(M_1(1, r, \eta)) \cong k$, $\eta = \infty$ or $\eta \in k$.

(c) $\text{End}_{D_4}(\Omega V(1, r)) \cong k$ and $\text{End}_{D_4}(\Omega^{-1}V(1, r)) \cong k$.

**Proof.** Parts (a) and (b) can be found in [8]. Part (c) follows from a straightforward computation. $\square$

2. Four-dimensional $D_4$-module algebras

In this section, we classify 4-dimensional Yetter–Drinfeld $H_4$-module algebras (or $D_4$-module algebras) according to their $D_4$-module structures. Let $A$ be a $D_4$-module algebra. The set of invariants $A^{D_4} = \{ x \in A \mid h \cdot x = \varepsilon(h)x, \forall h \in D_4 \}$ is a subalgebra of $A$. The identity $1$ is in $A^{D_4}$. Thus $A$ must contain a $D_4$-submodule isomorphic to $V(1, 0)$. If $A$ is a 4-dimensional $D_4$-module algebra, then as a $D_4$-module, $A$ is isomorphic to one of the following:
Proof. Let $A$ be a 4-dimensional $D_4$-module algebra. Then as a $D_4$-module, $A$ is not isomorphic to any of the following $D_4$-modules:

(E) $\Omega V(1, 0) \oplus V(1, 1),$

(G) $\Omega^{-1} V(1, 1) \oplus V(1, 1),$

(J) $M_1(1, 0, \eta) \oplus 2V(1, 1), \eta \in k \cup \{\infty\},$

(L) $M_1(1, 1, \eta_1) \oplus M_1(1, 0, \eta_2), \text{  where } \eta_1, \eta_2 \in k \cup \{\infty\} \text{ and } \eta_1 \neq \eta_2.$

Proof. Let $A$ be a 4-dimensional $D_4$-module algebra. If $A$ is isomorphic to $\Omega V(1, 0) \oplus V(1, 1)$ as a $D_4$-module, then there is a linear basis $\{x_1, x_2, x_3, 1\}$ in $A$ such that

$$a \cdot x_1 = a \cdot x_2 = 0, \quad a \cdot x_3 = 1, \quad d \cdot x_1 = d \cdot x_3 = 0, \quad d \cdot x_2 = -1,$$

$$b \cdot x_1 = c \cdot x_1 = -x_1, \quad b \cdot x_2 = c \cdot x_2 = -x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3,$$

where $kx_1 \cong V(1, 1)$ and span$\{x_2, x_3, 1\} \cong \Omega V(1, 0)$. Let $x_1x_3 = \alpha_0 1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \alpha_i \in k, 0 \leq i \leq 3.$ Since $b$ and $c$ fix $x_1 x_3$ and $b \cdot x_i = -x_i$ for $i = 1, 2, 3$, we have $x_1 x_3 = \alpha_0 1.$ This leads to $0 = a \cdot \alpha_0 = a \cdot (x_1 x_3) = (a \cdot x_1)(b \cdot x_3) + x_1(a \cdot x_3) = x_1,$ a contradiction! So $A \cong \Omega V(1, 0) \oplus V(1, 1)$ is impossible.
Assume that $A$ is isomorphic to $\Omega^{-1}V(1,1) \oplus V(1,1)$ as a $D_4$-module. Then there is a linear basis $\{x_1, x_2, x_3, x_4\}$ in $A$ such that span$\{x_1\} \cong V(1,1)$, span$\{x_2, x_3, x_4\} \cong \Omega^{-1}V(1,1)$, and

\[
\begin{align*}
    a \cdot x_1 &= a \cdot x_3 = a \cdot x_4 = 0, & a \cdot x_2 &= x_3, \\
    d \cdot x_1 &= d \cdot x_3 = d \cdot x_4 = 0, & d \cdot x_2 &= x_4, \\
    b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = -x_2, \\
    b \cdot x_3 &= c \cdot x_3 = x_3, & b \cdot x_4 &= c \cdot x_4 = x_4.
\end{align*}
\]

Since $b \cdot (x_1 x_2) = x_1 x_2$, we have $x_1 x_2 \in \text{span}\{x_3, x_4\} = A^{D_4}$. Hence $a \cdot (x_1 x_2) = 0$ and $d \cdot (x_1 x_2) = 0$. But

\[
a \cdot (x_1 x_2) = (a \cdot x_1)(b \cdot x_2) + x_1(a \cdot x_2) = x_1 x_3.
\]

It follows that $x_1 x_3 = 0$. Similarly, $x_1 x_4 = 0$. Thus we have shown that $x_1 A^{D_4} = 0$, which is impossible since $1 \in A^{D_4}$. So it is impossible that $A \cong \Omega^{-1}V(1,1) \oplus V(1,1)$.

Suppose that $A$ is isomorphic to $M_1(1,0,\eta) \oplus 2V(1,1)$ as a $D_4$-module with $\eta = \infty$ or $\eta \in k$. Then $A = I_1 \oplus I_2$ with $I_1 \cong M_1(1,0,\eta)$ and $I_2 \cong 2V(1,1)$. Clearly, the unity 1 is in $I_1$. Let $x$ be a non-zero element of $I_2$. Then $kx \cong V(1,1)$. Since the multiplication map $A \otimes A \rightarrow A$ is a $D_4$-module homomorphism and $V(1,1) \otimes M_1(1,0,\eta) \cong M_1(1,1,\eta)$, $x I_1$ is isomorphic to a quotient module of $M_1(1,1,\eta)$. However, $V(1,1) \cong kx \subseteq x I_1$ as $I_1$ contains 1. Since $M_1(1,1,\eta)$ has only one non-trivial submodule $V(1,1)$, the only non-trivial quotient module of $M_1(1,1,\eta)$ is isomorphic to $V(1,0)$. It follows that $x I_1 \cong M_1(1,1,\eta)$ and that $A$ contains a copy of $M_1(1,1,\eta)$. Contradiction! Therefore, $A$ cannot be isomorphic to $M_1(1,0,\eta) \oplus 2V(1,1)$.

Finally, assume that $A$ is isomorphic to $M_1(1,1,\eta_1) \oplus M_1(1,0,\eta_2)$ as a $D_4$-module, where $\eta_1, \eta_2 \in k \cup \{\infty\}$ and $\eta_1 \neq \eta_2$. Write $A = I_1 \oplus I_2$, where $I_1 \cong M_1(1,1,\eta_1)$ and $I_2 \cong M_1(1,0,\eta_2)$. Choose a non-zero element $x \in \text{soc}(I_1)$. Then $kx \cong V(1,1)$. Since $V(1,1) \otimes M_1(1,0,\eta_2) \cong M_1(1,1,\eta_2)$, the same reason as above assures that $x I_2$ is isomorphic to a quotient module of $M_1(1,1,\eta_2)$. On the other hand, $kx \subseteq x I_2$ as $1 \in I_2$. But the only non-trivial quotient module of $M_1(1,1,\eta_2)$ is isomorphic to $V(1,0)$. It follows that $x I_2 \cong M_1(1,1,\eta_2)$, which is impossible as $\eta_1 \neq \eta_2$. Thus $A$ is not isomorphic to $M_1(1,1,\eta_1) \oplus M_1(1,0,\eta_2)$ as a $D_4$-module. \[\square\]

In the following, we will show that each of 4-dimensional $D_4$-modules given on the list (12) admits a $D_4$-module algebra structure except those modules given in Lemma 2.1.

**Theorem 2.2.** Let $A$ be a 4-dimensional $D_4$-module algebra. Then $A$ is isomorphic to one of the $D_4$-module algebras described in the following lemmas from 2.3 to 2.31.

We begin with the semisimple $D_4$-modules of type (A), which eventually produce $\mathbb{Z}_2$-graded algebras. We point out that the classification of 4-dimensional algebras (with the trivial $D_4$-action) over an algebraically closed field has been done in [9]. The classification
of 4-dimensional \( \mathbb{Z}_2 \)-graded algebras has been completed (see [1]). There exist infinitely many \( \mathbb{Z}_2 \)-graded 4-dimensional algebras up to isomorphism. The following lemma is obvious.

**Lemma 2.3.** Let \( A \) be a \( D_4 \)-module algebra isomorphic to \( nV(1, 0) \oplus (4 - n)V(1, 1) \) as a \( D_4 \)-module, \( 1 \leq n \leq 4 \). Then \( A = A_0 \oplus A_1 \) is in fact a \( \mathbb{Z}_2 \)-graded algebra and the \( D_4 \)-action on \( A \) is given by

\[
a \cdot x = d \cdot x = 0, \quad b \cdot x = c \cdot x = (-1)^f x, \quad \forall x \in A_r, \ r \in \mathbb{Z}_2.
\]

There are 6 non-isomorphic \( D_4 \)-module algebras defined on the semisimple \( D_4 \)-modules \( V(2, r) \oplus 2V(1, 0), \ r = 0, 1 \).

**Lemma 2.4.** Let \( r \in \mathbb{Z}_2 \). Up to isomorphism, there exist three \( D_4 \)-module algebras \( A_i, \ i = 1, 2, 3 \), isomorphic to \( V(2, r) \oplus 2V(1, 0) \) as a \( D_4 \)-module. \( A_i, \ i = 1, 2, 3 \), has a \( k \)-linear basis \( \{x_1, x_2, x_3\} \) with the \( D_4 \)-action given by

\[
a \cdot x_1 = x_2, \quad a \cdot x_2 = a \cdot x_3 = 0, \quad d \cdot x_1 = d \cdot x_3 = 0, \quad d \cdot x_2 = 2x_1, \\
b \cdot x_1 = -c \cdot x_1 = (-1)^f x_1, \quad b \cdot x_2 = -c \cdot x_2 = (-1)^{f+1} x_2, \quad b \cdot x_3 = c \cdot x_3 = x_3.
\]

The multiplication of \( A_i, \ i = 1, 2, 3 \), are given by (2.4.1), (2.4.2) and (2.4.3) respectively.

\[
\begin{align*}
(2.4.1) \ x_i x_j &= 0, \ \forall 1 \leq i, j \leq 3. \\
(2.4.2) \ x_i x_j &= 0, \ x_3 x_i = x_i x_3 = x_i, \ x_3^2 = 1, \ \forall 1 \leq i, j \leq 2. \\
(2.4.3) \ x_i x_j &= 0, \ x_3 x_i = -x_i x_3 = x_i, \ x_3^2 = 1, \ \forall 1 \leq i, j \leq 2.
\end{align*}
\]

**Proof.** Without losing generality, we assume that \( A = V(2, r) \oplus 2V(1, 0) \) is a \( D_4 \)-module algebra. Choose a canonical basis \( \{x_1, x_2\} \) for \( V(2, r), \ r \in \mathbb{Z}_2 \), with the \( D_4 \)-action given by (3). Since \( 2V(1, 0) = A_{D_4} \), the linear basis of \( 2V(1, 0) \) can be chosen in a way that one basis element 1 is the identity of \( A \) and the other basis element \( x_1 \) satisfies \( x_1^2 = \alpha \) for some \( \alpha \in k \). Otherwise, if \( x_1^2 = \beta x_3 + \gamma \) we replace \( x_3 \) by \( x_3 - \frac{1}{2} \beta \). Since \( V(2, r) \) is projective, so is \( V(2, r) \otimes V(2, r) \) and \( V(2, r) \otimes V(2, r) \) is isomorphic to \( P(1, 1) \) by [8, Theorem 2.3]. Since the multiplication map \( A \otimes A \rightarrow A \) must be a \( D_4 \)-module endomorphism, \( V(2, r)^2 \) is a submodule of \( A \) and is isomorphic to a quotient module of \( P(1, 1) \). It is clear that \( \text{Hom}_{D_4}(P(1, 1), A) = 0 \). Hence \( V(2, r)^2 = 0 \), i.e., \( x_i x_j = 0, \) for all \( 1 \leq i, j \leq 2 \).

Observe that \( V(1, 0) \otimes V(2, r) \cong V(2, r) \otimes V(1, 0) \) (cf. [6, Lemma 3.3]) since \( D_4 \) is quasitriangular. So the left and right multiplication of \( V(2, r) \) by \( x_3 \) give two endomorphisms of \( V(2, r) \). But we know that \( \text{End}_{D_4}(V(2, r)) \cong k \) by Lemma 1.2. Thus we have that \( x_3 x_i = \alpha_1 x_i \) and \( x_i x_3 = \alpha_2 x_i \) for some \( \alpha_i \in k, \ i = 1, 2 \). From \( x_3^2 x_1 = x_3(x_3 x_1) \) and \( x_1 x_3^2 = (x_1 x_3)x_3 \), one obtains that \( \alpha = \alpha_1^2 \) and \( \alpha = \alpha_2^2 \). If \( \alpha = 0 \) then \( \alpha_1 = \alpha_2 = 0 \) and so \( A \) is the \( D_4 \)-module algebra described in (2.4.1).
Now suppose $\alpha \neq 0$. Then $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Replacing $x_3$ by $\alpha_1^{-1} x_3$, we may assume that $\alpha_1 = 1$ without changing the $D_4$-action. Then $\alpha = 1$ and $\alpha_2 = \pm 1$. If $\alpha_2 = 1$ then $A$ is the $D_4$-module algebra described in (2.4.2). If $\alpha_2 = -1$ then $A$ is the $D_4$-module algebra described in (2.4.3). \Box

Now we consider another class of $D_4$-module algebras defined on the semisimple $D_4$-modules $V(2, r) \oplus V(1, 1) \oplus V(1, 0)$.

**Lemma 2.5.** Let $r \in \mathbb{Z}_2$. Up to isomorphism, there exist two $D_4$-module algebras $A_i$, $i = 1, 2$, isomorphic to $V(2, r) \oplus V(1, 1) \oplus V(1, 0)$ as a $D_4$-module. $A_i$, $i = 1, 2$, has a $k$-linear basis $\{x_1, x_2, x_3, 1\}$ such that

\[
\begin{align*}
& a \cdot x_1 = x_2, \quad a \cdot x_2 = a \cdot x_3 = 0, \quad d \cdot x_1 = d \cdot x_3 = 0, \quad d \cdot x_2 = 2x_1, \\
& b \cdot x_1 = -c \cdot x_1 = (-1)^r x_1, \quad b \cdot x_2 = -c \cdot x_2 = (-1)^{r+1} x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3,
\end{align*}
\]

and the multiplication of $A_i$ are given by (2.5.1) and (2.5.2) respectively.

\[
\begin{align*}
& (2.5.1) \quad x_i x_j = 0, \ \forall 1 \leq i, j \leq 3, \\
& (2.5.2) \quad x_1^3 = 0, \quad x_2^2 = 0, \quad x_1 x_2 = (-1)^r x_2 x_1 = x_3.
\end{align*}
\]

**Proof.** Without loss of generality, we assume $A = V(2, r) \oplus V(1, 1) \oplus V(1, 0)$ is a $D_4$-module algebra. We may choose a linear basis $\{x_1, x_2, x_3, 1\}$ like we did in the proof of Lemma 2.4, such that $V(2, r) \cong k x_1 \oplus k x_2$, $V(1, 1) \cong k x_3$, $V(1, 0) \cong k 1$ with the $D_4$-actions given by (2) and (3). Since $V(1, 1) \otimes V(2, r) \cong V(2, r) \otimes V(1, 1) \cong V(2, r+1)$ by [6, Lemma 3.3], $V(2, r) V(1, 1) = V(1, 1) V(2, r) = 0$. Since $V(1, 1) \otimes V(1, 1) \cong V(1, 0)$, one gets $V(1, 1)^2 \subset k1$. It follows that $V(1, 1)^2 = 0$ as $V(2, r)V(1, 1) = 0$.

Now applying the same argument in the proof of Lemma 2.4, we obtain that $V(2, r)^2$ is isomorphic to a quotient module of $P(1, 1)$. Since $P(1, 1)/J \cdot P(1, 1) \cong V(1, 1)$ and $A$ is a semisimple $D_4$-module, $V(2, r)^2 \subset V(1, 1)$. It follows that $x_i^2 = \alpha_i x_3$, $i = 1, 2$. Observe that $b \cdot (x_i^2) = (b \cdot x_i)(b \cdot x_i) = x_i^2$ for $i = 1, 2$. But $b \cdot x_3 = -x_3$. Thus $x_i^2 = 0$ for $i = 1, 2$.

Now we have $0 = a \cdot (x_i^2) = (a \cdot x_1)(b \cdot x_1) + x_1(a \cdot x_1) = (-1)^r x_2 x_1 + x_1 x_2$. Hence $V(2, r)^2 = 0$ if and only if $x_1 x_2 = 0$ if and only if $x_2 x_1 = 0$. If $V(2, r)^2 = 0$ then $A$ is the $D_4$-module algebra with product given by (2.5.1). If $V(2, r)^2 \neq 0$, then replacing $x_3$ by $\alpha x_3$ for some suitable $\alpha \in k^*$ we may assume $x_1 x_2 = x_3$. In this case, $A$ is isomorphic to the $D_4$-module algebra given by (2.5.2). \Box

Next we will consider $D_4$-module algebra structures on the non-semisimple module $\Omega V(1, 0) \oplus V(1, 0)$. There are infinitely many $D_4$-module algebra structures on $\Omega V(1, 0) \oplus V(1, 0)$.

**Lemma 2.6.** Let $A$ be a $D_4$-module algebra isomorphic to $\Omega V(1, 0) \oplus V(1, 0)$ as a $D_4$-module. Then there exists a matrix $(\alpha_{ij}) \in M_2(k)$ such that $A$ is isomorphic to $A_{\Omega(1,0)}(\alpha_{ij})$ with $k$-linear basis $\{x_1, x_2, x_3, 1\}$ on which the $D_4$-action is given by
We have an element 

\[ (\alpha_{ij}) \]

We may assume that \( \alpha_{ij} \).

**Proof.** We may assume that \( A = \Omega V(1, 0) \otimes V(1, 0) \) as a \( D_4 \)-module without loss of generality. Since \( \Omega V(1, 0) \otimes V(1, 0) \cong \Omega V(1, 0) \), we know that \( \Omega V(1, 0) \cdot V(1, 0) \) is a submodule of \( A \) and it is isomorphic to a quotient module of \( \Omega V(1, 0) \). Let \( f \) be the composite map

\[ \Omega V(1, 0) \cong \Omega V(1, 0) \otimes V(1, 0) \rightarrow \Omega V(1, 0) \cdot V(1, 0) \rightarrow A. \]

Then \( \pi \circ f \) is a \( D_4 \)-module homomorphism from \( \Omega V(1, 0) \) to \( V(1, 0) \), where \( \pi \) is the projection of \( A \) onto \( V(1, 0) \). But \( \text{Hom}_{D_4}(\Omega V(1, 0), V(1, 0)) = 0 \). Thus we have \( \Omega V(1, 0) \cdot V(1, 0) \subseteq \Omega V(1, 0) \). This implies that \( 1 \notin \Omega V(1, 0) \). As the identity 1 must generate a copy of \( V(1, 0) \), we may assume \( 1 \in V(1, 0) \). Thus we may choose a basis \( \{x_1, x_2, x_3, 1\} \) for \( A \) so that \( \{x_1, x_2, x_3\} \) is a canonical basis for \( \Omega V(1, 0) \) with the \( D_4 \)-action given by (7) and \( V(1, 0) = k1 \) with 1 being the identity of \( A \). Since \( kx_3 \cong V(1, 0) \), the same reason as above assures that \( x_3 \Omega V(1, 0) \subseteq \Omega V(1, 0) \) and \( \Omega V(1, 0)x_3 \subseteq \Omega V(1, 0) \). Hence the left and right multiplication by the element \( x_3 \) give two \( D_4 \)-module endomorphisms of \( \Omega V(1, 0) \). Since \( \text{End}_{D_4}(\Omega V(1, 0)) \cong k \) and \( x_3x = x_3 \) for \( x = x_3 \), there exists an \( \alpha \in k \) such that \( x_3x = x_3x = \alpha x \) for all \( x \in \Omega V(1, 0) \). Clearly, \( x_3x \in \text{span}\{x_3, 1\} \) for any \( 1 \leq i, j \leq 2 \). Let \( x_3x = \alpha x + \beta_{ij} \) for some \( \alpha_{ij}, \beta_{ij} \in k \). Let \( 1 \leq i, j \leq 2 \). Now \( x_3(x_1x_i) = (x_3x_i)x_j \) implies that \( \alpha \beta_{ij} = \beta_{ij}x_3 \) for all \( i, j = 1, 2 \). Hence \( \beta_{ij} = 0 \) for all \( i, j = 1, 2 \). Since \( a \cdot x_3 = 0 \), we have \( a \cdot (x_1x_2) = a \cdot (x_1x_2) \). On the other hand, we also have \( a \cdot (x_1x_2) = (a \cdot x_1)(b \cdot x_2) + x_1(a \cdot x_2) = x_1x_3 = \alpha x_3 \). Hence \( \alpha = 0 \). This shows that \( A \) is isomorphic to \( A_{\Omega(1,0)}(\alpha_{ij}) \).

Now let \( A_{\Omega(1,0)}(\alpha_{ij}) \) and \( A_{\Omega(1,0)}(\beta_{ij}) \) be two \( D_4 \)-module algebras defined by \( (\alpha_{ij}), (\beta_{ij}) \in M_2(k) \) and suppose that \( \{x_1, x_2, x_3, 1\} \) and \( \{y_1, y_2, y_3, 1\} \) are two bases of \( A_{\Omega(1,0)}(\alpha_{ij}) \) and \( A_{\Omega(1,0)}(\beta_{ij}) \) given as in the lemma respectively. If \( f : A_{\Omega(1,0)}(\alpha_{ij}) \rightarrow A_{\Omega(1,0)}(\beta_{ij}) \) is an isomorphism of module algebras, then \( f \) is a module isomorphism. By the argument above, we have an element \( \delta \in k^\times \) such that \( f(x_i) = \delta y_i \) for all \( 1 \leq i \leq 3 \). Now for \( 1 \leq i, j \leq 2 \), we have \( \alpha_{ij} \delta y_3 = \alpha_{ij} f(x_3) = f(\alpha_{ij}x_3) = f(x_1x_j) = f(x_i)f(x_j) = \delta^2 y_i y_j = \delta^2 \beta_{ij} y_3 \), which implies \( \alpha_{ij} = \delta \beta_{ij} \).

Conversely, assume there is a \( \delta \in k^\times \) such that \( \alpha_{ij} = \delta \beta_{ij} \) for \( 1 \leq i, j \leq 2 \). Define a \( k \)-linear map \( f : A_{\Omega(1,0)}(\alpha_{ij}) \rightarrow A_{\Omega(1,0)}(\beta_{ij}) \) by \( f(x_i) = \delta y_i \), \( 1 \leq i \leq 3 \), and \( f(1) = 1 \). Then it is easy to see that \( f \) is a module algebra isomorphism. \( \square \)

Substituting \( \Omega V(1, 1) \) for \( \Omega V(1, 0) \) in Lemma 2.6, we have all \( D_4 \)-module algebra structures defined on \( \Omega V(1, 1) \oplus V(1, 0) \).
Lemma 2.7. Let $A$ be a $D_4$-module algebra isomorphic to $\Omega V(1, 1) \oplus V(1, 0)$ as a $D_4$-module. Then there exist elements $\alpha, \beta \in k$ such that $A$ is isomorphic to $A_{\Omega(1,1)}(\alpha, \beta)$ with a $k$-linear basis $\{x_1, x_2, x_3, 1\}$ on which the $D_4$-action is given by

$$
a \cdot x_1 = a \cdot x_3 = 0, \quad a \cdot x_2 = x_3, \quad d \cdot x_1 = -x_3, \quad d \cdot x_2 = d \cdot x_3 = 0, \quad b \cdot x_1 = c \cdot x_1 = x_1, \quad b \cdot x_2 = c \cdot x_2 = x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3,
$$

and the multiplication is given by

$$
x_1^2 = \alpha^2, \quad x_2^2 = \beta^2, \quad x_3^2 = 0, \quad x_1 x_3 = -x_3 x_1 = \alpha x_3, \quad x_2 x_3 = -x_3 x_2 = \beta x_3,
$$

$$
x_1 x_2 = -\beta x_1 + \alpha x_2 + \alpha \beta, \quad x_2 x_1 = \beta x_1 - \alpha x_2 + \alpha \beta, \quad \alpha, \beta \in k.
$$

Moreover, $A_{\Omega(1,1)}(\alpha, \beta) \cong A_{\Omega(1,1)}(\alpha', \beta')$ if and only if there exists an element $\delta \in k^\times$ such that $\alpha = \delta \alpha'$ and $\beta = \delta \beta'$.

Proof. Assume $A = \Omega V(1, 1) \oplus V(1, 0)$ as a $D_4$-module. Since $\text{soc}(\Omega V(1, 1)) \cong V(1, 1)$, the identity $1 \notin \Omega V(1, 1)$. We may choose a linear basis $\{x_1, x_2, x_3, 1\}$ such that $\{x_1, x_2, x_3\}$ is a canonical basis of $\Omega V(1, 1)$ with the $D_4$-action given by $(7)$ and $V(1, 0)$ is generated by the identity $1$. Let $x_1^2 = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3, \alpha_i \in k$. Since $a \cdot (x_1^2) = (a \cdot x_1)(b \cdot x_1) + x_1(a \cdot x_1) = 0$ and $b$ fixes $x_1^2$, we have $\alpha_2 = \alpha_3 = 0$ and hence $x_1^2 = \alpha_0 + \alpha_1 x_1$. Replacing $x_1$ with $x_1 + \delta$ for some suitable $\delta \in k$, we may assume $x_1^2 = \gamma_1$ for some $\gamma_1 \in k$. Similarly, we may assume $x_2^2 = \gamma_2$ for some $\gamma_2 \in k$. Since $k x_3 \cong V(1, 1)$ and $\Omega V(1, 1) \oplus V(1, 1) \cong \Omega V(1, 0), \Omega V(1, 1)x_3$ is a submodule of $A$ and it is isomorphic to a quotient module of $\Omega V(1, 0)$. Let $f$ be the composite map

$$
\Omega V(1, 0) \cong \Omega V(1, 1) \oplus k x_3 \to \Omega V(1, 1)x_3 \hookrightarrow A.
$$

Then $\pi \circ f$ is a $D_4$-module homomorphism from $\Omega V(1, 1)$ to $V(1, 0)$, where $\pi$ is the projection of $A$ onto $V(1, 0)$. Since $\text{Hom}_{D_4}(\Omega V(1, 0), V(1, 0)) = 0$, we have $\pi(\Omega V(1, 1)x_3) = 0$ and hence $\Omega V(1, 1)x_3 \subseteq \Omega V(1, 1)$. Thus the map $f$ is a $D_4$-morphism from $\Omega V(1, 0)$ to $\Omega V(1, 1)$. Since $\text{soc}(\Omega V(1, r)) = V(1, r)$ and $f$ sends the socle of $\Omega V(1, 0)$ into the socle of $\Omega V(1, 1)$, we have $\text{soc}(\Omega V(1, 0)) \subseteq \text{Ker}(f)$. Consequently, $\text{Im}(f) \subseteq \text{soc}(\Omega V(1, 1))$ because $\Omega V(1, 0)/\text{soc}(\Omega V(1, 0)) \cong 2V(1, 1)$ is semisimple. It follows that $\Omega V(1, 1)x_3 \subseteq \text{soc}(\Omega V(1, 1)) = k x_3$. Therefore, $x_3^2 = 0$ and there exist $\alpha$ and $\beta$ in $k$ such that $x_1 x_3 = \alpha x_3$ and $x_2 x_3 = \beta x_3$. It follows from the equations $x_1^2 = \gamma_1$ and $\gamma_1 x_3 = x_1^2 x_3 = x_1(x_1 x_3) = \alpha x_1 x_3 = \alpha^2 x_3$ that $\gamma_1 = \alpha^2$ and $x_1^2 = \alpha^2$. Similarly, one obtains that $\gamma_2 = \beta^2$ and $x_2^2 = \beta^2$.

The relation $x_3 x_1 = -x_1 x_3$ follows from $0 = d \cdot (\alpha^2) = d \cdot (x_1^2) = -x_3 x_1 + x_1 (-x_3)$ and the relation $x_2 x_3 = -x_3 x_2 = \beta x_3$ follows from $0 = a \cdot (\beta^2) = a \cdot (x_2^2) = x_3 x_2 + x_2 x_3$.

Finally, let $x_1 x_2 = \tau_1 x_1 + \tau_2 x_2 + \tau_3 x_3 + \tau_0$ and $x_2 x_1 = \tilde{\xi}_1 x_1 + \tilde{\xi}_2 x_2 + \tilde{\xi}_3 x_3 + \tilde{\xi}_0$ for some $\tau_i, \tilde{\xi}_i \in k, 0 \leq i \leq 3$. Observe that the element $b$ fixes $x_1 x_2$ and $x_2 x_1$. So $\tau_3 = \tilde{\xi}_3 = 0$. From the equation $a \cdot (x_1 x_2) = a \cdot (\tau_1 x_1 + \tau_2 x_2 + \tau_0)$ one gets $\tau_2 = \alpha$; from the equation
\[ d \cdot (x_1x_2) = d \cdot (\tau_1x_1 + \tau_2x_2 + \tau_0) \] one gets \( \tau_1 = -\beta \); from the equation \( x_1(x_1x_2) = (x_1x_1)x_2 \) one gets \( \tau_0 = \alpha\beta \). Similarly, one obtains that \( \xi_1 = \beta, \xi_2 = -\alpha \) and \( \xi_3 = \alpha\beta \). Hence \( x_1x_2 = -\beta x_1 + \alpha x_2 + \alpha\beta \) and \( x_2x_1 = \beta x_1 - \alpha x_2 + \alpha\beta \). Thus we have proved that \( A \) has the desired multiplication, i.e., \( A \cong A_{\Omega(1,1)}(\alpha, \beta) \).

Now let \( A_{\Omega(1,1)}(\alpha, \beta) \) and \( A_{\Omega(1,1)}(\alpha', \beta') \) be two \( D_4 \)-module algebras with \( \alpha, \beta, \alpha', \beta' \in k \) and suppose that \( \{x_1, x_2, x_3, 1\} \) and \( \{y_1, y_2, y_3, 1\} \) are two bases of \( A_{\Omega(1,1)}(\alpha, \beta) \) and \( A_{\Omega(1,1)}(\alpha', \beta') \) given as in the lemma respectively. If \( f \) is a module algebra isomorphism from \( A_{\Omega(1,1)}(\alpha, \beta) \) to \( A_{\Omega(1,1)}(\alpha', \beta') \), then \( f \) is a module isomorphism and \( f(1) = 1 \). Note that \( \text{Hom}_{D_4}(\Omega V (1, 1), V(1, 0)) \cong \text{Hom}_{D_4}(\Omega V (1, 1)/\text{soc}(\Omega V (1, 1)), V(1, 0)) \cong k(2) \).

Since \( \text{End}_{D_4}(\Omega V (1, 1)) \cong k \) by Lemma 1.2, we have elements \( \delta \in k^x \) and \( \gamma_1, \gamma_2 \in k \) such that \( f(x_i) = \delta y_i + \gamma_i, 1 \leq i \leq 2 \), and \( f(x_3) = \delta y_3 \). Now we have \( \alpha^2 = f(\alpha^2) = f(x_1^2) = f(\alpha y_1 + \gamma_1)^2 = 2\delta y_1 y_1 + \delta^2 \alpha^2 + \gamma_1^2 \), which implies \( \gamma_1 = 0 \). Similarly, we have \( \gamma_2 = 0 \). Then we have \( \alpha \delta y_3 = \alpha f(x_3) = f(\alpha x_3) = f(x_1x_3) = f(x_1)f(x_3) = \delta^2 y_1 y_3 = \delta^2 \alpha' y_3 \), which implies \( \alpha = \delta \alpha' \). Similarly, one can show that \( \beta = \delta \beta' \).

Conversely, assume there is an element \( \delta \in k^x \) such that \( \alpha = \delta \alpha' \) and \( \beta = \delta \beta' \). Define a \( k \)-linear map \( f : A_{\Omega(1,1)}(\alpha, \beta) \to A_{\Omega(1,1)}(\alpha', \beta') \) by \( f(x_i) = \delta y_i, 1 \leq i \leq 3 \), and \( f(1) = 1 \). Then it is easy to see that \( f \) is a module algebra isomorphism.

In order for determining the multiplication of chosen basis elements, we developed a technical argument based on the tensor products of indecomposable \( D_4 \)-modules in both proofs of Lemmas 2.6 and 2.7. The argument will be used frequently in the sequel. For convenience we will call it the MMBTP argument standing for “multiplication measured by tensor products.” The MMBTP argument can tell us where the identity element is about and where the product of two basis elements is about.

Considering the \( D_4 \)-module \( \Omega^{-1}V (1, 0) \oplus V (1, 0) \), we have all \( D_4 \)-module algebra structures defined on \( \Omega^{-1}V (1, 0) \oplus V (1, 0) \) in the following.

**Lemma 2.8.** Let \( A \) be a \( D_4 \)-module algebra isomorphic to \( \Omega^{-1}V (1, 0) \oplus V (1, 0) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( A_1 \) or \( A_{\Omega^{-1}(1,0)}(\alpha, \beta, \gamma) \), for some \( \alpha, \beta \) and \( \gamma \) in \( k \), both of which have a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
\begin{align*}
 a \cdot x_1 &= x_2, & a \cdot x_2 &= a \cdot x_3 &= 0, & d \cdot x_1 &= x_3, & d \cdot x_2 &= d \cdot x_3 &= 0, \\
 b \cdot x_1 &= c \cdot x_1 &= x_1, & b \cdot x_2 &= c \cdot x_2 &= -x_2, & b \cdot x_3 &= c \cdot x_3 &= -x_3.
\end{align*}
\]

The multiplication of \( A_1 \) and \( A_{\Omega^{-1}(1,0)}(\alpha, \beta, \gamma) \) are given by (2.8.1) and (2.8.2) respectively.

(2.8.1) \[
\begin{align*}
 x_1^2 &= 1, & x_1x_2 &= -x_2x_1 &= x_2, & x_1x_3 &= -x_3x_1 &= x_3, & x_2^2 &= 0, & x_3^2 &= 0, \\
x_2x_3 &= x_3x_2 &= 0.
\end{align*}
\]

(2.8.2) \[
\begin{align*}
 x_1^2 &= \alpha^2 + \beta \gamma, & x_1x_2 &= -x_2x_1 &= \alpha x_2 + \beta x_3, & x_1x_3 &= -x_3x_1 &= \gamma x_2 - \alpha x_3, & x_2^2 &= 0, & x_3^2 &= 0, \\
x_2x_3 &= x_3x_2 &= 0, & a, \beta, \gamma \in k.
\end{align*}
\]
Moreover, \( A_1 \not\cong A_{\Omega^{-1}(1,0)}(\alpha, \beta, \gamma) \) for any \( \alpha, \beta \) and \( \gamma \) in \( k \), and \( A_{\Omega^{-1}(1,0)}(\alpha, \beta, \gamma) \cong A_{\Omega^{-1}(1,0)}(\alpha', \beta', \gamma') \) if and only if there exists an element \( \delta \in k^\times \) such that \( \alpha = \delta\alpha' \), \( \beta = \delta\beta' \) and \( \gamma = \delta\gamma' \).

**Proof.** We may assume \( A = \Omega^{-1}V(1,0) \oplus V(1,0) \) as a \( D_4 \)-module. The identity \( 1 \not\in \Omega^{-1}V(1,0) \), since \( \text{soc}(\Omega^{-1}V(1,0)) \cong 2V(1,1) \). Hence we may choose a linear basis \( \{x_1, x_2, x_3, 1\} \) for \( A \) such that \( \{x_1, x_2, x_3\} \) is a canonical basis of \( \Omega^{-1}V(1,0) \) with the \( D_4 \)-action given by (8) and \( V(1,0) \) is generated by the identity 1. Let \( x_1^2 = \alpha_0 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 \), for some \( \alpha_i \in k \). If \( x_1 \neq 0 \), then \( kx_1 \cong V(1,1) \) for \( i = 2, 3 \) and \( V(1,1) \otimes V(1,1) \cong \Omega^{-1}(1,0) \). So we have \( kx_1 \cong V(1,1) \) isomorphic to a submodule of \( A \) isomorphic to a quotient module of \( V(1,0) \). It follows from \( A_4 = k1 \) that \( x_1x_j = \alpha_{ij}x_i \) for some \( \alpha_{ij} \in k \), \( 2 \leq i, j \leq 3 \). Hence \( \alpha_{ii}x_j = x_1^2x_j = x_i(x_1x_j) = \alpha_{ij}x_i \) and \( \alpha_{ij} = 0 \) for any \( 2 \leq i, j \leq 3 \). Thus we have \( x_1x_j = 0 \) for any \( 2 \leq i, j \leq 3 \). Now \( x_1^2 = \delta \) implies that \( 0 = a \cdot x_1^2 = x_2x_1 + x_1x_2 \) and \( 0 = d \cdot x_1^2 = x_3x_1 + x_1x_3 \). Consequently, we have the relations \( x_1x_2 = -x_2x_1 \) and \( x_1x_3 = -x_3x_1 \).

Next we compute the products of basis elements. Since \( b \cdot (x_1x_2) = -x_1x_2 \) and \( b \cdot (x_1x_3) = -x_1x_3 \), \( x_1x_2 \) and \( x_1x_3 \) must be contained in \( \text{span}\{x_2, x_3\} \). Hence there exist \( \alpha, \beta, \gamma \) and \( \eta \) in \( k \) such that

\[
x_1x_2 = \alpha x_2 + \beta x_3 \quad \text{and} \quad x_1x_3 = \gamma x_2 + \eta x_3.
\]

From the equations \( x_1^2x_2 = x_1(x_1x_2) \) and \( x_1^2x_3 = x_1(x_1x_3) \), one obtains that \( \delta = \alpha^2 + \beta\gamma = \beta\gamma + \eta^2 \) and \( \gamma (\alpha + \eta) = \beta (\alpha + \eta) \). If \( \alpha + \eta = 0 \), then \( \eta = -\alpha \). So \( A \) has the \( D_4 \)-module algebra structure described in (2.8.2), i.e., \( A \cong A_{\Omega^{-1}(1,0)}(\alpha, \beta, \gamma) \).

Now assume \( \alpha + \eta \neq 0 \). Then \( \beta = 0 \), \( \gamma = 0 \) and \( \eta \neq \alpha \neq 0 \) since \( \eta^2 = \alpha^2 \). Consequently, \( x_1^2 = \alpha^2x_2 \), \( x_1x_2 = \alpha x_2 \) and \( x_1x_3 = \alpha x_3 \). Since \( \alpha \neq 0 \), one may replace \( x_i \) with \( \alpha^{-1}x_i \) and obtains the \( D_4 \)-module algebra described in (2.8.1). That is, \( A \) is isomorphic to \( A_1 \) in this case.

The rest of the proof is similar to the proof of Lemma 2.7. \( \square \)

Substituting \( \Omega^{-1}V(1,1) \) for \( \Omega^{-1}V(1,0) \) in Lemma 2.8, we have the following \( D_4 \)-module algebra structures defined on \( \Omega^{-1}V(1,1) \oplus V(1,0) \).

**Lemma 2.9.** Let \( A \) be a \( D_4 \)-module algebra isomorphic to \( \Omega^{-1}V(1,1) \oplus V(1,0) \) as a \( D_4 \)-module. Then there exist elements \( \alpha, \beta \in k \) such that \( A \) is isomorphic to \( A_{\Omega^{-1}(1,1)}(\alpha, \beta) \) with a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) on which \( D_4 \)-acts as follows

\[
a \cdot x_1 = x_2, \quad a \cdot x_2 = a \cdot x_3 = 0, \quad d \cdot x_1 = x_3, \quad d \cdot x_2 = d \cdot x_3 = 0, \quad b \cdot x_1 = c \cdot x_1 = -x_1, \quad b \cdot x_2 = c \cdot x_2 = x_2, \quad b \cdot x_3 = c \cdot x_3 = x_3.
\]

The multiplication of \( A_{\Omega^{-1}(1,1)}(\alpha, \beta) \) is given by
Moreover, for $\alpha, \beta, \alpha', \beta' \in k$, $A_{\Omega^{-1}(1,1)}(\alpha, \beta) \cong A_{\Omega^{-1}(1,1)}(\alpha', \beta')$ if and only if there exists an element $\delta \in k^*$ such that $\alpha = \delta \alpha'$ and $\beta = \delta \beta'$.

**Proof.** Assume that $A = \Omega^{-1}V(1,1) \oplus V(1,0)$ is a $D_4$-module algebra. As $\Omega^{-1}V(1,1) \otimes V(1,0) \cong \Omega^{-1}V(1,1)$, we have that $\Omega^{-1}V(1,1) \cdot V(1,0)$ is a submodule of $A$ and it is isomorphic to a quotient module of $\Omega^{-1}V(1,1)$. Now we have $\text{Hom}_{D_4}(\Omega^{-1}V(1,1), V(1,0)) = 0$ by the structure of $\Omega^{-1}V(1,1)$. By the MMBTP argument we obtain that $\Omega^{-1}V(1,1) \cdot V(1,0) \subseteq \Omega^{-1}V(1,1)$. Similarly, we have $V(1,0) \cdot \Omega^{-1}V(1,1) \subseteq \Omega^{-1}V(1,1)$. It makes clear that the identity of $A$ is not in $\Omega^{-1}V(1,1)$. However, the identity $1$ must generate a copy of $V(1,0)$, we may assume $1 \in V(1,0)$. Thus we may choose a linear basis $\{x_1, x_2, x_3, 1\}$ for $A$ such that $\{x_1, x_2, x_3\}$ is a canonical basis for $\Omega^{-1}V(1,1)$ with the $D_4$-action given by (8) and $V(1,0) = k1$. Since $kx_2 \cong V(1,0)$, the MMBTP argument assures that $x_2\Omega^{-1}V(1,1) \subseteq \Omega^{-1}V(1,1)$ and $\Omega^{-1}V(1,1)x_2 \subseteq \Omega^{-1}V(1,1)$. Consequently, the left and right multiplications by the element $x_2$ give two $D_4$-module endomorphisms of $\Omega^{-1}V(1,1)$. Since $\text{End}_{D_4}(\Omega^{-1}V(1,1)) \cong k$ and $x_2$ is in $\Omega^{-1}V(1,1)$, there exists an $\alpha \in k$ such that $x_2x = x_2 = \alpha x$ for all $x \in \Omega^{-1}V(1,1)$. Similarly, there exists a $\beta \in k$ such that $x_3x = x_3 = \beta x$ for all $x \in \Omega^{-1}V(1,1)$. Now from $x_2 \in \Omega^{-1}V(1,1)$ and $x_3 \in \Omega^{-1}V(1,1)$, one gets $\alpha x_3 = x_3 = \beta x_2$. This implies that $\alpha = \beta = 0$. Thus we have proved that $x_2x_1 = x_1x_2 = x_3x_i = x_i = 0$ for any $1 \leq i \leq 3$.

Note that the element $b$ fixes $x_i^2$. So we have $x_i^2 = \alpha x_i + \beta x_3 + \gamma$ for some $\alpha, \beta, \gamma \in k$. Now $x_2^2 = x_1(x_1x_2)$ implies $\gamma = 0$, and so $x_i^2 = \alpha x_i + \beta x_3$. This shows that $A$ is isomorphic to $A_{\Omega^{-1}(1,1)}(\alpha, \beta)$. The rest of the proof is similar to the proof of Lemma 2.6. $\Box$

Now we consider $D_4$-module algebra structures on the non-semisimple modules $M_1(1, r, \eta) \oplus 2V(1,0)$, $r \in \mathbb{Z}_2$, $\eta \in k \cup \{\infty\}$. We first consider $D_4$-module algebra structures on $M_1(1,0, \infty) \oplus 2V(1,0)$. There are infinitely many $D_4$-module algebra structures defined on $M_1(1,0, \infty) \oplus 2V(1,0)$.

**Lemma 2.10.** Let $A$ be a $D_4$-module algebra isomorphic to $M_1(1,0, \infty) \oplus 2V(1,0)$ as a $D_4$-module. Then $A$ is isomorphic to $A_1A_2$ or $A_{+}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for some $\alpha_i \in k$, $i = 1, 2, 3, 4$, where $A_1$, $A_2$ and $A_{+}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ have a $k$-linear basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action given by

\[
\begin{align*}
 a \cdot x_1 &= a \cdot x_2 = a \cdot x_3 = 0, & d \cdot x_1 &= x_2, & d \cdot x_2 &= d \cdot x_3 = 0, \\
 b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = x_2, & b \cdot x_3 &= c \cdot x_3 = x_3,
\end{align*}
\]

and their multiplication are given respectively by (2.10.1), (2.10.2) and (2.10.3).

(2.10.1) $x_1^2 = x_3$, $x_2^2 = 0$, $x_3^2 = 0$, $x_i x_j = 0$, $\forall i \neq j$.
(2.10.2) $x_1^2 = x_3$, $x_2^2 = 0$, $x_3^2 = 1$, $x_1 x_2 = x_2 x_1 = 0$, $x_1 x_3 = -x_3 x_1 = x_1$,
$x_2 x_3 = -x_3 x_2 = x_2$. 

\[x_2 x_3 = -x_3 x_2 = x_2.\]
(2.10.3) \[ x_1^2 = \alpha_1 x_2, \quad x_2^2 = \alpha_2 x_2, \quad x_3^2 = \alpha_3 x_2 + \alpha_4^2 - \alpha_2 \alpha_3, \quad x_1 x_2 = x_2 x_1 = \alpha_2 x_1, \quad x_1 x_3 = x_3 x_1 = \alpha_4 x_1, \quad x_2 x_3 = x_3 x_2 = \alpha_4 x_2. \]

Moreover, \( A_1 \not\cong A_2, \) \( A_i \not\cong A_{+,i}^{\infty}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) for any \( i = 1, 2 \) and \( \alpha_j \in k, \) \( j = 1, 2, 3, 4, \) and \( A_{+,i}^\infty(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cong A_{+,j}^\infty(\beta_1, \beta_2, \beta_3, \beta_4) \) if and only if there exist \( \delta, \gamma \in k^\times \) and \( \theta \in k \) such that \( \phi_1 = \delta \beta_1, \phi_2 = \delta \beta_2, \phi_3 = \delta^{-1} \gamma^2 \beta_3 + \delta^{-1} (\theta^2 \beta_2 + 2 \gamma \theta \beta_4) \) and \( \phi_4 = \theta \beta_2 + \gamma \beta_4. \)

**Proof.** Assume \( A = M_1(1, 0, \infty) \oplus 2 V(1, 0) \) as a \( D_4 \)-module. Since \( M_1(1, 0, \infty) \otimes V(1, 0) \cong V(1, 0) \otimes M_1(1, 0, \infty) \cong M_1(1, 0, \infty) \) and \( \text{Hom}_{D_4}(M_1(1, 0, \infty), V(1, 0)) = 0 \) by the structure of \( M_1(1, 0, \infty) \), we have \( M_1(1, 0, \infty) \cdot (2 V(1, 0)) \subseteq M_1(1, 0, \infty) \) and \( (2 V(1, 0)) \cdot M_1(1, 0, \infty) \subseteq M_1(1, 0, \infty) \) by the MMBTP argument. This implies \( 1 \not\in M_1(1, 0, \infty) \). Therefore, we may assume \( 1 \in 2 V(1, 0) \) as the identity \( 1 \) generates a copy of \( V(1, 0) \). Thus we may choose a linear basis \( \{ x_1, x_2, x_3, 1 \} \) for \( A \) such that \( \{ x_1, x_2 \} \) is a canonical basis for \( M_1(1, 0, \infty) \) with the \( D_4 \)-action given by (5) and \( 2 V(1, 0) = kx_3 + k1 \). Note that \( A^{D_4} = \text{span}\{x_2, x_3, 1\} \). so we have \( x_3^2 \in A^{D_4} \). Replacing \( x_3 \) with \( x_3 + \alpha \) for some suitable \( \alpha \in k \), we may assume

\[ x_3^2 = \alpha_3 x_2 + \gamma, \quad \alpha_3, \gamma \in k. \]

By the choice of \( x_2 \) and \( x_3 \), we have \( k x_i \cong V(1, 0), i = 2, 3 \). The MMBTP argument yields that the left and right multiplication of \( M_1(1, 0, \infty) \) by the element \( x_i, i = 2, 3 \), give four \( D_4 \)-module endomorphisms of \( M_1(1, 0, \infty) \). But \( \text{End}(M_1(1, 0, \infty)) \cong k \) by Lemma 1.2. Thus the four endomorphisms are given by scalar multiplications, in which \( x_2 x = x x_2 \) for \( x = x_2 \in M(1, 0, \infty) \). Hence there exist \( \alpha_2, \alpha_4 \) and \( \delta \in k \) such that

\[ x_1 x_2 = x_2 x_1 = \alpha_2 x_1, \quad x_2^2 = \alpha_2 x_2, \]

\[ x_1 x_3 = \alpha_4 x_1, \quad x_2 x_3 = \alpha_4 x_2, \quad x_3 x_1 = \delta x_1, \quad x_3 x_2 = \delta x_2. \]

Since \( b \) fixes \( x_1^2 \) and \( b \cdot x_1 = -x_1, \) \( x_1^2 \in \text{span}\{x_2, x_3, 1\} = A^{D_4} \). Hence we may assume \( x_1^2 = \alpha_1 x_2 + \beta x_3 + \beta_0 \), where \( \alpha_1, \beta, \beta_0 \in k \). Now from the equation \( x_1^2 x_3 = x_1 (x_1 x_3) \) one gets \( \beta_2 = \beta_0 \alpha_4, \beta_3 = 0 \) and \( \beta_0 = \beta_4 \); from the equations \( x_1^2 x_2 = x_2 x_1^2 = (x_2 x_1) x_1 \) one gets \( \beta_0 \alpha_2 = 0, \beta_2 = 0, \beta_0 + \beta \alpha_4 = 0 \) and \( \beta_0 + \beta \delta = 0 \). It follows that \( \beta_0 = \beta \alpha_4 = \beta \delta = 0 \). Thus we have

\[ x_1^2 = \alpha_1 x_2 + \beta x_3 \quad \text{and} \quad \beta \alpha_2 = \beta \alpha_3 = \beta \alpha_4 = \beta \delta = \beta \gamma = 0. \]

Similarly, from \( x_1 x_3 = x_1 (x_3 x_1), \) \( x_2 x_3 = x_2 (x_3 x_1) \) and \( x_3^2 x_3 = x_3 x_3^2 \), one obtains

\[ \alpha_1 (\delta - \alpha_4) = \alpha_2 (\delta - \alpha_4) = \alpha_3 (\delta - \alpha_4) = 0. \]

And from \( x_2 x_3^2 = x_2 (x_3 x_3) \) and \( x_3^2 x_2 = x_3 (x_3 x_2) \), we get

\[ \alpha_4^2 = \alpha_2 \alpha_3 + \gamma = \delta^2. \]
If $\beta \neq 0$, then $\alpha_2 = \alpha_3 = \alpha_4 = \delta = \gamma = 0$. In this case, replacing $x_3$ with $\beta x_3 + \alpha_1 x_2$ one can see that $A$ is isomorphic to the $D_4$-module algebra $A_1$ described in (2.10.1). If $\beta = 0$ and $\delta \neq \alpha_4$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\delta = -\alpha_4 \neq 0$ and $\gamma = \alpha_4^2$. In this case, replacing $x_3$ with $\alpha_4^{-1} x_3$ one can see that $A$ is isomorphic to the $D_4$-module algebra $A_2$ given by (2.10.2). Finally, if $\beta = 0$ and $\delta = \alpha_4$, then $\gamma = \alpha_4^2 - \alpha_2 \alpha_3$ and so $A$ is exactly isomorphic to the $D_4$-module algebra $A_+^\infty(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ described in (2.10.3).

The proof of the last part is similar to the proof of Lemma 2.6. □

Substituting $M_1(1, 0, \eta)$ for $M_1(1, 0, \infty)$ in Lemma 2.10, $\eta \in k$, we have all $D_4$-module algebra structures defined on $M_1(1, 0, \eta) \oplus 2V(1, 0)$.

**Lemma 2.11.** Let $A$ be a $D_4$-module algebra isomorphic to $M_1(1, 0, \eta) \oplus 2V(1, 0)$, $\eta \in k$. Then $A$ is isomorphic to $A_1$, $A_2$ or $A_+^\eta(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for some $\alpha_i \in k$, $i = 1, 2, 3, 4$, where $A_1$, $A_2$ and $A_+^\eta(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ have a $k$-linear basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action given by

\[
\begin{align*}
  a \cdot x_1 &= x_2, & a \cdot x_2 &= a \cdot x_3 = 0, & d \cdot x_1 &= -\eta x_2, & d \cdot x_2 &= d \cdot x_3 = 0, \\
  b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = x_2, & b \cdot x_3 &= c \cdot x_3 = x_3,
\end{align*}
\]

and their multiplication are given by (2.11.1)–(2.11.3) respectively.

(2.11.1) $x_i^2 = x_3$, $x_i^2 = 0$, $x_i^2 = 0$, $x_i x_j = 0$, $\forall i \neq j$.

(2.11.2) $x_1^2 = 0$, $x_2^2 = 0$, $x_3^2 = 1$, $x_1 x_2 = x_2 x_1 = 0$, $x_1 x_3 = -x_3 x_1 = x_1$, $x_2 x_3 = -x_3 x_2 = x_2$.

(2.11.3) $x_1^2 = \alpha_1 x_2$, $x_2^2 = \alpha_2 x_2$, $x_3^2 = \alpha_3 x_2 + \alpha_4^2 - \alpha_2 \alpha_3$, $x_1 x_2 = x_2 x_1 = \alpha_2 x_1$, $x_1 x_3 = x_3 x_1 = \alpha_4 x_1$, $x_2 x_3 = x_3 x_2 = \alpha_4 x_2$.

Moreover, $A_1 \ncong A_2$, $A_i \ncong A_+^\eta(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for any $i = 1, 2$ and $\alpha_j \in k$, $j = 1, 2, 3, 4$, $A_+^\eta(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cong A_+^\eta(\beta_1, \beta_2, \beta_3, \beta_4)$ if and only if there exist $\delta, \gamma \in k^\times$ and $\theta \in k$ such that $\alpha_1 = \delta \beta_1$, $\alpha_2 = \delta \beta_2$, $\alpha_3 = \delta^{-1} \gamma^2 \beta_3 + \delta^{-1} (\theta^2 \beta_2 + 2 \gamma \theta \beta_4)$ and $\alpha_4 = \theta \beta_2 + \gamma \beta_4$.

**Proof.** The proof is similar to the proof of Lemma 2.10. □

Substituting $M_1(1, 1, \infty)$ for $M_1(1, 0, \infty)$ in Lemma 2.10, we obtain eleven non-isomorphic $D_4$-module algebra structures on $M_1(1, 1, \infty) \oplus 2V(1, 0)$.

**Lemma 2.12.** Up to isomorphism, there exist eleven $D_4$-module algebras $A_1$, $A_2$, $A_3$, $A_4(r)$, $A_5(r)$ and $A_6(r, s)$, $r, s \in \mathbb{Z}_2$, isomorphic to $M_1(1, 1, \infty) \oplus 2V(1, 0)$ as a $D_4$-module. The $D_4$-module algebras have a linear basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action given by

\[
\begin{align*}
  a \cdot x_1 &= a \cdot x_2 = a \cdot x_3 = 0, & d \cdot x_1 &= x_2, & d \cdot x_2 &= d \cdot x_3 = 0, \\
  b \cdot x_1 &= c \cdot x_1 = x_1, & b \cdot x_2 &= c \cdot x_2 = -x_2, & b \cdot x_3 &= c \cdot x_3 = x_3,
\end{align*}
\]
and their multiplication are given by (2.12.1), (2.12.2), (2.12.3), (2.12.4(r)), (2.12.5(r)) and (2.12.6(r, s)) respectively.

(2.12.1) \(x_i x_j = 0, 1 \leq i, j \leq 3\).
(2.12.2) \(x_1^2 = x_3, x_2^2 = 0, x_3^2 = 0, x_i x_j = 0, \forall i \neq j\).
(2.12.3) \(x_1^2 = \frac{1}{2}(x_3 + 1), x_2^2 = 0, x_3^2 = 1, x_1 x_2 = -x_2 x_1 = x_2, x_1 x_3 = x_3 x_1 = x_1,\)
\(x_2 x_3 = x_3 x_2 = x_2\).
(2.12.4(r)) \(x_1^2 = 0, x_2^2 = 0, x_3^2 = 1, x_1 x_2 = x_2 x_1 = 0, x_1 x_3 = -(1)^r x_3 x_1 = x_1, x_2 x_3 =-(1)^r x_3 x_2 = x_2\).
(2.12.5(r)) \(x_1^2 = x_3 + 1, x_2^2 = 0, x_3^2 = 0, x_1 x_2 = -x_2 x_1 = x_2, x_1 x_3 = -x_3 x_1 = -(1)^r x_3,\)
\(x_2 x_3 = x_3 x_2 = 0\).
(2.12.6(s, r)) \(x_1^2 = 1, x_2^2 = 0, x_3^2 = 0, x_1 x_2 = -x_2 x_1 = x_2, x_1 x_3 = -(1)^r x_3 x_1 = -(1)^r x_3,\)
\(x_2 x_3 = x_3 x_2 = 0\).

**Proof.** Assume that \(A = M_1(1, 1, \infty) \oplus 2V(1, 0)\) is a \(D_4\)-module algebra. Then \(A^{D_4} = 2V(1, 0)\) as \(\text{soc}(M_1(1, 1, \infty)) \cong V(1, 1)\). Hence \(1 \in 2V(1, 0)\), and we may choose a linear basis \(\{x_1, x_2, x_3\}\) for \(A\) such that \(\{x_1, x_2\}\) is a canonical basis for \(M_1(1, 1, \infty)\) with the \(D_4\)-action given by (5) and \(2V(1, 0) = k x_3 + k 1\) with 1 being the identity of \(A\). Since \(b\) fixes \(x_1^2\), we have \(x_1^2 \in \text{span}\{x_1, x_3, 1\}\). It is clear that \(x_3^2 \in A^{D_4} = \text{span}\{x_3, 1\}\). Thus, without changing \(D_4\)-action we may choose \(x_1\) and \(x_3\) such that

\[x_1^2 = \alpha x_3 + \alpha_0 \quad \text{and} \quad x_3^2 = \beta, \quad \alpha, \alpha_0, \beta \in k.\]

Now from the equation \(d \cdot (x_1^2) = d \cdot (\alpha x_3 + \alpha_0) = 0\) one gets that \(x_1 x_2 + x_2 x_1 = 0\). Since \(b \cdot (x_1 x_2) = -x_1 x_2\) and \(b\) fixes 1, \(x_1, x_3\), we have \(x_1 x_2 = \theta x_2\) for some \(\theta \in k\). Consequently, we have \(x_2^2 = -d \cdot (x_1 x_2) = -\theta d \cdot x_2 = 0\).

Since \(b\) fixes \(x_1 x_3\) and \(x_3 x_1\), we infer that \(x_1 x_3\) and \(x_3 x_1\) are in \(\text{span}\{x_1, x_3, 1\}\). There exist \(\delta, \delta', \delta_0, \gamma, \gamma'\) and \(\gamma_0 \in k\) such that \(x_1 x_3 = \delta x_1 + \delta' x_3 + \delta_0\) and \(x_3 x_1 = \gamma x_1 + \gamma' x_3 + \gamma_0\). Now from the equation \(d \cdot (x_1 x_3) = d \cdot (\delta x_1 + \delta' x_3 + \delta_0)\) one gets \(x_2 x_3 = -\delta x_2\).

Similarly, one obtains \(x_3 x_2 = \gamma x_2\). Thus, from the identities \(x_1 x_3^2 = (x_1 x_3)x_3\) and \(x_3 x_1^2 = x_3(x_3 x_1)\) we obtain that \(\beta = \delta^2 = \gamma^2\), \(\delta_0 = -\delta \delta'\) and \(\gamma_0 = -\gamma \gamma'\). Hence we have

\[x_1 x_3 = \delta x_1 + \delta' x_3 - \delta \delta', \quad x_3 x_1 = \gamma x_1 + \gamma' x_3 - \gamma \gamma',\]

\[x_2 x_3 = \delta x_2, \quad x_3 x_2 = \gamma x_2, \quad \beta = \delta^2 = \gamma^2.\]

Substituting \(\alpha x_3 + \alpha_0\) for \(x_1^2\), one gets

\[\alpha x_3 x_1 + \alpha_0 x_1 = x_1^2 x_1 = x_1 x_1^2 = \alpha x_1 x_3 + \alpha_0 x_1,\]

consequently, \(\alpha x_1 x_3 = \alpha x_1 x_3\) and

\[\alpha (\delta - \gamma) = 0 = \alpha (\delta' - \gamma').\]
On the other hand, we have $x_2x_1^2 = (x_2x_1)x_1$, $x_1^2x_3 = x_1(x_1x_3)$ and $x_3x_1^2 = (x_3x_1)x_1$, which implies that

$$\alpha_0 = \theta^2 - \alpha\delta = \delta^2 + \alpha\delta = \gamma' + \alpha\gamma.$$  

Similarly, from $(x_1x_3)x_1 = x_1(x_3x_1)$, $(x_1x_3)x_2 = x_1(x_3x_2)$ and $(x_3x_1)x_3 = x_3(x_1x_3)$, one obtains

$$(\delta - \gamma)(\delta' + \gamma') = (\delta - \gamma)(\theta - \delta') = (\delta + \gamma)(\delta' - \gamma') = 0.$$  

Now we first assume $\gamma \neq \delta$. Then $\gamma = -\delta \neq 0$, $\theta = \delta'$, $\gamma' = -\delta'$ and $\alpha_0 = \delta^2$. Using a new linear basis $\{x_1, x_1x_3, x_2, \delta^{-1}x_3, 1\}$ of $A$, one can easily check that $A$ is isomorphic to the $D_4$-module algebra $A_4(1)$ described in (2.12.4)(1).

Next, we assume $\gamma = \delta \neq 0$. Then $\gamma' = \delta'$ and $\alpha = \frac{1}{2}\delta^{-1}(\theta^2 - \delta^2)$. If $\theta = 0$, then we can consider the basis $\{x_1 + \frac{1}{2}\delta'(\delta^{-1}x_3 - 1), x_2, \delta^{-1}x_3, 1\}$ of $A$. It is not difficult to check that in this case is isomorphic to the $D_4$-module algebra $A_4(0)$ described in (2.12.4)(0). If $\theta \neq 0$, then we can consider the basis $\{\theta^{-1}x_1 + \frac{1}{2}\theta^{-1}\delta'(\delta^{-1}x_3 - 1), \theta^{-1}x_2, \delta^{-1}x_3, 1\}$ of $A$. In this case, $A$ is isomorphic to the $D_4$-module algebra $A_3$ described in (2.12.3).

Finally, assume $\gamma = \delta = 0$. Then $\beta = 0$, $\alpha_0 = \theta^2 = \delta^2 = \gamma'^2$ and $\alpha(\delta' - \gamma') = 0$. Hence, we have

$$x_1^2 = \alpha x_3 + \theta^2, \quad x_2^2 = x_3^2 = 0, \quad x_2x_3 = x_3x_2 = 0,$$

$$x_1x_2 = -x_2x_1 = \theta x_2, \quad x_1x_3 = \delta'x_3, \quad x_3x_1 = \gamma'x_3.$$  

Thus, $\delta' = \gamma' = 0$ if $\theta = 0$ and $\alpha = 0$. Therefore, $A$ is isomorphic to the $D_4$-module algebra $A_1$ described in (2.12.1) in this case. If $\theta = 0$ and $\alpha \neq 0$, then $\delta' = \gamma' = 0$. In this case, we may replace $x_3$ with $\alpha x_3$. Consequently, $A$ is isomorphic to the $D_4$-module algebra $A_2$ described in (2.12.2).

Now suppose $\theta \neq 0$. Replacing $x_1$ and $x_2$ with $\theta^{-1}x_1$ and $\theta^{-1}x_2$ respectively, we may assume that $\theta = 1$ and $\delta'^2 = \gamma'^2 = 1$ without losing generality. If $\delta' = \gamma'$, then $\delta' = \gamma' = (-1)^r$ for some $r \in \mathbb{Z}_2$. Let $x_1' = x_1 - \frac{(-1)^r}{2} \alpha x_3$. With the new basis $\{x_1', x_2, x_3, 1\}$ of $A$, it is easy to see that $A$ is isomorphic to the $D_4$-module algebra $A_6(0, r)$ described in (2.12.6(0, $r$)). If $\delta' \neq \gamma'$, then $\delta' = -\gamma' = (-1)^r$ for some $r \in \mathbb{Z}_2$. Hence, we have

$$x_1^2 = \alpha x_3 + 1, \quad x_2^2 = x_3^2 = 0, \quad x_2x_3 = x_3x_2 = 0,$$

$$x_1x_2 = -x_2x_1 = x_2, \quad x_1x_3 = -x_3x_1 = (-1)^r x_3.$$  

Thus, $\alpha = 0$ implies that $A$ is isomorphic to the $D_4$-module algebra $A_6(1, r)$ given by (2.12.6(1, $r$)). If $\alpha \neq 0$, then replacing $x_3$ with $\alpha x_3$, we obtain that $A$ is isomorphic to the $D_4$-module algebra $A_5(0, r)$ given by (2.12.5($r$)).

The verification of the last statement is straightforward. \[\square\]

Substituting $M_1(1, 1, \eta)$ for $M_1(1, 1, \infty)$ in Lemma 2.12, $\eta \in k$, we obtain eleven $D_4$-module algebra structures on $M_1(1, 1, \eta) \oplus 2V(1, 0)$.
Lemma 2.13. Let \( \eta \in k \). Up to isomorphism, there exist eleven \( D_4 \)-module algebras \( A_1, A_2, A_3, A_4(r), A_5(r) \) and \( A_6(r, s) \), \( r, s \in \mathbb{Z}_2 \), isomorphic to \( A = M_1(1, 1, \eta) \oplus 2V(1, 0) \) as a \( D_4 \)-module. The \( D_4 \)-module algebras have a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
\begin{align*}
  a \cdot x_1 &= x_2, & a \cdot x_2 &= a \cdot x_3 = 0, & d \cdot x_1 &= -\eta x_2, & d \cdot x_2 &= d \cdot x_3 = 0, \\
  b \cdot x_1 &= c \cdot x_1 = x_1, & b \cdot x_2 &= c \cdot x_2 = -x_2, & b \cdot x_3 &= c \cdot x_3 = x_3,
\end{align*}
\]

and their multiplication are given by (2.13.1), (2.13.2), (2.13.3), (2.13.4(\( r \))), (2.13.5(\( r \))) and (2.13.6(\( s, r \))) respectively.

**Proof.** The proof is similar to the proof of Lemma 2.12. \( \square \)

Now we consider \( D_4 \)-module algebra structures on the modules of form \( M_1(1, r, \eta) \oplus V(1, 1) \oplus V(1, 0), r \in \mathbb{Z}_2, \eta \in k \cup \{\infty\} \). It turns out that on each module there are infinitely many non-isomorphic \( D_4 \)-module algebra structures.

Lemma 2.14. There exist three infinite families of non-isomorphic \( D_4 \)-module algebras \( A_1^{(0,\infty)}(\alpha, \beta), A_2^{(0,\infty)}(\alpha, \delta) \) and \( A_3^{(0,\infty)}(\alpha, \beta, \gamma) \), \( \alpha, \beta, \gamma, \delta \in k \), isomorphic to \( M_1(1, 0, \infty) \oplus V(1, 1) \oplus V(1, 0) \) as a \( D_4 \)-module. All of them have a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
\begin{align*}
  a \cdot x_1 &= a \cdot x_2 = a \cdot x_3 = 0, & d \cdot x_1 &= x_2, & d \cdot x_2 &= d \cdot x_3 = 0, \\
  b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = x_2, & b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]

The multiplication of \( A_1^{(0,\infty)}(\alpha, \beta), A_2^{(0,\infty)}(\alpha, \delta) \) and \( A_3^{(0,\infty)}(\alpha, \beta, \gamma) \) are given respectively by (2.14.1), (2.14.2) and (2.14.3).

\[
\begin{align*}
(2.14.1) \quad & x_1^2 = \alpha x_1, \quad x_2^2 = x_2, \quad x_3^2 = \beta x_2 - \beta, \quad x_1 x_2 = x_2 x_1 = x_1, \quad x_1 x_3 = x_3 x_1 = 0, \quad x_2 x_3 = x_3 x_2 = 0, \\
(2.14.2) \quad & x_1^2 = \alpha x_1 + \delta, \quad x_2^2 = 0, \quad x_3^2 = 0, \quad x_1 x_2 = x_2 x_1 = x_3, \quad x_1 x_3 = x_3 x_1 = \delta x_2, \quad x_2 x_3 = x_3 x_2 = 0.
\end{align*}
\]
Moreover, among each family we have

(i) \( A_1^{(0,\infty)}(\alpha, \beta) \cong A_1^{(0,\infty)}(\alpha', \beta') \) if and only if \( \alpha = \alpha' \) and \( \beta = \theta^2 \beta' \) for some \( \theta \in k^\times \);

(ii) \( A_2^{(0,\infty)}(\alpha, \delta) \cong A_2^{(0,\infty)}(\alpha', \delta') \) if and only if \( \alpha = \theta \alpha' + \lambda \delta' \) and \( \delta = \theta^2 \delta' \) for some \( \theta \in k^\times \) and \( \lambda \in k \);

(iii) \( A_3^{(0,\infty)}(\alpha, \beta, \delta, \gamma) \cong A_3^{(0,\infty)}(\alpha', \beta', \delta', \gamma') \) if and only if \( \alpha = \theta \alpha' + \lambda (\delta' + \gamma') + \theta^{-1} \lambda^2 \beta' \), \( \beta = \theta^{-1} \xi^2 \beta' \), \( \delta = \xi \delta' + \theta^{-1} \lambda \xi \beta' \) and \( \gamma = \xi \gamma' + \theta^{-1} \lambda \xi \beta' \) for some \( \theta, \xi \in k^\times \) and \( \lambda \in k \).

**Proof.** Let \( A \) be a \( D_4 \)-module algebra and assume that \( A = M_1(1, 0, \infty) \oplus V(1, 1) \oplus V(1, 0) \) as a \( D_4 \)-module. We work out the multiplication of \( A \). Since \( M_1(1, 0, \infty) \otimes V(1, 0) \cong M_1(1, 0, \infty) \), we have that \( M_1(1, 0, \infty) \) is a submodule of \( A \) and it is isomorphic to a quotient module of \( M_1(1, 0, \infty) \). If \( I \in M_1(1, 0, \infty) \), then \( V(1, 0) \subseteq M_1(1, 0, \infty) V(1, 0) \). Hence \( M_1(1, 0, \infty) V(1, 0) \cong M_1(1, 0, \infty) \) since \( M_1(1, 0, \infty) \) has only one non-trivial submodule \( V(1, 0) \), and the only non-trivial quotient module of \( M_1(1, 0, \infty) \) is isomorphic to \( V(1, 1) \). On the other hand, if \( I \) is a submodule of \( A \) isomorphic with \( M_1(1, 0, \infty) \), then \( \text{soc}(I) = \text{soc}(M_1(1, 0, \infty)) \). It follows that \( V(1, 0) \subseteq M(1, 0, \infty) \), a contradiction. Hence, \( 1 \notin M(1, 0, \infty) \), and we may assume \( 1 \in V(1, 0) \) as the identity 1 generates a copy of \( V(1, 0) \). Thus we may choose a linear basis \( \{x_1, x_2, x_3, x_1' \} \) for \( A \) such that \( \{x_1, x_2\} \) is a canonical basis of \( M_1(1, 0, \infty) \) with the \( D_4 \)-action given by (5), \( V(1, 1) = k x_3 \) and \( V(1, 0) = k 1 \) with 1 being the identity of \( A \).

Since \( b \) fixes \( x_1^2 \), we have \( x_1^2 \in k x_2 + k 1 \), say, \( x_1^2 = \alpha x_2 + \alpha' \) for some \( \alpha \) and \( \alpha' \) in \( k \). Then \( x_1 x_2 - x_2 x_1 = d \cdot (x_1^2) = d \cdot (\alpha x_2 + \alpha') = 0 \). So \( x_1 \) and \( x_2 \) commute. Since \( b \cdot (x_1 x_2) = -x_1 x_2 \), we have \( x_1 x_2 = t x_1 + t' x_3 \) for some \( t, t' \in k \). From \( d \cdot (x_1 x_2) = d \cdot (t x_1 + t' x_3) \) one gets \( x_2^2 = t x_2 \). Similarly, since \( b \) fixes \( x_2^2 \), \( x_1 x_3 \) and \( x_2 x_3 \), we have \( x_2^2 = \beta x_2 + \beta' \), \( x_1 x_3 = \delta x_2 + \delta' \) and \( x_3 x_1 = \gamma x_2 + \gamma' \) for some \( \beta, \beta', \delta, \delta', \gamma, \gamma' \in k \). Let the element \( d \) act on the foregoing equations. We obtain that \( x_2 x_3 = x_3 x_2 = 0 \). Now replacing \( x_1 x_3, x_3 x_1, x_1^2 \) and \( x_3^2 \) with their linear expressions in the equations: \( x_3 (x_1 x_3) = (x_3 x_1) x_3, x_1^2 x_3 = x_1 (x_1 x_3), x_3 x_3^2 = (x_3 x_1) x_1 \) and \( x_3 x_1^2 = (x_1 x_3) x_3 \), we obtain that \( \gamma' = \delta', \alpha' = \delta t' = \gamma t' \), \( \delta' = -\delta t = -\gamma t \) and \( \beta' = \beta t' \). Consequently, we have

\[
\begin{align*}
x_1^2 &= \alpha x_2 + \delta t', \\
x_2^2 &= t x_2, \\
x_3^2 &= \beta x_2 - \beta t, \\
x_1 x_2 &= x_2 x_1 = t x_1 + t' x_3, \\
x_1 x_3 &= \delta x_2 - \delta t, \\
x_3 x_1 &= \gamma x_2 - \delta t, \\
x_2 x_3 &= x_3 x_2 = 0,
\end{align*}
\]

and

\[
(\delta - \gamma) t' = 0, \\
\delta t = \gamma t = -\beta t'.
\]

If \( t \neq 0 \), then \( \gamma = \delta = -\beta t' t^{-1} \). Let \( y_1 = t^{-1} x_1 + t^{-2} t' x_3, y_2 = t^{-1} x_2 \) and \( y_3 = x_3 \). Then a straightforward computation yields that \( y_1^2 = a_0 y_2, y_2^2 = y_2, y_3^2 = \beta_0 y_2 - \beta_0, y_1 y_2 = y_2 y_1 = y_1 \) and \( y_1 y_3 = y_3 y_1 = y_2 y_3 = y_3 y_2 = 0 \), where \( a_0 = t^{-1} \alpha - t^{-3} t^2 \beta \) and
\(\beta_0 = \beta t\). Thus, \(A \cong A_1^{(0,\infty)}(\alpha_0, \beta_0)\) has the \(D_4\)-module algebra structure described in (2.14.1).

Moreover, we have
\[ x_1^2 = \alpha x_2 + \delta t', \quad x_2 = x_3 = 0, \quad x_1 x_2 = x_1 x_3 = x_2 x_1 = \delta x_2 \quad \text{and} \quad x_2 x_3 = x_3 x_2 = 0. \]

Replacing \(x_3\) with \(t' x_3\), we see that \(A\) is isomorphic to \(A_2^{(0,\infty)}(\alpha, \delta t')\), the \(D_4\)-module algebra given by (2.14.2). Finally, if \(t = t' = 0\) then \(A\) is obviously isomorphic to \(A_3^{(0,\infty)}(\alpha, \beta, \delta, \gamma)\), the \(D_4\)-module algebra given by (2.14.3).

The verification of the last three isomorphisms is straightforward. \(\square\)

Substituting \(M_1(1, 0, \eta)\) for \(M_1(1, 0, \infty)\) in Lemma 2.14, \(\eta \in k\), we obtain three infinite families of \(D_4\)-module algebra structures on \(M_1(1, 0, \eta) \oplus V(1, 1) \oplus V(1, 0)\).

**Lemma 2.15.** Let \(\eta \in k\). There exist three infinite families of non-isomorphic \(D_4\)-module algebras \(A_1^{(0,\eta)}(\alpha, \beta)\), \(A_2^{(0,\eta)}(\alpha, \delta)\) and \(A_3^{(0,\eta)}(\alpha, \beta, \delta, \gamma)\) are given respectively by (2.15.1), (2.15.2) and (2.15.3).

\[
\begin{align*}
(2.15.1) \quad & x_1^2 = \alpha x_2, \quad x_2^2 = x_2, \quad x_3^2 = \beta x_2 - \alpha x_2 x_1, \quad x_1 x_2 = x_2 x_1 = x_1 x_3 = x_3 x_1 = 0, \quad x_2 x_3 = x_3 x_2 = 0. \quad \\
(2.15.2) \quad & x_1^2 = \alpha x_2 + \delta, \quad x_2^2 = 0, \quad x_3^2 = 0, \quad x_1 x_2 = x_2 x_1 = x_3, \quad x_1 x_3 = x_3 x_1 = \delta x_2, \quad x_2 x_3 = x_3 x_2 = 0. \quad \\
(2.15.3) \quad & x_1^2 = \alpha x_2, \quad x_2^2 = 0, \quad x_3^2 = \beta x_2, \quad x_1 x_2 = x_2 x_1 = 0, \quad x_1 x_3 = \delta x_2, \quad x_3 x_1 = \gamma x_2, \quad x_2 x_3 = x_3 x_2 = 0.
\end{align*}
\]

Moreover, we have

(i) \(A_1^{(0,\eta)}(\alpha, \beta) \cong A_1^{(0,\eta)}(\alpha', \beta')\) if and only if \(\alpha = \alpha'\) and \(\beta = \theta^2 \beta'\) for some \(\theta \in k^\times\);

(ii) \(A_2^{(0,\eta)}(\alpha, \delta) \cong A_2^{(0,\eta)}(\alpha', \delta')\) if and only if \(\alpha = \theta \alpha' + \lambda \delta'\) and \(\delta = \theta^2 \delta'\) for some \(\theta \in k^\times\) and \(\lambda \in k\);

(iii) \(A_3^{(0,\eta)}(\alpha, \beta, \delta, \gamma) \cong A_3^{(0,\eta)}(\alpha', \beta', \delta', \gamma')\) if and only if \(\alpha = \theta \alpha' + \lambda (\delta' + \gamma') + \theta^{-1} \lambda^2 \beta'\), \(\beta = \theta^{-1} \xi^2 \beta'\), \(\delta = \xi \delta' + \theta^{-1} \lambda \xi \beta'\) and \(\gamma = \xi \gamma' + \theta^{-1} \lambda \xi \beta'\) for some \(\theta, \xi \in k^\times\) and \(\lambda \in k\).

**Proof.** Similar to the proof of Lemma 2.14. \(\square\)

Replacing \(M_1(1, 0, \infty)\) with \(M_1(1, 1, \infty)\) in Lemma 2.14, we obtain three families of \(D_4\)-module algebra structures on \(M_1(1, 1, \infty) \oplus V(1, 1) \oplus V(1, 0)\).
Lemma 2.16. There are three families (two infinite and one finite) of non-isomorphic $D_4$-module algebras $A_1^{(1,\infty)}(\alpha)$, $A_2^{(1,\infty)}(\alpha, \beta)$ and $A_3^{(1,\infty)}[s, r]$, $\alpha, \beta \in k$, $s, r \in \mathbb{Z}_2$, isomorphic to $M_1(1,1,\infty) \oplus V(1,1) \oplus V(1,0)$ as a $D_4$-module. $A_1^{(1,\infty)}(\alpha)$, $A_2^{(1,\infty)}(\alpha, \beta)$ and $A_3^{(1,\infty)}[s, r]$ have a linear basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action given by

\[
a \cdot x_1 = a \cdot x_2 = a \cdot x_3 = 0, \quad d \cdot x_1 = x_2, \quad d \cdot x_2 = d \cdot x_3 = 0,
\]
\[
b \cdot x_1 = c \cdot x_1 = x_1, \quad b \cdot x_2 = c \cdot x_2 = -x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3.
\]

The multiplication of $A_1^{(1,\infty)}(\alpha)$, $A_2^{(1,\infty)}(\alpha, \beta)$ and $A_3^{(1,\infty)}[s, r]$ are given respectively by (2.16.1), (2.16.2) and (2.16.3).

(2.16.1) $x_1^2 = \alpha$, $x_2^2 = 0$, $x_3^2 = 0$, $x_1 x_2 = -x_2 x_1 = x_3$, $x_1 x_3 = -x_3 x_1 = \alpha x_2$, $x_2 x_3 = x_3 x_2 = 0$.

(2.16.2) $x_1^2 = 0$, $x_2^2 = 0$, $x_3^2 = 0$, $x_1 x_2 = x_2 x_1 = 0$, $x_1 x_3 = \alpha x_2$, $x_3 x_1 = \beta x_2$, $x_2 x_3 = x_3 x_2 = 0$.

(2.16.3) $x_1^2 = 1$, $x_2^2 = 0$, $x_3^2 = 0$, $x_1 x_2 = -x_2 x_1 = x_2$, $x_1 x_3 = (-1)^s x_3 x_1 = (-1)^r x_3$, $x_2 x_3 = x_3 x_2 = 0$.

Moreover, we have

(i) for any $\alpha, \alpha' \in k$, $A_1^{(1,\infty)}(\alpha) \cong A_1^{(1,\infty)}(\alpha')$ if and only if $\alpha = \delta^2 \alpha'$ for some $\delta \in k^\times$;

(ii) for any $\alpha$, $\beta$, $\alpha'$ and $\beta' \in k$, $A_2^{(1,\infty)}(\alpha, \beta) \cong A_2^{(1,\infty)}(\alpha', \beta')$ if and only if $\alpha = \delta \alpha'$ and $\beta = \delta \beta'$ for some $\delta \in k^\times$;

(iii) for any $s$, $r$, $s'$ and $r' \in \mathbb{Z}_2$, $A_3^{(1,\infty)}[s, r] \cong A_3^{(1,\infty)}[s', r']$ if and only if $s = s'$ and $r = r'$.

Proof. Let $A$ be a $D_4$-module algebra and assume that $A = M_1(1,1,\infty) \oplus V(1,1) \oplus V(1,0)$ as a $D_4$-module. Since $A^{D_4} = M_1(1,1,\infty) \oplus V(1,1) \oplus V(1,0)^{D_4} = V(1,0)$, we have $k1 = V(1,0)$. Hence we may choose a linear basis $\{x_1, x_2, x_3, 1\}$ for $A$ such that $\{x_1, x_2\}$ is a canonical basis for $M_1(1,1,\infty)$ with the $D_4$-action given by (5), $V(1,1) = kx_3$ and $V(1,0) = k1$ with 1 being the identity of $A$. Since $b \cdot (x_1^2) = x_1^2$, $x_1^2 \in kx_1 + k1$. Replacing $x_1$ with $x_1 + \delta$ for some suitable $\delta \in k$, we may assume that $x_1^2 = \alpha$ for some $\alpha \in k$. Hence $0 = d \cdot \alpha = d \cdot (x_1^2) = (d \cdot x_1)(c \cdot x_1) + x_1(d \cdot x_1) = x_2 x_1 + x_1 x_2$, and so $x_2 x_1 = -x_1 x_2$. Since $b \cdot (x_1 x_2) = -x_1 x_2$, $x_1 x_2 \in kx_2 \oplus kx_3$. Hence $x_2^2 = -d \cdot (x_1 x_2) = 0$ as $d \cdot x_2 = d \cdot x_3 = 0$. Now we are going to prove the statement in two cases where $x_1 x_2 \in M_1(1,1,\infty)$ or $x_1 x_2 \notin M_1(1,1,\infty)$. Note that $kx_2 = \text{soc}(M_1(1,1,\infty))$.

Case 1. $x_1 x_2 \notin M_1(1,1,\infty)$. In this case, $x_1 x_2 \neq 0$ and it generates a submodule of $A$ isomorphic to $V(1,1)$. Hence, we may choose $x_3 = x_1 x_2$ without changing anything else. Thus we have $x_1^2 = \alpha$, $x_2^2 = 0$, $x_1 x_2 = -x_2 x_1 = x_3$. Then $x_1 x_3 = x_2^2 x_2 = \alpha x_2$, $x_3 x_1 = -x_2^2 x_1 = -\alpha x_2$, $x_2 x_3 = -x_2^2 x_1 = 0$, $x_3 x_2 = x_1 x_2^2 = 0$ and $x_3^2 = -x_1 x_2^2 = 0$. Therefore, $A$ is isomorphic to $A_1^{(1,\infty)}(\alpha)$, the $D_4$-module algebra described in (2.16.1).
Case 2. \(x_1, x_2 \in M_1(1, 1, \infty)\). Then \(x_1 x_2 = -x_2 x_1 = \beta x_2\) for some \(\beta \in k\). Since \(b \cdot (x_1 x_3) = -x_1 x_3\), we have \(x_1 x_3 = \delta x_2 + \delta' x_3\) for some \(\delta, \delta' \in k\). Similarly, we have \(x_3 x_1 = \gamma x_2 + \gamma' x_3\) for some \(\gamma, \gamma' \in k\). Hence \(d \cdot (x_1 x_3) = d \cdot (\delta x_2 + \delta' x_3) = 0\). On the other hand, \(d \cdot (x_1 x_3) = (d \cdot x_1)(c \cdot x_3) + x_1(d \cdot x_3) = -x_2 x_3\). Thus we have \(x_2 x_3 = 0\). Similarly, we obtain that \(x_3 x_2 = 0\). Since \(V(1, 1) \otimes V(1, 1) \cong V(1, 0)\), it is clear that \(V(1, 1)^2 \subseteq V(1, 0) = A^{D_4}\). This implies \(x_1^2 \in k \cdot 1\). But \(x_1^2 = x_3(x_3 x_2) = 0\). It follows that \(x_2^2 = 0\).

Now from the equations \(x_1^2 x_2 = (x_1 x_2) x_1 = (x_1 x_2) = 0\) and \(x_1^2 x_3 = (x_1 x_3) x_1 = (x_1 x_3) = 0\), one gets \(\alpha = \beta^2 = \delta^2\) and \(\delta(\beta + \delta') = 0\); from the equations \(x_3 x_2^2 = (x_3 x_2) x_1 = (x_3 x_2) = 0\), we have \(x_3 x_2 = x_3 x_1 = x_3\). If \(\alpha = \gamma^2\), \(\gamma(\beta - \gamma') = 0\) and \(\delta(\beta + \delta') = \gamma(\delta' - \beta)\).

If \(\beta = 0\), then \(\alpha = \delta = \gamma = 0\). Hence, we have \(x_1^2 = x_2^2 = x_3^2 = 0\), \(x_1 x_2 = -x_2 x_1 = 0\), \(x_2 x_3 = x_3 x_2 = 0\), \(x_1 x_3 = \delta x_2\) and \(x_3 x_1 = \gamma x_2\). We have showed that in this case \(A \cong A_2^{(1, \infty)}(\delta, \gamma)\), the \(D_4\)-module algebra described in (2.16.2).

If \(\beta \neq 0\), then we may assume \(\beta = 1\) by replacing \(x_1\) and \(x_2\) with \(\beta^{-1} x_1\) and \(\beta^{-1} x_2\) respectively. Thus we have \(\alpha = \delta^2 = \gamma^2 = 1\), \(\delta(1 + \delta') = \gamma(1 - \gamma') = 0\) and \(\delta(1 + \gamma') = \gamma(\delta' - 1)\). Hence \(\delta' = \pm 1\) and \(\gamma' = \pm 1\). We also have

\[
\begin{align*}
x_1^2 &= 1, \\
x_2^2 &= 0, \\
x_3^2 &= 0, \\
x_1 x_2 &= -x_2 x_1 = x_2, \\
x_1 x_3 &= \delta x_2 + \delta' x_3, \\
x_3 x_1 &= \gamma x_2 + \gamma' x_3, \\
x_2 x_3 &= x_3 x_2 = 0.
\end{align*}
\]

If \(\delta' = \gamma' = 1\), then \(\delta = 0\), \(x_1 x_3 = x_3\) and \(x_3 x_1 = \gamma x_2 + x_3\). Let \(x_1 x_3 = x_2^2 + \frac{1}{2} \gamma x_2\). Then \(x_1 x_3 = x_3'\) and \(x_2 x_3 = x_3'\). With the new basis \(\{x_1, x_2, x_3', 1\}\) of \(A\), it is easy to see that \(A\) is isomorphic to \(A_3^{(1, \infty)}[0, 0]\) in this case.

If \(\delta' = \gamma' = -1\), then \(\gamma = 0\), \(x_1 x_3 = \delta x_2 - x_3\) and \(x_3 x_1 = -x_3\). Let \(x_1 x_3 = x_3 - \frac{1}{2} \delta x_2\). Then one can check that \(x_1 x_3' = x_2 x_3' = -x_3'\) and \(A\) is isomorphic to \(A_3^{(1, \infty)}[0, 1]\) in this case.

If \(\delta' = 1\) and \(\gamma' = -1\), then \(\delta = \gamma = 0\), and hence \(A\) is isomorphic to \(A_3^{(1, \infty)}[1, 0]\) in this case. If \(\delta' = -1\) and \(\gamma' = 1\), then \(\gamma = -\delta\) and \(x_1 x_3 = -x_3 x_1 = \delta x_2 - x_3\). Let \(x_1 x_3 = x_3 - \frac{1}{2} \delta x_2\). It is easy to see that \(x_1 x_3' = -x_3 x_1 = -x_3'\). Consequently, \(A\) is isomorphic to \(A_3^{(1, \infty)}[1, 1]\) in this case.

The rest of the proof is straightforward. \(\square\)

Substituting \(M_1(1, 1, \eta)\) for \(M_1(1, 1, \infty)\) in Lemma 2.16, \(\eta \in k\), we have all \(D_4\)-module algebra structures defined on \(M_1(1, 1, \eta) \oplus V(1, 1) \oplus V(1, 0)\).

Lemma 2.17. Let \(\eta \in k\). There exist three families of non-isomorphic \(D_4\)-module algebras \(A_1^{(1, \eta)}(\alpha), A_2^{(1, \eta)}(\alpha, \beta)\) and \(A_3^{(1, \eta)}[s, r], \alpha, \beta \in k, s, r \in \mathbb{Z}_2\), isomorphic to \(M_1(1, 1, \eta) \oplus V(1, 1) \oplus V(1, 0)\) as a \(D_4\)-module. All of them have a \(k\)-linear basis \(\{x_1, x_2, x_3, 1\}\) with the \(D_4\)-action given by

\[
\begin{align*}
a \cdot x_1 &= x_2, \\
a \cdot x_2 &= a \cdot x_3 = 0, \\
d \cdot x_1 &= -\eta x_2, \\
d \cdot x_2 &= d \cdot x_3 = 0, \\
b \cdot x_1 &= c \cdot x_1 = x_1, \\
b \cdot x_2 &= c \cdot x_2 = -x_2, \\
b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]
The multiplication of $A_1^{(1,\eta)}(\alpha)$, $A_2^{(1,\eta)}(\alpha, \beta)$ and $A_3^{(1,\eta)}[s, r]$ are given respectively by (2.17.1), (2.17.2) and (2.17.3).

(2.17.1) $x_1^2 = \alpha$, $x_2^2 = 0$, $x_3^2 = 0$, $x_1x_2 = -x_2x_1 = x_3$, $x_1x_3 = -x_3x_1 = x_2x_2 = 0$.

(2.17.2) $x_1^2 = 0$, $x_2^2 = 0$, $x_3^2 = 0$, $x_1x_2 = x_2x_1 = 0$, $x_1x_3 = \alpha x_2$, $x_3x_1 = \beta x_2$, $x_2x_3 = x_3x_2 = 0$.

(2.17.3) $x_1^2 = 1$, $x_2^2 = 0$, $x_3^2 = 0$, $x_1x_2 = -x_2x_1 = x_2$, $x_1x_3 = (-1)^s x_3x_1 = (-1)^r x_3$,

Moreover, we have

(i) for any $\alpha, \alpha' \in k$, $A_1^{(1,\eta)}(\alpha) \cong A_1^{(1,\eta)}(\alpha')$ if and only if $\alpha = \delta^2 \alpha'$ for some $\delta \in k^\times$;

(ii) for any $\alpha, \beta, \alpha'$ and $\beta' \in k$, $A_2^{(1,\eta)}(\alpha, \beta) \cong A_2^{(1,\eta)}(\alpha', \beta')$ if and only if $\alpha = \delta \alpha'$ and $\beta = \delta \beta'$ for some $\delta \in k^\times$; and

(iii) for any $s, r, s'$ and $r' \in \mathbb{Z}_2$, $A_3^{(1,\eta)}[s, r] \cong A_3^{(1,\eta)}[s', r']$ if and only if $s = s'$ and $r = r'$.

**Proof.** Similar to the proof of Lemma 2.16. \qed

Recall that for elements $\alpha, \beta$ and $\gamma$ in $k$, the generalized quaternion algebra $(\frac{a, b, c}{k})$ is generated by two elements $u$ and $v$ subject to the relations:

$$u^2 = \alpha, \quad v^2 = \beta, \quad \text{and} \quad uv + vu = \gamma.$$  

The algebra $(\frac{a, b, c}{k})$ is $4$-dimensional, and it has a canonical basis $\{u, v, uv, 1\}$.

For $\alpha \in k$, let $k(\sqrt{\alpha})$ denote the $k$-algebra generated by one generator $x$ subject to the relation $x^2 = \alpha$. That is, $k(\sqrt{\alpha})$ is the quotient algebra $k[X]/(X^2 - \alpha)$ of the polynomial algebra $k[X]$ modulo the principal ideal $(X^2 - \alpha)$. The algebra $k(\sqrt{\alpha})$ is of dimension 2 with a $k$-basis $\{x, 1\}$. There is a canonical $D_4$-action on $k(\sqrt{\alpha})$ given by

$$a \cdot x = 0, \quad d \cdot x = 1, \quad b \cdot x = c \cdot x = -x,$$  \hspace{1cm} (13)

making $k(\sqrt{\alpha})$ into a $D_4$-module algebra. Denote by $k(\sqrt{\alpha})_\infty$ the $D_4$-module algebra $k(\sqrt{\alpha})$ with the $D_4$-action (13). It is not difficult to see that $k(\sqrt{\alpha})_\infty \cong M_1(1, 0, \infty)$ as $D_4$-modules and $k(\sqrt{\alpha})_\infty \cong k(\sqrt{\beta})_\infty$ as $D_4$-module algebras if and only if $\alpha = \beta \in k$.

The underlying algebra $k(\sqrt{\alpha})_\infty$ admits another $D_4$-module algebra structure given by

$$a \cdot x = 1, \quad d \cdot x = -\eta, \quad b \cdot x = c \cdot x = -x,$$  \hspace{1cm} (14)

where $\eta \in k$. It is easy to see $k(\sqrt{\alpha}) \cong M_1(1, 0, \eta)$ as $D_4$-modules. Denote by $k(\sqrt{\alpha})_\eta$ the $D_4$-module algebra $k(\sqrt{\alpha})$ with $D_4$-action (14). For any $\alpha, \alpha', \eta, \eta' \in k$, $k(\sqrt{\alpha})_\eta \cong k(\sqrt{\alpha'})_\eta'$ if and only if $\alpha = \alpha'$ and $\eta = \eta'$.

Now we consider the $D_4$-module algebra structures on $V(2, r) \oplus M_1(1, 0, \eta)$, $r \in \mathbb{Z}_2$, $\eta \in k \cup \{\infty\}$. Let us first look at the module $V(2, 0) \oplus M_1(1, 0, \infty)$. It turns out that there are infinitely many $D_4$-module algebra structures defined on $V(2, 0) \oplus M_1(1, 0, \infty)$. \hfill \hfill \hfill \hfill
Lemma 2.18. Let $A$ be a $D_4$-module algebra isomorphic to $V(2,0) \oplus M_1(1,0,\infty)$ as a $D_4$-module. Then $A$ is isomorphic to $(\frac{\alpha,0,0}{k})^\infty_+$ for some $\alpha \in k$, where $(\frac{\alpha,0,0}{k})^\infty_+ = (\frac{\alpha,0,0}{k})_+$ as an algebra, and the $D_4$-module structure is given by
\[
\begin{align*}
a \cdot u &= 2uv, & a \cdot v &= 0, & d \cdot u &= 0, & d \cdot v &= 1, \\
b \cdot u &= -c \cdot u = u, & b \cdot v &= c \cdot v = -v.
\end{align*}
\]
Moreover, for any $\alpha, \beta \in k$, $(\frac{\alpha,0,0}{k})^\infty_+ \cong (\frac{\beta,0,0}{k})^\infty_+$ if and only if $\alpha = \delta^2 \beta$ for some $\delta \in k^\times$.

Proof. It is easy to check that the $D_4$-module algebra $(\frac{\alpha,0,0}{k})^\infty_+$ is well defined for any $\alpha \in k$. Now assume that $A = V(2,0) \oplus M_1(1,0,\infty)$ is a $D_4$-module algebra. It is clear that $A^{D_4} = \text{soc}(M_1(1,0,\infty)) = k1$. Hence we may choose a linear basis $\{x_1, x_2, x_3, 1\}$ for $A$ such that $\{x_1, x_2\}$ is a canonical basis for $V(2,0)$ with the $D_4$-action given by (3) and $\{x_3, 1\}$ is a canonical basis for $M_1(1,0,\infty)$ with the $D_4$-action given by (5) and 1 being the identity of $A$. Thus we have the following relations:
\[
\begin{align*}
a \cdot x_1 &= x_2, & a \cdot x_2 &= a \cdot x_3 = 0, & d \cdot x_1 &= 0, & d \cdot x_2 &= 2x_1, & d \cdot x_3 &= 1, \\
b \cdot x_1 &= -c \cdot x_1 = x_1, & b \cdot x_2 &= -c \cdot x_2 = -x_2, & b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]
Observe that the set $\{x \in A | b \cdot x = c \cdot x = x\}$ is equal to $k1$. Since $b \cdot (x_i^2) = c \cdot (x_i^2) = x_i^2$ for $1 \leq i \leq 3$, we have $x_i^2 = \alpha$, $x_i^2 = \beta$ and $x_i^2 = \gamma$ for some $\alpha, \beta, \gamma \in k$. Since $b \cdot (x_1x_3) = -x_1x_3$ and $c \cdot (x_1x_3) = x_1x_3$, we have $x_1x_3 = \delta x_2$ for some $\delta \in k$. Hence $a \cdot (x_1x_3) = a \cdot (\delta x_2) = 0$. On the other hand, we have $a \cdot (x_1x_3) = (a \cdot x_1)(b \cdot x_3) + x_1(a \cdot x_3) = -x_1x_3$. It follows that $x_2x_3 = 0$. Similarly, one can show that $x_3x_1 = \delta' x_2$ for some $\delta' \in k$ and $x_3x_2 = 0$. Now we have $v \cdot x_2 = x_3x_2 = x_3(x_3x_2) = 0$. This implies $v = 0$. From the equations $d \cdot (x_1x_3) = d \cdot (\delta x_2)$ and $d \cdot (x_3x_1) = d \cdot (\delta' x_2)$, one gets that $\delta = \frac{1}{2}$ and $\delta' = -\frac{1}{2}$. It follows that there is a $D_4$-module algebra isomorphism:
\[
f : A \to (\frac{\alpha,0,0}{k})^\infty_+, \quad x_1 \mapsto u, \quad x_2 \mapsto 2uv, \quad x_3 \mapsto v, \quad 1 \mapsto 1.
\]
The verification of the last isomorphism is straightforward. \( \square \)

Substituting $M_1(1,0,\eta)$ for $M_1(1,0,\infty)$ in Lemma 2.18, $\eta \in k$, we obtain all $D_4$-module quaternion algebra structures on $V(2,0) \oplus M_1(1,0,\eta)$.

Lemma 2.19. Let $\beta \in k$ and $A$ be a $D_4$-module algebra isomorphic to $V(2,0) \oplus M_1(1,0,-2\beta)$ as a $D_4$-module. Then there exists an $\alpha \in k$ such that $A \cong (\frac{\alpha,\beta,0}{k})^\infty_+$, where $(\frac{\alpha,\beta,0}{k})^\infty_+ = (\frac{\alpha,\beta,0}{k})_+$ as an algebra, and the $D_4$-action is given by
\[
\begin{align*}
a \cdot u &= 0, & a \cdot v &= 1, & d \cdot u &= 2uv, & d \cdot v &= 2\beta, \\
b \cdot u &= -c \cdot u = -u, & b \cdot v &= c \cdot v = -v.
\end{align*}
\]
Moreover, for any \( \alpha, \beta, \alpha', \beta' \in k \), \( \left( \frac{\alpha, \beta, 0}{k} \right)_+ \cong \left( \frac{\alpha', \beta', 0}{k} \right)_+ \) if and only if \( \beta = \beta' \) and \( \alpha = \delta^2 \alpha' \) for some \( \delta \in k^\times \).

**Proof.** The proof is similar to the one of Lemma 2.18. The \( D_4 \)-module algebra \( \left( \frac{\alpha, \beta, 0}{k} \right)_+ \) is well defined for any \( \alpha, \beta \in k \) and has been studied in [15].

Assume that \( A = V(2, 0) \oplus M_1(1, 0, -2\beta) \) is a \( D_4 \)-module algebra, \( \beta \in k \). Then \( A^{D_4} = \text{soc}(M_1(1, 0, -2\beta)) = k1 \), and hence we may choose a linear basis \( \{x_1, x_2, x_3, 1\} \) for \( A \) such that \( \{x_1, x_2\} \) is a canonical basis for \( V(2, 0) \) with the \( D_4 \)-action given by (3) and \( \{x_3, 1\} \) is a canonical basis for \( M_1(1, 0, -2\beta) \) with the \( D_4 \)-action given by (6) and 1 being the identity of \( A \). Thus we have

\[
\begin{align*}
  a \cdot x_1 &= x_2, & a \cdot x_2 &= 0, & a \cdot x_3 &= 1, \\
  d \cdot x_1 &= 0, & d \cdot x_2 &= 2x_1, & d \cdot x_3 &= 2\beta, \\
  b \cdot x_1 &= -c \cdot x_1 = x_1, & b \cdot x_2 &= -c \cdot x_2 = -x_2, & b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]

Now applying the same argument in the proof of Lemma 2.18, we have \( x_2^2 = \alpha \) and \( x_3^2 = \gamma \) for some \( \alpha \) and \( \gamma \in k \) since \( b \cdot (x_2^2) = c \cdot (x_2^2) = x_2^2 \) for \( 1 \leq i \leq 3 \). We also have \( x_1x_3 = \delta x_2 \) and \( x_3x_1 = \delta' x_2 \) for some \( \delta, \delta' \in k \) as \( b \cdot (x_1x_3) = -c \cdot (x_1x_3) = -x_1x_3 \) and \( b \cdot (x_3x_1) = -c \cdot (x_3x_1) = -x_3x_1 \). Now from the equations \( a \cdot (x_1x_3) = a \cdot (\delta x_2) \) and \( a \cdot (x_3x_1) = a \cdot (\delta' x_2) \) one gets \( x_2x_3 = x_1 \) and \( x_3x_2 = -x_1 \); from the equation \( d \cdot (x_1x_3) = d \cdot (\delta x_2) \) one gets \( \delta = \beta \).

On the other hand, we also have \( \gamma x_2 = x_2x_3^2 = (x_2x_3)x_3 = x_1x_3 = \delta x_2 \). Hence \( \gamma = \delta = \beta \). This shows that the mapping \( x_2 \mapsto u, x_3 \mapsto v \) induces a \( D_4 \)-module algebra isomorphism from \( A \) to \( \left( \frac{\alpha, \beta, 0}{k} \right)_+ \). The verification of the last statement is straightforward. \( \square \)

The Yetter–Drinfeld \( H_4 \)-module algebra \( \left( \frac{\alpha, \beta, 0}{k} \right)_+ \) has an \( H_4 \)-comodule structure induced by a quasitriangular structure \( R_\beta \), cf. Section 3, and it is an \( H_4 \)-Galois object if \( \alpha \neq 0 \). The reader can find more study of \( \left( \frac{\alpha, \beta, 0}{k} \right)_+ \) in [16,19].

Substituting \( V(2, 1) \) for \( V(2, 0) \) in Lemma 2.18, we have all \( D_4 \)-module algebra structures defined on \( V(2, 1) \oplus M_1(1, 0, \infty) \).

**Lemma 2.20.** Let \( A \) be a \( D_4 \)-module algebra isomorphic to \( V(2, 1) \oplus M_1(1, 0, \infty) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( A^\infty (\alpha) \) for some \( \alpha \in k \). \( A^\infty (\alpha) \) has a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
\begin{align*}
  a \cdot x_1 &= 2x_2, & a \cdot x_2 &= a \cdot x_3 = 0, & d \cdot x_1 &= 0, & d \cdot x_2 &= x_1, & d \cdot x_3 &= 1, \\
  b \cdot x_1 &= -c \cdot x_1 = -x_1, & b \cdot x_2 &= -c \cdot x_2 = x_2, & b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]

The multiplication of \( A^\infty (\alpha) \) is given by

\[
\begin{align*}
  x_1^2 &= \alpha, & x_2^2 &= 0, & x_3^2 &= 0, & x_1x_2 &= x_2x_1 = \alpha x_3, \\
  x_1x_3 &= x_3x_1 = x_2, & x_2x_3 &= x_3x_2 = 0.
\end{align*}
\]
Moreover, for any \( \alpha, \beta \in k \), \( A^\infty(\alpha) \cong A^\infty(\beta) \) if and only if \( \alpha = \delta^2 \delta \beta \) for some \( \delta \in k^\times \).

**Proof.** Assume that \( A = V(2, 1) \oplus M_1(1, 0, \infty) \) as a \( D_4 \)-module. Then \( A^{D_4} = \text{soc}(M_1(1, 0, \infty)) = k1 \), and hence we may choose a linear basis \( \{x_1, x_2, x_3, 1\} \) for \( A \) such that \( \{x_1, 2x_2\} \) is a canonical basis for \( V(2, 1) \) with the \( D_4 \)-action given by (3) and \( \{x_3, 1\} \) is a canonical basis for \( M_1(1, 0, \infty) \) with the \( D_4 \)-action given by (5) and 1 being the identity of \( A \). The same reason as in the proof of Lemma 2.18 assures that \( x_1^2 = \alpha \) and \( x_2^2 = \beta \) for some \( \alpha, \beta \in k \). Hence \( a \cdot (x_1^2) = a \cdot \alpha = 0 \). This implies \( x_1 x_2 = x_2 x_1 \). Since \( b \cdot (x_1 x_2) = c \cdot (x_1 x_2) = -x_1 x_2 \), one can see \( x_1 x_2 = x_2 x_1 = \delta x_3 \) for some \( \delta \in k \). Moreover, we have \( 0 = a \cdot (\delta x_3) = a \cdot (x_1 x_2) = 2x_2^2 \), and so \( x_2^2 = 0 \). Applying the same argument in the proof of Lemma 2.18, one obtains that \( x_1 x_3 = \theta x_2 \) and \( x_3 x_1 = \theta' x_2 \) for some \( \theta, \theta' \in k \). Hence we have \( 0 = a \cdot (\theta x_2) = a \cdot (x_1 x_3) = -2x_2 x_3 \), and so \( x_2 x_3 = 0 \). Similarly, \( x_3 x_2 = 0 \). On the other hand, from the equations \( d \cdot (x_1 x_3) = d \cdot (\theta x_2) \) and \( d \cdot (x_3 x_1) = d \cdot (\theta' x_2) \), one gets \( \theta = \theta' = 1 \). Therefore, \( x_1 x_3 = x_3 x_1 = x_2 \). Since \( x_2^2 = \beta \), we have \( \beta x_2 = x_2 x_3^2 = (x_2 x_3) x_3 = 0 \). This shows that \( x_3^2 = 0 \). Finally, from \( \alpha x_2 = x_1^2 x_2 = x_1 (x_1 x_2) = \delta x_1 x_3 = \delta x_2 \) one gets \( \delta = \alpha \). Thus we have proved that \( A \) has the desired \( D_4 \)-module algebra structure. \( \square \)

Substituting \( M_1(1, 0, \eta), \eta \in k \), for \( M_1(1, 0, \infty) \) in Lemma 2.20, we have all \( D_4 \)-module algebra structures defined on \( V(2, 1) \oplus M_1(1, 0, \eta) \).

**Lemma 2.21.** Let \( \beta \in k \) and \( A \) be a \( D_4 \)-module algebra isomorphic to \( V(2, 1) \oplus M_1(1, 0, -2\beta) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( A_-(\alpha, \beta) \) for some \( \alpha \in k \), where \( A_-(\alpha, \beta) \) has a linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
a \cdot x_1 = x_2, \quad a \cdot x_2 = 0, \quad a \cdot x_3 = 1, \\
da \cdot x_1 = 0, \quad d \cdot x_2 = 2x_1, \quad d \cdot x_3 = 2\beta, \\
b \cdot x_1 = -c \cdot x_1 = -x_1, \quad b \cdot x_2 = -c \cdot x_2 = x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3.
\]

The multiplication of \( A_-(\alpha, \beta) \) is given by

\[
x_1^2 = \alpha \beta, \quad x_2^2 = \alpha, \quad x_3^2 = \beta, \quad x_1 x_2 = x_2 x_1 = \alpha x_3, \\
x_1 x_3 = x_3 x_1 = \beta x_2, \quad x_2 x_3 = x_3 x_2 = x_1.
\]

Moreover, for any \( \alpha, \beta, \beta', \beta' \in k \), \( A_-(\alpha, \beta) \cong A_-(\alpha', \beta') \) if and only if \( \beta = \beta' \) and \( \alpha = \delta^2 \alpha' \) for some \( \delta \in k^\times \).

**Proof.** Assume that \( A = V(2, 1) \oplus M_1(1, 0, -2\beta) \) is a \( D_4 \)-module algebra, \( \beta \in k \). Then \( A^{D_4} = k1 = \text{soc}(M_1(1, 0, -2\beta)) \), and hence \( A \) has a linear basis \( \{x_1, x_2, x_3, 1\} \) such that \( \{x_1, x_2\} \) is a canonical basis for \( V(2, 1) \) with the \( D_4 \)-action given by (3) and \( \{x_3, 1\} \) is a canonical basis for \( M_1(1, 0, -2\beta) \) with the \( D_4 \)-action given by (6) and 1 being the identity of \( A \). Now using the same argument in the proof of Lemma 2.20, one can show that \( x_1^2 = \delta, \ x_2^2 = \gamma, \ x_1 x_2 = x_2 x_1 = \alpha x_3, \ x_1 x_3 = \theta x_2, \ x_3 x_1 = \theta' x_2 \) for some \( \delta, \gamma, \alpha, \theta \) and
\[ \theta' \in k. \text{ Now from the equation } a \cdot (x_1 x_2) = a \cdot (\alpha x_3), \text{ one gets } x_3^2 = \alpha; \text{ from the equations } a \cdot (x_1 x_3) = a \cdot (\theta x_2) \text{ and } a \cdot (x_3 x_1) = a \cdot (\theta' x_2), \text{ one gets that } x_2 x_3 = x_1 \text{ and } x_3 x_2 = x_1. \]

On the other hand, from the equations \( d \cdot (x_1 x_3) = d \cdot (\theta x_2) \) and \( d \cdot (x_3 x_1) = d \cdot (\theta' x_2), \) one obtains \( \theta = \theta' = \beta. \) Therefore, we have \( x_1 x_3 = x_3 x_1 = \beta x_2. \) We also have \( \gamma x_2 = x_3 x_2 = x_3 x_1 = \beta x_2. \) It follows that \( \gamma = \beta, \) and so \( x_2^2 = \beta. \) Finally, from \( \delta x_2 = x_3 x_2 = x_1 x_2 x_2 = \alpha x_1 x_3 = \alpha \beta x_2, \) we obtain \( \delta = \alpha \beta, \) and hence \( x_1^2 = \alpha \beta. \) We have proved that \( A \) is isomorphic to \( A_-(\alpha, \beta). \) The verification of the last statement is straightforward. \( \square \)

Now we consider \( D_4 \)-module algebra structures on the modules \( M_1(1, 0, \eta_1) \oplus M_1(1, 0, \eta_2), \) where \( \eta_1, \eta_2 \in k \cup \{ \infty \} \) with \( \eta_1 \neq \eta_2. \) We first consider the case where one of \( \eta_1 \) and \( \eta_2 \) is \( \infty, \) and then consider the other case.

**Lemma 2.22.** Let \( \eta \in k \) and \( A \) be a \( D_4 \)-module algebra isomorphic to \( M_1(1, 0, \eta) \oplus M_1(1, 0, \infty) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( k(\sqrt{\alpha})_\eta \times k(\sqrt{\beta})_\infty \) for some \( \alpha, \beta \in k. \)

Moreover, for any \( \alpha, \beta, \alpha', \beta', \eta \in k, k(\sqrt{\alpha})_\eta \times k(\sqrt{\beta})_\infty \cong k(\sqrt{\alpha'})_\eta \times k(\sqrt{\beta'})_\infty \) as \( D_4 \)-module algebras if and only if \( \alpha = \alpha' \) and \( \beta = \beta'. \)

**Proof.** Assume that \( A = M_1(1, 0, \eta) \oplus M_1(1, 0, \eta) \) is a \( D_4 \)-module algebra, \( \eta \in k. \) Then \( A \) has a linear basis \( \{x_1, x_2, x_3, x_4\} \) such that \( \{x_1, x_2\} \) is a canonical basis for \( M_1(1, 0, \eta) \) with the \( D_4 \)-action given by (6) and \( \{x_3, x_4\} \) is a canonical basis for \( M_1(1, 0, \infty) \) with the \( D_4 \)-action given by (5). Hence we have

\[
\begin{align*}
  a \cdot x_1 &= x_2, & a \cdot x_2 &= a \cdot x_3 = a \cdot x_4 = 0, \\
  d \cdot x_1 &= -\eta x_2, & d \cdot x_3 &= x_4, & d \cdot x_2 &= d \cdot x_4 = 0, \\
  b \cdot x_1 &= c \cdot x_1 = x_1, & b \cdot x_2 &= c \cdot x_2 = x_2, \\
  b \cdot x_3 &= c \cdot x_3 = -x_3, & b \cdot x_4 &= c \cdot x_4 = x_4.
\end{align*}
\]

Clearly, \( A^{D_4} = \text{soc}(M_1(1, 0, \eta)) + \text{soc}(M_1(1, 0, \infty)) = k x_2 + k x_4. \) If \( x_2 x_4 \neq 0, \) then \( x_2 M_1(1, 0, \infty) \neq 0. \) Hence \( x_2 M_1(1, 0, \infty) \) is a submodule of \( A \) and has a submodule \( k x_2 x_4 \cong V(1, 0). \) However, \( x_2 M_1(1, 0, \infty) \) is isomorphic to a quotient module of \( M_1(1, 0, \infty) \) as \( k x_2 \cong V(1, 0). \) It follows that \( x_2 M_1(1, 0, \infty) \cong M_1(1, 0, \infty). \) Similarly, \( M_1(1, 0, \eta) x_4 \neq 0 \) and \( M_1(1, 0, \eta) x_4 \cong M_1(1, 0, \eta). \) Therefore, \( M_1(1, 0, \eta) x_4 = M_1(1, 0, \eta) \) and \( x_2 M_1(1, 0, \infty) = M_1(1, 0, \infty) \) because there is no non-zero module homomorphism between \( M_1(1, 0, \infty) \) and \( M_1(1, 0, \eta). \) Thus one gets \( x_2 x_4 \in M_1(1, 0, \eta) \cap M_1(1, 0, \infty), \) a contradiction. Hence we have \( x_2 x_4 = 0, \) and \( x_4 x_2 = 0 \) similarly. Then by \( 1 \in A^{D_4} = \text{span}\{x_2, x_4\} \) it follows that \( 1 = \alpha x_2 + \alpha x_4 \) with \( \alpha_2 \neq 0 \) and \( \alpha_4 \neq 0. \) Replacing the basis \( \{x_1, x_2, x_3, x_4\} \) by \( \{\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4\}, \) we may assume \( \alpha_2 = \alpha_4 = 1 \) without losing generality. Hence \( x_2 + x_4 = 1, \) \( x_2^2 = x_2 \) and \( x_4^2 = x_4. \) Since \( x_2^2 = x_2, \) we have \( M_1(1, 0, \eta) x_2 = x_2 M_1(1, 0, \eta) \) by the same argument as above. Hence the left and right multiplication by the element \( x_2 \) give two \( D_4 \)-module endomorphisms of \( M_1(1, 0, \eta). \) Since \( \text{End}_{D_4}(M_1(1, 0, \eta)) \cong k \) by Lemma 1.2(b) and \( x_2 x = x x_2 = x \) holds for \( x = x_2, \) we know that \( x_1 x_2 = x_2 x_1 = x_1. \) So \( x_2 \) is a local unit of
Lemma 2.23. Let \( \eta_1, \eta_2 \in k \) with \( \eta_1 \neq \eta_2 \) and \( A \) be a \( D_4 \)-module algebra isomorphic to \( M_1(1, 0, \eta_1) \oplus M_1(1, 0, \eta_2) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( k(\sqrt{\alpha_1})_{\eta_1} \times k(\sqrt{\alpha_2})_{\eta_2} \) for some \( \alpha_1, \alpha_2 \in k \).

Moreover, for any \( \alpha_1, \alpha_2, \eta_1, \eta_2, \alpha'_1, \alpha'_2, \eta'_1, \eta'_2 \in k \) with \( \eta_1 \neq \eta_2 \) and \( \eta'_1 \neq \eta'_2 \),

\[
k(\sqrt{\alpha_1})_{\eta_1} \times k(\sqrt{\alpha_2})_{\eta_2} \cong k(\sqrt{\alpha'_1})_{\eta'_1} \times k(\sqrt{\alpha'_2})_{\eta'_2}
\]

as \( D_4 \)-module algebras if and only if one of the following holds:

(i) \( \alpha_1 = \alpha'_1, \eta_1 = \eta'_1, \alpha_2 = \alpha'_2 \) and \( \eta_2 = \eta'_2 \);

(ii) \( \alpha_1 = \alpha'_2, \eta_1 = \eta'_2, \alpha_2 = \alpha'_1 \) and \( \eta_2 = \eta'_1 \).

Proof. Similar to the proof of Lemma 2.22. \( \square \)

Now we consider \( D_4 \)-module algebra structures on the module \( 2M_1(1, 0, \eta) \), \( \eta \in k \cup \{\infty\} \). We will first consider the case where \( \eta = \infty \). There are infinitely many \( D_4 \)-module algebra structures on \( 2M_1(1, 0, \infty) \).

Lemma 2.24. Let \( A \) be a \( D_4 \)-module algebra isomorphic to \( 2M_1(1, 0, \infty) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( A_\infty^\infty(\alpha, \beta, \gamma) \) for some \( \alpha, \beta, \gamma \in k \). The algebra \( A_\infty^\infty(\alpha, \beta, \gamma) \) has a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
\begin{align*}
    a \cdot x_1 &= a \cdot x_2 = a \cdot x_3 = 0, & d \cdot x_1 &= x_2, & d \cdot x_2 &= 0, & d \cdot x_3 &= 1, \\
    b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = x_2, & b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]

The multiplication of \( A_\infty^\infty(\alpha, \beta, \gamma) \) is given by

\[
x_1^2 = \alpha x_1 x_2 + \alpha \beta, \quad x_2^2 = \alpha, \quad x_3^2 = \gamma x_2 + \beta, \quad x_1 x_2 = x_2 x_1 = \alpha x_3, \\
x_1 x_3 = x_3 x_1 = \beta x_2 + \alpha \gamma, \quad x_2 x_3 = x_3 x_2 = x_1.
\]

Moreover, for any \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in k \), \( A_\infty^\infty(\alpha, \beta, \gamma) \cong A_\infty^\infty(\alpha', \beta', \gamma') \) if and only if \( \alpha = \delta^2 \alpha', \beta = \beta' \) and \( \gamma = \delta \gamma' \) for some \( \delta \in k^\times \).
Proof. Assume that $A = 2M_1(1, 0, \infty)$ as a $D_4$-module. Then one can choose a linear basis $\{x_1, x_2, x_3, 1\}$ for $A$ such that $\{x_1, x_2\}$ and $\{x_3, 1\}$ are canonical bases for the two copies of $M_1(1, 0, \infty)$, respectively, with the $D_4$-action given by (5) and $1$ being the identity of $A$. Observe that $A^{D_4} = kx_2 + k1$, and hence $x_2^3 \in A^{D_4}$. Replacing $x_1$ and $x_2$ with $x_1 + \delta x_3$ and $x_2 + \delta$ for some suitable $\delta \in k$, we may assume that $x_2^2 = \alpha$ for some $\alpha \in k$. Since $b$ fixes $x_1^2$, we have $x_1^2 = tx_2 + t'$ for some $t, t' \in k$. Then from the equation $d \cdot (x_1^2) = d \cdot (tx_2 + t')$ one gets $x_1 x_2 = x_2 x_1$. Since $b \cdot (x_1 x_2) = -x_1 x_2$, we have $x_1 x_2 = \theta x_1 + \theta' x_3$ for some $\theta, \theta' \in k$. Now by the equation $d \cdot (x_1 x_2) = d \cdot (\theta x_1 + \theta' x_3)$ one obtains $\theta = 0$ and $\theta' = \alpha$. Hence $x_1 x_2 = x_2 x_1 = \alpha x_3$. Similarly, we have $x_1 x_3 = \beta x_2 + \beta'$ and $x_3 x_1 = \beta_1 x_2 + \beta_1'$ for some $\beta, \beta', \beta_1, \beta_1' \in k$. So $x_2 x_3 = x_3 x_2 = 1$. We also have $x_3^2 = \gamma x_2 + \gamma'$ for some $\gamma, \gamma' \in k$. From the equation $x_3 (x_1 x_3) = (x_3 x_1) x_3$, one gets $\beta_1 = \beta$ and $\beta_1' = \beta'$. Hence we have $x_1 x_3 = x_3 x_1 = \beta x_2 + \beta'$. Then from the equations $x_1^2 x_3 = (x_1 x_3) x_3$ and $(x_1 x_2) x_3 = x_1 (x_2 x_3)$, one gets $t = \beta' = \alpha \gamma$ and $t' = \alpha \beta = \alpha \gamma'$. Finally, by the equation $x_1 x_3^2 = (x_1 x_3) x_3$, we obtain that $\gamma' = \beta$ and $\beta' = \alpha \gamma$. Consequently, $x_1^2 = \alpha \gamma x_2 + \alpha \beta$, $x_1 x_3 = x_3 x_1 = \beta x_2 + \alpha \gamma$ and $x_3^2 = \gamma x_2 + \beta$. So $A \cong A^\infty(\alpha, \beta, \gamma)$ as desired. The verification of the last statement is straightforward. \qed

Substituting $M_1(1, 0, \eta)$ for $M_1(1, 0, \infty)$ in Lemma 2.24, $\eta \in k$, we have all $D_4$-module algebra structures defined on $2M_1(1, 0, \eta)$.

Lemma 2.25. Let $\eta \in k$ and $A$ be a $D_4$-module algebra isomorphic to $2M_1(1, 0, \eta)$ as a $D_4$-module. Then $A$ is isomorphic to $A^\eta(\alpha, \beta, \gamma)$ for some $\alpha, \beta, \gamma \in k$. The algebra $A^\eta(\alpha, \beta, \gamma)$ has a linear basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action given by

\[
\begin{align*}
  a \cdot x_1 &= x_2, & a \cdot x_2 &= 0, & a \cdot x_3 &= 1, \\
  d \cdot x_1 &= -\eta x_2, & d \cdot x_2 &= 0, & d \cdot x_3 &= -\eta, \\
  b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = x_2, & b \cdot x_3 &= c \cdot x_3 = -x_3.
\end{align*}
\]

The multiplication of $A^\eta(\alpha, \beta, \gamma)$ is given by

\[
\begin{align*}
  x_1^2 &= \alpha \gamma x_2 + \alpha \beta, & x_2^2 &= \alpha, & x_3^2 &= \gamma x_2 + \beta, & x_1 x_2 &= x_2 x_1 = \alpha x_3, \\
  x_1 x_3 &= x_3 x_1 = \beta x_2 + \alpha \gamma, & x_2 x_3 &= x_3 x_2 = x_1.
\end{align*}
\]

Moreover, for any $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in k$, $A^\eta(\alpha, \beta, \gamma) \cong A^\eta(\alpha', \beta', \gamma')$ if and only if $\alpha = \delta^2 \alpha'$, $\beta = \beta'$ and $\gamma = \delta \gamma'$ for some $\delta \in k^\times$.

Proof. Similar to the proof of Lemma 2.24. \qed

Now we turn our attention to the $D_4$-module algebra structures on the modules $M_1(1, 1, \eta) \oplus M_1(1, 0, \eta)$, $\eta \in k \cup \{\infty\}$, which will provide many $H_4$-Azumaya algebras. We first consider the case of $\eta = \infty$. There are infinitely many $D_4$-module algebra structures on $M_1(1, 1, \infty) \oplus M_1(1, 0, \infty)$. 

Lemma 2.26. Let $A$ be a $D_4$-module algebra isomorphic to $M_1(1,1,\infty) \oplus M_1(1,0,\infty)$ as a $D_4$-module. Then $A$ is isomorphic to $\left(\frac{0}{0},\frac{1}{1}\right) \oplus \left(\frac{a \cdot b \cdot 0}{k}\right)$ for some $\alpha, \beta \in k$, where $\left(\frac{a \cdot b \cdot \gamma}{k}\right) = \left(\frac{a \cdot b \cdot \gamma}{k}\right)$ as an algebra, $\alpha, \beta, \gamma \in k$, and the $D_4$-module structure is given by

$$a \cdot u = 0, \quad a \cdot v = 0, \quad d \cdot u = 0, \quad d \cdot v = 1,$$

$$b \cdot u = c \cdot u = -u, \quad b \cdot v = c \cdot v = -v.$$

Moreover, $\left(\frac{0}{0},\frac{1}{1}\right) \neq \left(\frac{a \cdot b \cdot 0}{k}\right)$ for any $\alpha, \beta \in k$, and $\left(\frac{a \cdot b \cdot 0}{k}\right) \cong \left(\frac{\alpha', \beta', \gamma'}{k}\right)$ if and only if $\alpha = \delta^2 \alpha'$ and $\beta = \beta'$ for some $\delta \in k^\times$.

Proof. It is easy to check that $\left(\frac{a \cdot b \cdot \gamma}{k}\right)$ given in the lemma is indeed a $D_4$-module algebra for any $\alpha, \beta, \gamma \in k$. Now assume that $A = M_1(1,1,\infty) \oplus M_1(1,0,\infty)$ is a $D_4$-module algebra. Since $\text{soc}(M_1(1,r,\infty)) \cong V(1,r)$ for any $r \in \mathbb{Z}_2$, the invariant subalgebra $A^{D_4} = \text{soc}(M_1(1,0,\infty)) = k1$. We are allowed to choose a linear basis $\{x_1, x_2, x_3, 1\}$ for $A$ such that $\{x_1, x_2\}$ is a canonical basis for $M_1(1,1,\infty)$ and $\{x_3, 1\}$ is a canonical basis for $M_1(1,0,\infty)$ with the $D_4$-actions given by (5) and 1 being the identity of $A$. This means the following relations:

$$a \cdot x_1 = ax_2 = ax_3 = 0, \quad d \cdot x_1 = x_2, \quad d \cdot x_2 = 0, \quad d \cdot x_3 = 1,$$

$$b \cdot x_1 = c \cdot x_1 = x_1, \quad b \cdot x_2 = c \cdot x_2 = -x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3.$$

Since $b$ fixes $x_2^2$, we have $x_2^2 = \alpha_1 x_1 + \alpha$ for some $\alpha_1, \alpha \in k$. Then the action of the element $d$ on the equation $x_2^2 = \alpha_1 x_1 + \alpha$ renders $\alpha_1 = 0$, and consequently, $x_2^2 = \alpha$. Similarly, $x_3^2 = \beta$ for some $\beta \in k$. Since $b \cdot (x_2 x_3) = x_2 x_3, x_2 x_3 = \delta_1 x_1 + \delta$ for some $\delta_1, \delta \in k$. Now from the equation $d \cdot (x_2 x_3) = d \cdot (\delta_1 x_1 + \delta)$, one gets $\delta_1 = 1$, and so $x_2 x_3 = x_1 + \delta$. Similarly, $x_3 x_2 = -x_1 + \theta$ for some $\theta \in k$. Let $\gamma = \delta + \theta$. Replacing $x_1$ with $x_1 + \delta$, we may assume $x_2 x_3 = x_1$ and $x_3 x_2 = -x_1 + \gamma$. Therefore, we have $A \cong \left(\frac{a \cdot b \cdot \gamma}{k}\right)$ given by the mapping $x_1 \mapsto u v, x_2 \mapsto u, x_3 \mapsto v$.

In case $\gamma = 0$, we have $A \cong \left(\frac{a \cdot b \cdot 0}{k}\right)$. If $\alpha \neq 0$, let $x_3' = x_3 - \frac{1}{2} \alpha^{-1} \gamma x_2$. Then $x_3'^2 = \beta - \frac{1}{4} \alpha^{-1} \gamma^2$ and $x_2 x_2' + x_2' x_2 = 0$. It follows that $A \cong \left(\frac{a \cdot b \cdot 0}{k}\right)$ in this case, where $\beta' = \beta - \frac{1}{4} \alpha^{-1} \gamma^2$. If $\alpha = 0$ and $\gamma \neq 0$, let $x_3' = \gamma^{-1} x_1, x_2' = \gamma^{-1} x_2$ and $x_3' = x_3 - \beta' \gamma^{-1} x_2$. Then $x_3'^2 = 0, x_2'^2 = 0, x_3^2 x_2' = x'_1$ and $x_3' x_2 = -x'_1 + 1$. It follows that $A \cong \left(\frac{0}{0},\frac{1}{0}\right)$ in this case.

The verification of the last isomorphism is straightforward. \qed

Lemma 2.27. Let $\eta \in k$ and $A$ be a $D_4$-module algebra isomorphic to $M_1(1,1,\eta) \oplus M_1(1,0,\eta)$ as a $D_4$-module. Then $A$ is isomorphic to $\left(\frac{0}{0,\frac{1}{1}}\right) \oplus \left(\frac{a \cdot b \cdot 0}{k}\right)$ for some $\alpha, \beta \in k$, where $\left(\frac{a \cdot b \cdot \gamma}{k}\right) = \left(\frac{a \cdot b \cdot \gamma}{k}\right)$ as an algebra, $\alpha, \beta, \gamma \in k$, and the $D_4$-module structure is given by

$$a \cdot u = 0, \quad a \cdot v = 1, \quad d \cdot u = 0, \quad d \cdot v = -\eta,$$

$$b \cdot u = c \cdot u = -u, \quad b \cdot v = c \cdot v = -v.$$
Moreover, \( \left( \frac{0,0,1}{k} \right)_\eta \not\cong \left( \frac{\alpha,\beta,0}{k} \right)_\eta \) for any \( \alpha, \beta, \eta \in k \), and \( \left( \frac{\alpha,\beta,0}{k} \right)_\eta \cong \left( \frac{\alpha',\beta',0}{k} \right)_\eta \) if and only if \( \alpha = \delta^2 \alpha' \) and \( \beta = \beta' \) for some \( \delta \in k^\times \).

**Proof.** Similar to the proof of Lemma 2.26. \( \square \)

Now we consider \( D_4 \)-module algebra structures on the modules \( M_2(1,0,\eta) \), \( \eta \in k \cup \{\infty\} \). We will only give the proof for \( D_4 \)-module algebra structures defined on \( M_2(1,0,\infty) \). The proof for \( M_2(1,0,\eta) \), \( \eta \in k \), is similar.

**Lemma 2.28.** Let \( A \) be a \( D_4 \)-module algebra isomorphic to \( M_2(1,0,\infty) \) as a \( D_4 \)-module. Then \( A \) is isomorphic to \( A_{(2,2)}\infty(\alpha,\beta) \) for some \( \alpha, \beta \in k \). The algebra \( A_{(2,2)}\infty(\alpha,\beta) \) has a \( k \)-linear basis \( \{x_1, x_2, x_3, 1\} \) with the \( D_4 \)-action given by

\[
\begin{align*}
    a \cdot x_1 &= 0, & a \cdot x_2 &= x_3, & a \cdot x_3 &= 0, & d \cdot x_1 &= x_3, & d \cdot x_2 &= 1, & d \cdot x_3 &= 0, \\
    b \cdot x_1 &= c \cdot x_1 = -x_1, & b \cdot x_2 &= c \cdot x_2 = -x_2, & b \cdot x_3 &= c \cdot x_3 = x_3.
\end{align*}
\]

The multiplication of \( A_{(2,2)}(\alpha, \beta) \) is given by

\[
\begin{align*}
    x_1^2 &= 0, & x_2^2 &= \alpha x_3 + \beta, & x_3^2 &= 0, & x_1 x_2 &= x_2 x_1 = \beta x_3, \\
    x_1 x_3 &= x_3 x_1 = 0, & x_2 x_3 &= x_3 x_2 = x_1.
\end{align*}
\]

Moreover, for any \( \alpha, \beta, \alpha' \) and \( \beta' \in k \), \( A_{(2,2)}(\alpha, \beta) \cong A_{(2,2)}(\alpha', \beta') \) if and only if \( \alpha = \alpha' \) and \( \beta = \beta' \).

**Proof.** Assume that \( A = M_2(1,0,\infty) \) as a \( D_4 \)-module. Then \( A \) has a canonical basis \( \{x_1, x_2, x_3, x_4\} \) with the \( D_4 \)-action given by (9). Hence \( A' = kx_1 + kx_3 \) is the unique submodule of \( (1,1) \)-type in \( A \), and \( A' \cong M_1(1,0,\infty) \). We also have \( A^{D_4} = \text{soc}(A) = kx_3 + kx_4 \cong 2V(1,0) \). Thus we have that \( x_4 \in A^{D_4} \), but not in \( A' \). It is clear that \( kx_4 \cong V(1,0) \) as \( D_4 \)-modules, and \( x_4 A' \) is a submodule of \( A \). Since \( V(1,0) \otimes M_1(1,0,\infty) \cong M_1(1,0,\infty) \), \( x_4 A' \) is isomorphic to a quotient module of \( M_1(1,0,\infty) \).

If \( 1 \in A' \), then \( kx_4 \subseteq x_4 A' \). From an argument we used many times before, it follows that \( x_4 A' \cong M_1(1,0,\infty) \). Since \( A' \) is the unique submodule of \( (1,1) \)-type in \( A \), we have \( x_4 A' = A' \). This implies \( x_4 \in A' \), a contradiction. Consequently, \( 1 \notin A' \). From the equation \( 1 \in A^{D_4} = kx_3 + x_4 \) we know that \( 1 = \alpha x_3 + \beta x_4 \) for some \( \alpha, \beta \in k \) with \( \beta \neq 0 \). Replacing the basis \( \{x_1, x_2, x_3, x_4\} \) with \( \{\beta x_1, \alpha x_1 + \beta x_2, \beta x_3, \alpha x_3 + \beta x_4\} \), we may assume \( x_4 = 1 \), the identity of \( A \).

Since \( b \) fixes \( x_2^2 \), we have \( x_2^2 \in A^{D_4} \), \( 1 \leq i \leq 3 \). Hence \( a \cdot (x_i^2) = d \cdot (x_i^2) = 0 \). Thus, from the equations \( d \cdot (x_i^2) = 0 \) and \( a \cdot (x_i^2) = 0 \) one gets \( x_1 x_3 = x_3 x_1 \) and \( x_2 x_3 = x_3 x_2 \). Since \( b \cdot (x_2 x_3) = -x_2 x_3 \), we obtain that \( x_2 x_3 = t_1 x_1 + t_2 x_2 \) for some \( t_1, t_2 \in k \). Applying the action of \( d : d \cdot (x_i x_j) = d \cdot (t_1 x_1 + t_2 x_2) \), one gets \( t_1 = 1 \) and \( t_2 = 0 \). Hence we have \( x_2 x_3 = x_1 \). Then by \( a \cdot (x_2 x_3) = a \cdot x_1 \) one gets \( x_3^2 = 0 \), and so \( x_1^2 = x_2^2 x_3^2 = 0 \) and \( x_1 x_3 = (x_2 x_3) x_3 = x_2 x_3^2 = 0 \). Thus we have

\[
\begin{align*}
    x_1^2 &= x_3^2 = 0, & x_1 x_3 &= x_3 x_1 = 0, & x_2 x_3 &= x_3 x_2 = x_1.
\end{align*}
\]
Since $x_2^2 \in A^{D_4}$, $x_2^2 = \alpha x_3 + \beta$ for some $\alpha, \beta \in k$. It follows that $x_1 x_2 = x_2 x_1 = x_2^2 x_3 = \alpha x_3^2 + \beta x_3 = \beta x_3$. Therefore, $A \cong A^\infty_{(2,2)}(\alpha, \beta)$ as desired.

It is straightforward to check the last statement. □

**Lemma 2.29.** Let $\eta \in k$ and $A$ be a $D_4$-module algebra isomorphic to $M_2(1, 0, \eta)$ as a $D_4$-module. Then $A$ isomorphic to $A^\eta_{(2,2)}(\alpha, \beta)$, $\alpha, \beta \in k$. The algebra $A^\eta_{(2,2)}(\alpha, \beta)$ has a linear basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action on $A$ given by

$$a \cdot x_1 = x_3, \quad a \cdot x_2 = 1, \quad a \cdot x_3 = 0,$$

$$d \cdot x_1 = -\eta x_3, \quad d \cdot x_2 = -x_3 - \eta, \quad d \cdot x_3 = 0,$$

$$b \cdot x_1 = c \cdot x_1 = -x_1, \quad b \cdot x_2 = c \cdot x_2 = -x_2, \quad b \cdot x_3 = c \cdot x_3 = x_3,$$

and its multiplication is given by

$$x_1^2 = 0, \quad x_2^2 = \alpha x_3 + \beta, \quad x_3^2 = 0, \quad x_1 x_2 = x_2 x_1 = \beta x_3,$$

$$x_1 x_3 = x_3 x_1 = 0, \quad x_2 x_3 = x_3 x_2 = x_1.$$

Moreover, for any $\alpha, \beta, \alpha', \beta' \in k$, $A^\eta_{(2,2)}(\alpha, \beta) \cong A^\eta_{(2,2)}(\alpha', \beta')$ if and only if $\alpha = \alpha'$ and $\beta = \beta'$.

**Proof.** Similar to the proof of Lemma 2.28. □

Now we consider $D_4$-module algebra structures on the modules $M_2(1, 0, \delta, \eta)$, where $\delta, \eta \in k$ and $X^2 + \eta X + \delta$ is an irreducible quadratic polynomial in $k[X]$.

**Lemma 2.30.** Let $A$ be a $D_4$-module algebra isomorphic to $M_2(1, 0, \delta, \eta)$ as a $D_4$-module for some $\delta, \eta \in k$ such that $X^2 + \eta X + \delta$ is irreducible over $k$. Then $A$ is isomorphic to $A_{(2,2)}(\alpha, \beta)$ for some $\alpha, \beta \in k$. The algebra $A_{(2,2)}(\alpha, \beta)$ has a $k$-linear basis $\{x_1, x_2, x_4, 1\}$ with the $D_4$-action given by

$$a \cdot x_1 = x_3, \quad a \cdot x_2 = 1, \quad a \cdot x_3 = 0,$$

$$d \cdot x_1 = 1, \quad d \cdot x_2 = -\delta x_3 + \eta, \quad d \cdot x_3 = 0,$$

$$b \cdot x_1 = c \cdot x_1 = -x_1, \quad b \cdot x_2 = c \cdot x_2 = -x_2, \quad b \cdot x_3 = c \cdot x_3 = x_3,$$

and its multiplication is given by

$$x_1^2 = (\alpha \delta^{-2} \eta^2 - \alpha \delta^{-1} + \beta \delta^{-1} \eta)x_3 - (\alpha \delta^{-2} \eta + \beta \delta^{-1}), \quad x_2^2 = \alpha x_3 + \beta,$$

$$x_3^2 = \delta^{-1} \eta x_3 - \delta^{-1}, \quad x_1 x_2 = x_2 x_1 = (\alpha \delta^{-1} \eta + \beta)x_3 - \alpha \delta^{-1},$$

$$x_1 x_3 = x_3 x_1 = \delta^{-1} \eta x_1 - \delta^{-1} x_2, \quad x_2 x_3 = x_3 x_2 = x_1.$$
Moreover, if $\alpha, \beta, \delta, \eta, \alpha', \beta', \delta', \eta' \in k$ such that $X^2 + \eta X + \delta$ and $X^2 + \eta' X + \delta'$ are irreducible in $k[X]$, then $A(2,2)(\alpha, \beta, \delta, \eta) \cong A(2,2)(\alpha', \beta', \delta', \eta')$ if and only if $\alpha = \alpha'$, $\beta = \beta'$, $\delta = \delta'$ and $\eta = \eta'$.

**Proof.** We may assume that $A = M_2(1, 0, \delta, \eta)$ as a $D_4$-module. Since $A$ has a submodule $k1$ isomorphic to $V(1, 0)$, it follows from Lemma 1.1 and its proof that there is a canonical basis $\{x_1, x_2, x_3, 1\}$ for $A = M_2(1, 0, \delta, \eta)$ with the $D_4$-action given by (11) and 1 being the identity of $A$.

Since $b$ fixes $x_2^2, x_2^3 \in A_0$ and so $x_2^2 = \alpha x_3 + \beta$ for some $\alpha, \beta \in k$. Hence we have $d \cdot (x_2^2) = d \cdot (\alpha x_3 + \beta) = 0$. This implies $x_2 x_3 = x_3 x_2$. Similarly, we have $a \cdot (x_1 x_2) = 0$, which implies $x_3 x_2 = x_1$. Then from the equation $d \cdot (x_3 x_2) = d \cdot x_1$ one gets $x_3^3 = \delta^{-1} \eta x_3 - \delta^{-1}$. Therefore, $x_2 x_3 = x_3 x_2 = x_1$, and so $x_1 x_2 = x_2 x_1 = x_2^2 x_3 = \alpha x_3 + \beta x_3 = (\alpha \delta^{-1} \eta + \beta) x_3 - \alpha \delta^{-1}$ and $x_1 x_3 = x_3 x_1 = x_2 x_3^2 = \delta^{-1} \eta x_2 x_3 - \delta^{-1} x_2 = \delta^{-1} \eta x_3 - \delta^{-1} x_2$. Next, we may assume $x_1^2 = t x_3 + t', t, t' \in k$ since $b$ fixes $x_1^2$. Then by $x_2^2 = x_1(x_1 x_2), we get $t = \alpha \delta^{-2} \eta - \alpha \delta^{-1} + \beta \delta^{-1}$ and $t' = -\alpha \delta^{-2} \eta - \beta \delta^{-1}$. Hence we have $x_1^2 = (\alpha \delta^{-2} \eta^2 - \alpha \delta^{-1} + \beta \delta^{-1}) x_3 - (\alpha \delta^{-2} \eta + \beta \delta^{-1})$. Thus we have proved that $A$ has the claimed structure, i.e., $A \cong A(2,2)(\alpha, \beta, \delta, \eta)$.

The rest of the proof is straightforward. □

Finally, we consider $D_4$-module algebra structures on the module $P(1, 0)$. There are infinitely many $D_4$-module algebra structures on $P(1, 0)$, of which many are $H_4$-Azumaya algebras.

**Lemma 2.31.** Let $A$ be a $D_4$-module algebra isomorphic to $P(1, 0)$ as a $D_4$-module. Then there exist $\alpha, \beta, \gamma \in k$ such that $A$ is isomorphic to $\left(\frac{\alpha, \beta, \gamma}{k}\right)_P$, where $\left(\frac{\alpha, \beta, \gamma}{k}\right)_P = \left(\frac{\alpha, \beta, \gamma}{k}\right)$ as an algebra, and the $D_4$-module action is given by

\[
\begin{align*}
    a \cdot u & = 0, \quad a \cdot v = 1, \quad d \cdot u = -1, \quad d \cdot v = 0, \\
    b \cdot u & = c \cdot u = -u, \quad b \cdot v = c \cdot v = -v.
\end{align*}
\]

Moreover, for any $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in k, \left(\frac{\alpha, \beta, \gamma}{k}\right)_P \cong \left(\frac{\alpha', \beta', \gamma'}{k}\right)_P$ if and only if $\alpha = \alpha'$, $\beta = \beta'$ and $\gamma = \gamma'$.

**Proof.** It is easy to check that the $D_4$-module algebra $\left(\frac{\alpha, \beta, \gamma}{k}\right)_P$ is well defined for any $\alpha, \beta, \gamma \in k$. Assume that $A = P(1, 0)$ as a $D_4$-module. Since $\text{soc}(P(1, 0)) \cong V(1, 0)$, we know that $A^{D_4} = \text{soc}(A) = k1$. Hence $A = P(1, 0)$ has a canonical basis $\{x_1, x_2, x_3, 1\}$ with the $D_4$-action given by (4) and 1 being the identity of $A$. Hence we have,

\[
\begin{align*}
    a \cdot x_1 & = x_2, \quad a \cdot x_2 = 0, \quad a \cdot x_3 = 1, \\
    d \cdot x_1 & = x_3, \quad d \cdot x_2 = -1, \quad d \cdot x_3 = 0, \\
    b \cdot x_1 & = c \cdot x_1 = x_1, \quad b \cdot x_2 = c \cdot x_2 = -x_2, \quad b \cdot x_3 = c \cdot x_3 = -x_3.
\end{align*}
\]
Since $b \cdot (x_i^2) = x_i^2$ for $1 \leq i \leq 3$, the elements $x_2^2$ and $x_3^2$ are in span$\{x_1, 1\}$. There exist $\alpha, \alpha', \beta$ and $\beta'$ in $k$ such that $x_2^2 = \alpha + \alpha'x_1$ and $x_3^2 = \beta + \beta'x_1$. From the equations $a \cdot (x_2^2) = a \cdot (\alpha + \alpha'x_1)$ and $d \cdot (x_2^2) = d \cdot (\beta + \beta'x_1)$ it follows that $\alpha' = \beta' = 0$. Hence, we have $x_2^2 = \alpha$ and $x_3^2 = \beta$. Similarly, there exist $\delta, \delta', \theta$ and $\theta'$ in $k$ such that $x_2x_3 = \delta + \delta'x_1$ and $x_3x_2 = \theta + \theta'x_1$. Then from the equations $a \cdot (x_2x_3) = a \cdot (\delta + \delta'x_1)$ and $a \cdot (x_3x_2) = a \cdot (\theta + \theta'x_1)$ one gets $\delta' = 1$ and $\theta' = -1$. Hence $x_2x_3 = x_1 + \delta$ and $x_3x_2 = -x_1 + \theta$. Let $\gamma = \delta + \theta$. Then $x_2x_3 + x_3x_2 = \gamma$. Thus one can see that the correspondence $x_2 \mapsto u$, $x_3 \mapsto v$ induces a $D_4$-module algebra isomorphism from $A$ to $(\frac{a, \beta, \gamma}{k})_P$.

The rest of the proof is straightforward. \qed

3. Four-dimensional $D_4$-Azumaya algebras

In this section, we investigate the $H_4$-Azumaya algebra structures on 4-dimensional Yetter–Drinfeld $H_4$-modules and classify all 4-dimensional $H_4$-Azumaya algebras. The method we will employ in this section is to test the 29 types of 4-dimensional $D_4$-module algebras classified in Section 2. Recall that a Yetter–Drinfeld $H_4$-module algebra is $H_4$-Azumaya if and only if it is $D_4$-Azumaya with respect to the canonical quasitriangular structure

$$\mathcal{R} = \frac{1}{2}(1 \otimes 1 + b \otimes 1 + 1 \otimes c - b \otimes c + a \otimes d + ab \otimes d + a \otimes cd - ab \otimes cd).$$

Thus it is sufficient to consider whether the 4-dimensional $D_4$-module algebras described in Section 2 are $D_4$-Azumaya (with respect to $\mathcal{R}$). In testing a non-$D_4$-Azumaya algebra, we will often use the property that any $D_4$-Azumaya algebra is $D_4$-central and $D_4$-simple. We will first exclude those non-$H_4$-Azumaya algebras $A$.

Lemma 3.1. Let $A$ be a 4-dimensional $D_4$-module algebra whose $D_4$-module structures are not of type $(A), (O)$ or $M_1(1, 1, \eta) \oplus M_1(1, 0, \eta)$ of type $(L), \eta \in k \cup \{\infty\}$. Then $A$ is not an $H_4$-Azumaya algebra.

Proof. Our proof will be based on case by case. Assume that $A$ is isomorphic to $V(2, r) \oplus 2V(1, 0)$ as a $D_4$-module for some $r \in \mathbb{Z}_2$. By Lemma 2.4, $A$ has a non-trivial $D_4$-stable ideal $I = \text{span}\{x_1, x_2\}$, and $I \cong V(2, r)$ as $D_4$-modules. So $A$ is not $D_4$-simple and hence not $D_4$-Azumaya.

If $A$ is isomorphic to $V(2, r) \oplus V(1, 1) \oplus V(1, 0)$ as a $D_4$-module for some $r \in \mathbb{Z}_2$, then $A$ has a non-trivial $D_4$-stable ideal $I = \text{span}\{x_1, x_2, x_3\}$ by Lemma 2.5, and $I \cong V(2, r) \oplus V(1, 1)$ as $D_4$-modules. So $A$ is not $D_4$-Azumaya in this case.

Suppose that $A$ is isomorphic to $\Omega V(1, 0) \oplus V(1, 0)$ (or $\Omega V(1, 1) \oplus V(1, 0)$) as a $D_4$-module. By Lemma 2.6 (or Lemma 2.7), $A$ has a $D_4$-stable ideal $kx_3$, which is isomorphic to $V(1, 0)$ (or $V(1, 1)$) as a $D_4$-module. Hence $A$ is not $D_4$-Azumaya in both cases.

If $A$ is isomorphic to $\Omega^{-1} V(1, 0) \oplus V(1, 0)$ (or $\Omega^{-1} V(1, 1) \oplus V(1, 0)$) as a $D_4$-module, then $A$ has a $D_4$-stable ideal $I = \text{span}\{x_2, x_3\}$ by Lemma 2.8 (or Lemma 2.9), and $I \cong 2V(1, 1)$ (or $I \cong 2V(1, 0)$). In both cases, $A$ is not $D_4$-Azumaya.
Assume that $A$ is isomorphic to $M_1(1, 0, \infty) \oplus 2V(1, 0)$. By Lemma 2.10, $A$ is isomorphic to the $D_4$-module algebra given by (2.10.1) or (2.10.2) or (2.10.3). If $A$ is the $D_4$-module algebra given by (2.10.1), then $kx_2$ is a $D_4$-stable ideal of $A$. If $A$ is the $D_4$-module algebra given by (2.10.2) or (2.10.3), then span{$x_1, x_2$} is a $D_4$-stable ideal of $A$. Thus, in any case $A$ is not $D_4$-Azumaya.

Similarly, if $A$ is isomorphic to $M_1(1, 0, \eta) \oplus 2V(1, 0)$ as a $D_4$-module for some $\eta \in \mathbb{k}$, then by Lemma 2.11, $A$ has a non-trivial $D_4$-stable ideal and $A$ is not $D_4$-Azumaya.

Suppose that $A$ is isomorphic to $M_1(1, 1, \eta) \oplus 2V(1, 0)$ as a $D_4$-module, where $\eta = \infty$ or $\eta \in \mathbb{k}$. Then by Lemmas 2.12 and 2.13, $A$ has a $D_4$-stable ideal $kx_2$. Hence $A$ is not $D_4$-Azumaya.

If $A$ is isomorphic to $M_1(1, 0, \eta) \oplus V(1, 1) \oplus V(1, 0)$ as a $D_4$-module, where $\eta = \infty$ or $\eta \in \mathbb{k}$, then by Lemmas 2.14 and 2.15, $A$ is isomorphic to $A_1^{(0, \eta)}(\alpha, \beta)$, or to $A_2^{(0, \eta)}(\alpha, \delta)$, or to $A_3^{(0, \eta)}(\alpha, \beta, \delta, \gamma)$, where $\alpha, \beta, \delta, \gamma \in \mathbb{k}$. However, $A_1^{(0, \eta)}(\alpha, \beta)$ has a $D_4$-stable ideal spanned by $x_1$ and $x_2$; the algebra $A_2^{(0, \eta)}(\alpha, \delta)$ has a $D_4$-stable ideal spanned by $x_2$ and $x_3$; the algebra $A_3^{(0, \eta)}(\alpha, \beta, \delta, \gamma)$ has a $D_4$-stable ideal $kx_2$. Therefore, in any case, $A$ is not $D_4$-Azumaya.

Similarly, if $A$ is isomorphic to $M_1(1, 1, \eta) \oplus V(1, 1) \oplus V(1, 0)$ as a $D_4$-module, where $\eta = \infty$ or $\eta \in \mathbb{k}$, then by Lemmas 2.16 and 2.17, $A$ has a non-trivial $D_4$-stable ideal. So $A$ is not $D_4$-Azumaya.

Now let $r \in \mathbb{Z}_2$, $\eta \in \mathbb{k} \cup \{\infty\}$, and assume that $A$ is isomorphic to $V(2, r) \oplus M_1(1, 0, \eta)$ as a $D_3$-module. We claim that $A \neq \bar{A}$ and $\text{End}(A)$ are not isomorphic as $D_4$-modules. It is not hard (but tedious) to check the following $D_4$-module isomorphisms:

- $V(2, r)^* \cong V(2, r + 1), r \in \mathbb{Z}_2$.
- $M_1(1, r, \eta)^* \cong M_1(1, r + 1, \eta), \eta \in \mathbb{k} \cup \{\infty\}$.
- $V(2, r) \otimes V(2, s) \cong P(1, r + s + 1), r, s \in \mathbb{Z}_2$.
- $M_1(1, r, \eta) \otimes M_1(1, s, \eta) \cong M_1(1, 0, \eta) \oplus M_1(1, 1, \eta), r, s \in \mathbb{Z}_2, \eta \in \mathbb{k} \cup \{\infty\}$.
- $V(2, r) \otimes M_1(1, s, \eta) \cong M_1(1, s, \eta) \otimes V(2, r) \cong V(2, 0) \oplus V(2, 1), r, s \in \mathbb{Z}_2, \eta \in \mathbb{k} \cup \{\infty\}$.

Then we have

$$A \otimes A \cong \left(V(2, r) \oplus M_1(1, 0, \eta)\right) \otimes \left(V(2, r) \oplus M_1(1, 0, \eta)\right)$$

$$\cong V(2, r) \otimes V(2, r) \otimes 2V(2, r) \otimes M_1(1, 0, \eta) \otimes M_1(1, 0, \eta)$$

$$\cong P(1, 1) \oplus 2V(2, 0) \oplus 2V(2, 1) \oplus M_1(1, 0, \eta) \oplus M_1(1, 1, \eta)$$

and

$$A \otimes A^* \cong \left(V(2, r) \oplus M_1(1, 0, \eta)\right) \otimes \left(V(2, r + 1) \oplus M_1(1, 1, \eta)\right)$$

$$\cong V(2, r) \otimes V(2, r + 1) \oplus V(2, r) \otimes M_1(1, 1, \eta)$$

$$\oplus M_1(1, 0, \eta) \otimes V(2, r + 1) \oplus M_1(1, 0, \eta) \oplus M_1(1, 1, \eta)$$

$$\cong P(1, 0) \oplus 2V(2, 0) \oplus 2V(2, 1) \oplus M_1(1, 0, \eta) \oplus M_1(1, 1, \eta).$$
Thus, $A \# \overline{A} = A \otimes A \not\cong A \otimes A^* = \text{End}(A)$ as $D_4$-modules. It follows that $A$ is not $D_4$-Azumaya in this case.

If $A$ is isomorphic to $2M_1(1,0,\eta)$ as a $D_4$-module, where $\eta = \infty$ or $\eta \in k$, then by Lemmas 2.24 and 2.25, $A$ is a commutative algebra. Therefore, $A^{D_4} \subseteq A^A \cap A^A$. Since $\dim A^{D_4} = 2$, the $D_4$-center of $A$ is not trivial. So $A$ is not $D_4$-Azumaya.

Similarly, if $A$ is of $(2,2)$-type as a $D_4$-module, then by Lemmas 2.28–2.30, we know that $A$ is a commutative algebra. Since the invariant subalgebra $A^{D_4}$ is of dimension 2 and is contained in the $D_4$-center of $A$, $A$ is not $D_4$-Azumaya.

Finally, assume that $A$ is isomorphic to $M_1(1,0,\eta) \oplus M_1(1,0,\infty)$ or to $M_1(1,0,\eta_1) \oplus M_1(1,0,\eta_2)$ as a $D_4$-module, where $\eta, \eta_1, \eta_2 \in k$ with $\eta \neq \eta_2$. By Lemmas 2.22 and 2.23, $A$ is a direct sum of two non-trivial $D_4$-stable ideals. So $A$ is not $D_4$-Azumaya. \hfill $\square$

Following Lemma 3.1, we have three types of $D_4$-modules, types (A), (L) and (O), on which possible Azumaya $D_4$-module algebra structures may exist. Let us first consider the 4-dimensional $D_4$-module algebras of type (A). In this case, $A$ is $D_4$-Azumaya if and only if $A$ is $\mathbb{Z}_2$-graded Azumaya. The $\mathbb{Z}_2$-grading is given by the action of $b$ or $c$, i.e.,

$$A_0 = \{x \in A \mid b \cdot x = x\} \quad \text{and} \quad A_1 = \{x \in A \mid b \cdot x = -x\}.$$ 

The following result might be well known. But we give a direct proof for reader’s convenience.

**Lemma 3.2.** Let $A$ be a 4-dimensional $\mathbb{Z}_2$-graded Azumaya algebra. Then $A$ is a quaternion algebra $(\frac{\alpha\beta}{k})$ for some $\alpha, \beta \in k^\times$ with

(a) the trivial grading, or
(b) the canonical grading: $A_0 = k1 \oplus kuv$ and $A_1 = ku \oplus kv$, where $u, v$ are two generators. We denote it by $(\frac{\alpha\beta}{k})_{\mathbb{Z}_2}$.

**Proof.** Let $Z(A) = k \oplus Z_1$ be the center of $A$, $Z_1 \subseteq A_1$. If $Z_1 = 0$, then $A$ is said of even type. Otherwise, it is of odd type. Recall from [18] or [12, Chapter IV] that $A \cong A_0 \otimes k(\sqrt{\alpha})$ for some $\alpha \in k^\times$ and $A_0$ is central simple if $A$ is of odd type. Since $\dim k(\sqrt{\alpha}) = 2$, we have $\dim A_0 = 2$. But there is no central simple algebras of dimension 2. So $A$ cannot be of odd type.

Suppose that $A$ is of even type. In this case $A$ is a central simple algebra. Thus, $A \cong M_n(D)$ for some division algebra. This implies that $A \cong M_2(k)$ or $A = D$ is a 4-dimensional central division algebra. In any case, $A$ is a quaternion algebra, say, $A = (\frac{\alpha\beta}{k})$ for some $\alpha, \beta \in k^\times$.

By the Skolem–Noether theorem, the action of $b$ is given by an inner automorphism. There exists a unit $w \in A$ such that $b \cdot x = wxw^{-1}$ and $w^2 = \lambda \in k^\times$. If $w \in k = Z(A)$, then $A$ has the trivial grading. Now assume that $w \notin Z(A)$. Then $\dim A_0 \geq 2$ and $A_1 \neq 0$. We claim that $A_0 = k \oplus kw$ and $A_1$ is of dimension 2. If, otherwise, $\dim A_1 = 1$, then $A_1^2 \subseteq A_0$ and $\dim A_1^2 \leq 1$. It follows that $I = A_1^2 \oplus A_1$ is a proper non-trivial $D_4$-stable ideal of $A$ and hence $A$ is not (graded) Azumaya.
Next we show that \( A = k(\sqrt{\alpha}) \# k(\sqrt{\beta}) \) is a graded product of two graded quadratic extension. Let \( A_1 = ku \oplus kv \). By definition, \( ux = -xw \) for any \( 0 \neq x \in A_1 \). Since \( x^2 \in A_0 \), so \( x^2 = \tau + \theta w \) for some \( \tau, \theta \in k \). However, \( x^3 = xx^2 = \tau x + \theta wx \) and \( x^3 = xx^2 = \tau x + \theta x w \). This implies that \( \theta wx = \theta x w \). Thus, \( \theta = 0 \) and we have \( x^2 = \tau \).

Now let \( x = u, v \) and \( u + v \) respectively, we obtain \( \alpha, \beta, \gamma \in k \) such that \( u^2 = \alpha, v^2 = \beta \) and \( uv + vu = (u + v)^2 - u^2 - v^2 = \gamma \).

Since \( A \) is Azumaya, we have \( A = A_1^2 + A_1 \). So \( A \) is generated by \( u \) and \( v \). Thus \( A = (\alpha, \beta, \gamma) \) as a graded algebra, where \( \gamma^2 - 4\alpha\beta \neq 0 \). We may choose the element \( u \) such that \( u^2 = \alpha \neq 0 \) since at least one of \( \alpha, \beta \) and \( \gamma \) is not zero. Let \( v' = v - \frac{1}{2}\alpha^{-1}\gamma u \). Then \( v' \in A_1 \) and \( v'^2 = \beta - \frac{1}{2}\alpha^{-1}\gamma^2 = \beta' \neq 0 \). Moreover,

\[
uv' + v'u = (uv + vu) - \alpha^{-1}\gamma u^2 = \gamma - \alpha^{-1}\gamma\alpha = 0.
\]

It follows that \( A = k(\sqrt{\alpha}) \# k(\sqrt{\beta'}) \) for \( \alpha, \beta' \in k^\times \). \( \square \)

Now let \( A \) be the 4-dimensional \( D_4 \)-module algebras of type (L) defined on \( M_1(1, 1, \eta) \oplus M_1(1, 0, \eta) \), \( \eta \in k \). There are only generalized quaternion algebras defined on \( M_1(1, 1, \eta) \oplus M_1(1, 0, \eta) \), cf. Lemma 2.27. The \( D_4 \)-module structure on \( \frac{\alpha, \beta, \gamma}{k} \) is given by the standard \( H_4 \)-Galois action on \( \frac{\alpha, \beta, \gamma}{k} \):

\[
b \cdot u = -u, \quad b \cdot v = -v, \quad a \cdot u = 0, \quad a \cdot v = 1
\]

and the \( H_4 \)-comodule structure is induced by the (quasi-)triangular structure

\[
R_{-\eta} = \frac{1}{2}(1 \otimes 1 + 1 \otimes b + b \otimes 1 - b \otimes b) - \frac{\eta}{2}(a \otimes a + a \otimes ba + ba \otimes ba - ba \otimes a)
\]

via (1) for some \( \eta \in k \). The following lemma generalizes [15, Proposition 5] and [16, Proposition 3.1].

**Lemma 3.3.** For \( \alpha, \beta, \gamma, \eta \in k \), the generalized quaternion algebra \( \frac{\alpha, \beta, \gamma}{k} \) is \( D_4 \)-Azumaya if and only if \( \gamma^2 - 4\alpha\beta - 2\alpha\eta \neq 0 \).

**Proof.** The proof is essentially similar to the one of [16, Proposition 3.1]. Let \( A = \frac{\alpha, \beta, \gamma}{k} \). Then one can easily check that \( \overline{A} \cong \frac{-\alpha, -\beta, -\eta, -\gamma}{k} \). Moreover, \( A \# \overline{A} \) is generated as an algebra by \( u \# \overline{v}, v \# \overline{u}, 1 \# \overline{u} \) and \( 1 \# v \). Identifying \( u, v, \overline{u} \) and \( \overline{v} \) with \( u \# \overline{v}, v \# \overline{u}, 1 \# \overline{u} \) and \( 1 \# v \) respectively, we get the relations:

\[
\begin{align*}
u^2 &= \alpha, \\
u'v' &= \beta, \\
\overline{u}^2 &= -\alpha, \\
\overline{v}^2 &= -\beta - \eta,
\end{align*}
\]

\[
\begin{align*}
uv + vu &= \gamma, \\
u\overline{u} + \overline{u}u &= 0, \\
u\overline{v} + \overline{v}u &= 0, \\
v\overline{v} + \overline{v}v &= -\eta, \\
\overline{u}\overline{v} + \overline{v}\overline{u} &= -\gamma.
\end{align*}
\]
These relations define a bilinear form on the four-dimensional space \( V = \text{span}\{u, v, \bar{u}, \bar{v}\} \). It follows that \( A \# \bar{A} \) is isomorphic to the Clifford algebra \( C(V) \) whose symmetric matrix is

\[
M = \begin{pmatrix}
2\alpha & \gamma & 0 & 0 \\
\gamma & 2\beta & 0 & -\eta \\
0 & 0 & -2\alpha & -\gamma \\
-\eta & -\gamma & -2(\beta + \eta)
\end{pmatrix}.
\]

It is clear that the Clifford algebra \( C(V) \) is Azumaya if and only if the determinant \( |M| = (\gamma^2 - 4\alpha\beta - 2\alpha\eta)^2 \neq 0 \) if and only if \( \gamma^2 - 4\alpha\beta - 2\alpha\eta \neq 0 \). Similarly, \( \bar{A} \# A \) is an Azumaya algebra if and only if \( \gamma^2 - 4\alpha\beta - 2\alpha\eta \neq 0 \). It follows that \( A \) is \( D_4 \)-Azumaya if and only if \( \gamma^2 - 4\alpha\beta - 2\alpha\eta \neq 0 \).

Next we consider the 4-dimensional \( D_4 \)-module algebras \( A \) of type (L) defined on the \( D_4 \)-module \( M_1(1, 1, \infty) \oplus M_1(1, 0, \infty) \) (cf. Lemma 2.26). There are only generalized quaternion algebras defined on \( M_1(1, 1, \infty) \oplus M_1(1, 0, \infty) \). If we let \( H_4 = k\langle a, b \rangle \) and \( H_4^* = k\langle c, d \rangle \), then the corresponding Yetter–Drinfeld \( H_4 \)-module algebra \( A \) has a standard \( H_4 \)-coaction

\[
\rho(u) = u \otimes b, \quad \rho(v) = v \otimes b + 1 \otimes a
\]

and the \( H_4 \)-module structure is induced by the cotriangular structure \( R_0 \in (H_4 \otimes H_4)^* \):

| \( R_0 \) | 1 \( a \) \( b \) \( ab \) |
|-----|-----|-----|-----|
| 1   | 1   | 1   | 0   |
| \( a \) | 0   | 0   | 0   | 0   |
| \( b \) | 1   | 0   | -1  | 0   |
| \( ab \) | 0   | 0   | 0   | 0   |

That is, \( h \cdot x = \sum x_{(0)} R_0(x_{(1)} \otimes h) \) for \( h \in H_4 \) and \( x \in A \). It is easy to see that the triangular structure on \( H_4^* \) dual to \( R_0 \) on \( H_4 \) is given by

\[
R_0 = \frac{1}{2} (1 \otimes 1 + 1 \otimes c + c \otimes 1 - c \otimes c).
\]

Note that a Yetter–Drinfeld \( H_4 \)-module algebra \( A \) with \( H_4 \)-action induced by \( H_4 \)-coaction via the cotriangular structure \( R_0 \) is a Yetter–Drinfeld \( H_4^* \)-module algebra with \( H_4^* \)-coaction induced by \( H_4^* \)-action via the triangular structure \( R_0 \). Moreover, \( A \) is \( H_4 \)-Azumaya with respect to \( R_0 \) if and only if \( A \) is \( H_4^* \)-Azumaya with respect to \( R_0 \). Since \( H_4 \) is a self-dual Hopf algebra, Lemma 3.3 or [15, Proposition 5] applies in the following.

**Lemma 3.4.** The generalized quaternion algebra \( (\alpha, \beta, \gamma) \) is \( D_4 \)-Azumaya if and only if \( \gamma^2 - 4\alpha\beta \neq 0 \).
The last class of 4-dimensional Azumaya $D_4$-module algebras are generalized quaternion algebras defined on the $D_4$-module $P(1, 0)$ of type (O). We give the necessary and sufficient condition for \( (\alpha, \beta, \gamma)_P \) to be $D_4$-Azumaya.

**Lemma 3.5.** Let $\alpha, \beta, \gamma \in k$. Then \( (\alpha, \beta, \gamma)_P \) is $D_4$-Azumaya if and only if $\gamma(\gamma + 1) - 4\alpha\beta \neq 0$.

**Proof.** Let $A = (\alpha, \beta, \gamma)_P$. Then one can check that $\overline{A} \cong (\frac{-\alpha, -\beta, -\gamma(\gamma + 1)}{k})_P$. Moreover, $A \# \overline{A}$ is generated as an algebra by $u \# \overline{1}, v \# \overline{1}, 1 \# \overline{u}$ and $1 \# \overline{v}$. Identifying $u, v, \overline{u}$ and $\overline{v}$ with $u \# 1, v \# 1, 1 \# u$ and $1 \# v$ respectively, we get the relations:

\[
\begin{align*}
    u^2 &= \alpha, & v^2 &= \beta, & \overline{u}^2 &= -\alpha, & \overline{v}^2 &= -\beta, \\
    uv + vu &= \gamma, & u\overline{u} + \overline{u}u &= 0, \\
    u\overline{v} + \overline{v}u &= -1, & v\overline{u} + \overline{u}v &= 0, \\
    v\overline{v} + \overline{v}v &= 0, & \overline{u}\overline{v} + \overline{v}\overline{u} &= -(\gamma + 1).
\end{align*}
\]

An similar argument to the proof of Lemma 3.3 shows that $A \# \overline{A}$ is an Azumaya algebra if and only if

\[
\det \begin{pmatrix}
    2\alpha & \gamma & 0 & -1 \\
    \gamma & 2\beta & 0 & 0 \\
    0 & 0 & -2\alpha & -\gamma(\gamma + 1) \\
    -1 & 0 & -(\gamma + 1) & -2\beta
\end{pmatrix} = (\gamma(\gamma + 1) - 4\alpha\beta)^2 \neq 0,
\]

which is equivalent to $\gamma(\gamma + 1) - 4\alpha\beta \neq 0$. Similarly, $\overline{A} \# A$ is an Azumaya algebra if and only if $\gamma(\gamma + 1) - 4\alpha\beta \neq 0$. It follows that $A$ is $D_4$-Azumaya if and only if $\gamma(\gamma + 1) - 4\alpha\beta \neq 0$. \qed

Now we can summarize this section by listing all non-isomorphic 4-dimensional $D_4$-Azumaya algebras. Their underlying algebras are generalized quaternion algebras.

**Theorem 3.6.** Let $A$ be a 4-dimensional $D_4$-module algebra. Then $A$ is $D_4$-Azumaya if and only if $A$ is isomorphic to one of the following:

(a) a quaternion algebra with the trivial $D_4$-action,

(b) \( (\alpha, \beta)_Z \), $\alpha, \beta \in k^\times$,

(c) \( (\alpha, 0, 0)_\infty \),

(d) \( (\alpha, \beta, 0)_\infty \), $\alpha, \beta \in k^\times$,

(e) \( (0, 0, 1)_\eta \), $\eta \in k$,

(f) \( (\alpha, \beta, 0)_\eta \), where $\alpha \in k^\times$, $\beta, \eta \in k$ with $2\beta + \eta \neq 0$,

(g) \( (\alpha, \beta, \gamma)_P \), where $\alpha, \beta, \gamma \in k$ with $\gamma(\gamma + 1) - 4\alpha\beta \neq 0$. 


The Azumaya algebras in class (a) represent torsion two elements in \( \text{Br}(k) \) and form the subgroup \( \text{Br}_2(k) \). A \( D_4 \)-Azumaya algebra in class (b) is the graded product of two graded quadratic extensions. So they represent torsion 2 elements in the subgroup \( \text{BW}(k) \), the Brauer–Wall group. The \( D_4 \)-Azumaya algebras in (c) and (e) are elementary Azumaya algebras representing the identity of \( \text{BQ}(k, H_4) \). The representatives of \( D_4 \)-Azumaya algebras in class (d) form a subgroup of \( \text{BC}(k, H_4, R_0) \) (\( \cong \text{BM}(k, H_4, R_0) \)) isomorphic to the semi-direct product \( k^+ \times (k^\times/k^\times 2) \) (see [19, Proposition 5.8]). The \( D_4 \)-Azumaya algebras in class (f) represent elements in the equivariant Brauer group \( \text{BM}(k, H_4, R) \).

The non-elementary \( D_4 \)-Azumaya algebras in class (g) do not represent elements in any equivariant Brauer group \( \text{BM}(k, H_4, R) \) or \( \text{BC}(k, H_4, R) \). Thus there exists only one class of \( H_4 \)-Azumaya algebra structures on \( M_2(k) \) (e.g., Theorem 4.10) whose underlying algebras are isomorphic to \( M_2(k) \). It is not difficult to work out that only four types of \( D_4 \)-modules admit the matrix algebra structure. They are \( D_4 \)-modules of types (A), (K), (L) and (O).

We begin with the easiest case, \( D_4 \)-module algebra structures on \( M_2(k) \) of type (A), i.e., \( \mathbb{Z}_2 \)-graded algebra structures on \( M_2(k) \). The classification of \( \mathbb{Z}_2 \)-graded algebra structures on \( M_2(k) \) has been done in [2,10]. Since a \( \mathbb{Z}_2 \)-graded algebra structure on \( M_2(k) \) yields a \( \mathbb{Z}_2 \)-graded Azumaya algebra, we can easily obtain the classification [2, Corollary 3.6] from Lemma 3.2 if we let the quaternion algebra \( \left( \frac{a,b}{k} \right) \) be \( M_2(k) \). We reformulate the result [2, Corollary 3.6] in terms of quaternion algebras.

**Theorem 4.1.** Suppose that \( M_2(k) \) is a \( \mathbb{Z}_2 \)-graded algebra. Then either

(a) the grading on \( M_2(k) \) is trivial, or 
(b) \( M_2(k) \) is isomorphic to \( \left( \frac{1-\alpha}{k} \right)_{\mathbb{Z}_2} \) for some \( \alpha \in k^\times \).

Moreover, for any \( \alpha, \beta \in k^\times \), \( \left( \frac{1-\alpha}{k} \right)_{\mathbb{Z}_2} \equiv \left( \frac{1-\beta}{k} \right)_{\mathbb{Z}_2} \) if and only if \( \alpha = \delta^2 \beta \) for some \( \delta \in k^\times \).

Note that in [2,10] a good grading on \( M_2(k) \) was introduced and studied, which is equivalent to an elementary (Azumaya) structure on a Matrix algebra. In this context, a good grading appears on \( M_2(k) \) as a quaternion algebra \( \left( \frac{1-\alpha}{k} \right)_{\mathbb{Z}_2} \), where \( \alpha \) is in \( k^2 \). They are all isomorphic. Thus we have a one-to-one correspondence between the set of non-isomorphic \( \mathbb{Z}_2 \)-graded algebra structures on \( M_2(k) \) and the set \( k^\times/k^\times 2 \cup \{1\} \), where the trivial graded algebra corresponds to 1. Since \( \left( \frac{1-\alpha}{k} \right)_{\mathbb{Z}_2} = k\langle \sqrt{1} \rangle \# k\langle \sqrt{-\alpha} \rangle \), we know that \( \left( \frac{1-\alpha}{k} \right)_{\mathbb{Z}_2} \) and \( \left( \frac{1-\beta}{k} \right)_{\mathbb{Z}_2} \) are Brauer equivalent if and only if \( \alpha \) and \( \beta \) represent the same class in the
group $k^{×}/k^{×2}$. Thus, up to Brauer equivalence, there exist $k^{×}/k^{×2}$ graded Azumaya algebra structures on $M_{2}(k)$. In other words, the graded Azumaya algebras with underlying algebra $M_{2}(k)$ form a subgroup $k^{×}/k^{×2}$ of the Brauer–Wall group $BW(k)$.

It is well known that the generalized quaternion algebra $(\alpha, \beta, \gamma, k)$, $\beta \neq 0$. In addition, $(\alpha, \beta, \gamma, k)$ is isomorphic to $M_{2}(k)$ if and only if $(\alpha, \beta, 0, k)$ is isomorphic to $(\alpha', \beta', 0, k)$ for some non-zero $\alpha'$ and $\beta'$ if $\gamma^{2} - 4\alpha\beta \neq 0$.

Lemma 4.2 [12, Theorem III.2.7]. Let $\alpha, \beta \in k^{×}$. Then $(\alpha, \beta, 0, k) \cong M_{2}(k)$ if and only if one of the following conditions is satisfied:

(a) $\alpha \in k^{×2}$;
(b) $\alpha \notin k^{×2}$ and $\beta = \theta^{2} - \alpha\delta^{2}$ for some $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$.

Remark 4.3.

(a) Let $\alpha, \beta \in k^{×}$. Then an algebra isomorphism from $(\alpha, \beta, 0, k)$ to $M_{2}(k)$ is given by

$$u \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}. \quad (\alpha \neq 0).$$

(b) Let $\alpha \in k^{×}$ with $\alpha \notin k^{×2}$, and let $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$. Then $\theta^{2} - \alpha\delta^{2} \neq 0$. An algebra isomorphism from $(\alpha, \beta, 0, k)$ to $M_{2}(k)$ is given by

$$u \mapsto \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} -\theta & -\delta \\ \alpha\delta & \theta \end{pmatrix}. \quad (\alpha, \beta, 0, k) \cong M_{2}(k)$$

As a result of Lemmas 2.19 and 4.2, we have the following.

Corollary 4.4. Assume that $M_{2}(k)$ is a $D_{4}$-module algebra isomorphic to $V(2, 0) \oplus M_{1}(1, 0, -2\beta)$ as a $D_{4}$-module for some $\beta \in k$. Then one of the following holds:

(a) $\beta \neq 0$ and $M_{2}(k) \cong (1, \beta, 0, k);$
(b) there exist an $\alpha \in k^{×}$ with $\alpha \notin k^{×2}$, and $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$ such that $\beta = \theta^{2} - \alpha\delta^{2} (\neq 0)$ and $M_{2}(k) \cong (\alpha, \beta, 0, k)_{+}.$

Corollary 4.5. Assume that $M_{2}(k)$ is a $D_{4}$-module algebra isomorphic to $M_{1}(1, 1, \infty) \oplus M_{1}(1, 0, \infty)$ as a $D_{4}$-module. Then $M_{2}(k)$ is isomorphic to one of the following:

(a) $(0, 0, 1, k)_{\infty}$,
(b) $(1, \beta, 0, k)_{\infty}$, where $\beta \in k^{×}$.
(c) \((\frac{\alpha \delta^2 - \alpha \delta^2}{k})_\infty\), where \(\alpha \in k^*\) with \(\alpha \notin k^{*2}\), and \(\theta, \delta \in k\) with \(\theta \neq 0\) or \(\delta \neq 0\).

**Proof.** It follows from Lemmas 2.26 and 4.2 as \((\frac{0.0.1}{k}) \cong M_2(k)\). □

**Corollary 4.6.** Assume that \(M_2(k)\) is a \(D_4\)-module algebra isomorphic to \(M_1(1, 1, \eta) \oplus M_1(1, 0, \eta)\) as a \(D_4\)-module for some \(\eta \in k\). Then the \(D_4\)-module algebra \(M_2(k)\) is isomorphic to one of the following:

(a) \((\frac{0.0.1}{k})\) as \(\eta'\),

(b) \((\frac{1.\beta.0}{k})\) where \(\beta \in k^*\),

(c) \((\frac{\alpha . \delta^2 - \alpha \delta^2}{k})\) as \(\eta\), where \(\alpha \in k^*\) with \(\alpha \notin k^{*2}\), and \(\theta, \delta \in k\) with \(\theta \neq 0\) or \(\delta \neq 0\).

**Proof.** It follows from Lemmas 2.27 and 4.2 since \((\frac{0.0.1}{k}) \cong M_2(k)\). □

**Lemma 4.7.** Let \(\alpha, \beta, \gamma \in k\) with \(\gamma \neq 0\). Then \((\frac{\alpha \beta \gamma}{k}) \cong M_2(k)\) if and only if one of the following conditions is satisfied:

(a) \(\alpha = 0\),

(b) \(\alpha = \delta^2\) and \(\beta = \frac{1}{4} \delta^{-2}(\gamma^2 - \lambda)\), where \(\delta, \lambda \in k^*\),

(c) \(\alpha \in k^*\) but \(\alpha \notin k^{*2}\), and \(\beta = \frac{1}{4} \alpha^{-1}[\gamma^2 - (\theta^2 - \alpha \delta^2)]\), where \(\theta, \delta \in k\) with \(\theta \neq 0\) or \(\delta \neq 0\).

**Proof.** As we noted before, \((\frac{\alpha \beta \gamma}{k}) \cong M_2(k)\) if and only if \(\gamma^2 - 4\alpha \beta \neq 0\) and \((\frac{\alpha \beta \gamma}{k})\) has a non-zero nilpotent element. Let \(x = t_1 u + t_2 v + t_3 u v + t_4 \in (\frac{\alpha \beta \gamma}{k})\), where \(t_i \in k\), \(1 \leq i \leq 4\). Then it is straightforward to check that \(x^2 = 0\) if and only if \(t_4 = -\frac{1}{2} \gamma^2 t_3\) and \(t_1^2 + \beta t_2^2 + \gamma t_1 t_2 + (\frac{\gamma^2}{4} - \alpha \beta) t_3^2 = 0\). Hence \((\frac{\alpha \beta \gamma}{k}) \cong M_2(k)\) if and only if \(\gamma^2 - 4\alpha \beta \neq 0\) and the equation

\[
\alpha x_1^2 + \beta x_2^2 + \gamma x_1 x_2 + \left(\frac{1}{4} \gamma^2 - \alpha \beta\right) x_3^2 = 0
\]

has a non-zero solution in \(k^3\).

If \(\alpha = 0\), then \(\gamma^2 - 4\alpha \beta = \gamma^2 \neq 0\) and \((x_1, x_2, x_3) = (1, 0, 0)\) is a non-zero solution of Eq. (17). If there exist \(\delta\) and \(\lambda\) in \(k^*\) such that \(\alpha = \delta^2\) and \(\beta = \frac{1}{4} \delta^{-2}(\gamma^2 - \lambda)\), then one can easily check that \(\gamma^2 - 4\alpha \beta = \lambda \neq 0\) and \((x_1, x_2, x_3) = (-\gamma, 2\delta^2, 2\delta)\) is a non-zero solution of Eq. (17). Now assume \(\alpha \in k^*\) but \(\alpha \notin k^{*2}\), and assume \(\beta = \frac{1}{4} \alpha^{-1}[\gamma^2 - (\theta^2 - \alpha \delta^2)]\), where \(\theta, \delta \in k\) with \(\theta \neq 0\) or \(\delta \neq 0\). Then \(\theta^2 - \alpha \delta^2 \neq 0\), and hence \(\gamma^2 - 4\alpha \beta = \theta^2 - \alpha \delta^2 \neq 0\). Furthermore, it is straightforward to check that \((x_1, x_2, x_3) = (\alpha^{-1}(\theta^2 - \alpha \delta^2 - \gamma \theta), 2\theta, 2\delta)\) is a non-zero solution of Eq. (17).

Conversely, assume \(\gamma^2 - 4\alpha \beta \neq 0\) and assume Eq. (17) has a non-zero solution \((t_1, t_2, t_3)\) in \(k^3\). If \(\alpha = 0\), then there is nothing to prove. Hence we may assume \(\alpha \neq 0\) in the following. Let \(\lambda = \gamma^2 - 4\alpha \beta\). Then \(\lambda \neq 0\) and \(\beta = \frac{1}{4} \alpha^{-1}(\gamma^2 - \lambda)\). If \(\alpha \in k^{*2}\), then
there exists a $\delta \in k^\times$ such that $\alpha = \delta^2$ and $\beta = \frac{1}{4}\delta^{-2}(\gamma^2 - \lambda)$, which is exactly the condition (b). Now suppose $\alpha \notin k^{\times 2}$. Then we first have $\alpha t_1^2 + \beta t_2^2 + \gamma t_1 t_2 + \frac{1}{4}\lambda t_3^2 = 0$. Then rearranging the equality, one gets

$$\alpha(2t_1 + \alpha^{-1}\gamma t_2)^2 = \lambda[\alpha^{-1}t_2^2 - t_3^2].$$

We claim that $2t_1 + \alpha^{-1}\gamma t_2 \neq 0$. In fact, if $2t_1 + \alpha^{-1}\gamma t_2 = 0$, then $\alpha^{-1}t_2^2 = t_3^2$ since $\lambda \neq 0$. Since $(t_1, t_2, t_3) \neq (0, 0, 0)$, we have $t_2 \neq 0$ and $t_3 \neq 0$. Hence $\alpha = (t_2t_3^{-1})^2 \in k^{\times 2}$, a contradiction. This shows the claim. It follows that $\alpha^{-1}t_2^2 - t_3^2 \neq 0$. Hence $t_2 \neq 0$ or $t_3 \neq 0$.

Next, let $\theta = t_2(2t_1 + \alpha^{-1}\gamma t_2)(\alpha^{-1}t_2^2 - t_3^2)^{-1}$ and $\delta = t_3(2t_1 + \alpha^{-1}\gamma t_2)(\alpha^{-1}t_2^2 - t_3^2)^{-1}$. Then $\theta \neq 0$ or $\delta \neq 0$, and $\lambda = \theta^2 - \alpha\delta^2$. It follows that $\beta = \frac{1}{4}\alpha^{-1}[\gamma^2 - (\theta^2 - \alpha\delta^2)]$. That is, the condition (c) is satisfied. \hfill \Box

**Remark 4.8.**

(a) Let $\beta \in k$ and $\gamma \in k^\times$. Then an algebra isomorphism from $(0, \frac{\beta \gamma^2}{k})$ to $M_2(k)$ is given by

$$u \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & \beta \gamma^{-1} \\ \gamma & 0 \end{pmatrix}.$$

(b) Let $\alpha, \beta, \gamma \in k^\times$. Then an algebra isomorphism from $(\frac{\alpha^2 + \frac{1}{4}\alpha^{-2}(\gamma^2 - \beta)}{k}, \gamma)$ to $M_2(k)$ is given by

$$u \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad v \mapsto \frac{1}{2}\alpha^{-1}\begin{pmatrix} \gamma & 1 \\ -\beta & -\gamma \end{pmatrix}.$$

(c) Let $\alpha, \gamma \in k^\times$ with $\alpha \notin k^{\times 2}$, and let $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$. Then $\theta^2 - \alpha\delta^2 \neq 0$ and an algebra isomorphism from $(\frac{\alpha + \frac{1}{4}\alpha^{-1}(\gamma^2 - (\theta^2 - \alpha\delta^2))}{k}, \gamma)$ to $M_2(k)$ is given by

$$u \mapsto \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \quad v \mapsto \frac{1}{2}\begin{pmatrix} \delta & \alpha^{-1}(\gamma + \theta) \\ \gamma - \theta & -\delta \end{pmatrix}.$$

Combining Lemmas 2.31, 4.2 and 4.7, we obtain all $D_4$-module algebra structures on $M_2(k)$ of $D_4$-module type $P(1, 0)$. 

**Corollary 4.9.** Assume that $M_2(k)$ is a $D_4$-module algebra isomorphic to $P(1, 0)$ as a $D_4$-module. Then the $D_4$-module algebra $M_2(k)$ is isomorphic to one of the following:

(a) $(\frac{\alpha^2, \beta, 0}{k})_p$, $\alpha, \beta \in k^\times$,

(b) $(\frac{\alpha^2 - \theta, \alpha\delta^2, 0}{k})_p$, $\alpha \in k^\times$ with $\alpha \notin k^{\times 2}$, and $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$,

(c) $(\frac{0, \beta, \gamma}{k})_p$, $\beta \in k$ and $\gamma \in k^\times$.  

Theorem 4.10. The following $D_4$-module algebras are mutually non-isomorphic. Any $D_4$-module algebra structure on $M_2(k)$ is isomorphic to one of them.

(a) $M_2(k)$ with the trivial $D_4$-action,
(b) $\left(1, \frac{\beta}{k}\right)_{\mathbb{Z}_2^2}$, $\beta \in k^\times$,
(c) $\left(1, \frac{\beta,0}{k}\right)_+$, $\beta \in k^\times$,
(d) $\left(\frac{\alpha,0^2-\alpha \delta^2,0}{k}\right)_+$, $\alpha \in k^\times$ with $\alpha \not\in k^{\times^2}$, and $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$,
(e) $\left(\frac{0,0,1}{k}\right)_+$, $\beta \in k^\times$,
(f) $\left(\frac{\alpha,0,0}{k}\right)_+$, $\beta \in k^\times$,
(g) $\left(\frac{\alpha,0^2-\alpha \delta^2,0}{k}\right)_+$, $\alpha \in k^\times$ with $\alpha \not\in k^{\times^2}$, and $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$,
(h) $\left(\frac{0,0,1}{k}\right)_+$, $\eta, \eta \in k$,
(i) $\left(\frac{\alpha,0,0}{k}\right)_+$, $\beta \in k^\times$ and $\eta \in k$,
(j) $\left(\frac{\alpha,0^2-\alpha \delta^2,0}{k}\right)_+$, $\alpha \in k^\times$ with $\alpha \not\in k^{\times^2}$, and $\eta, \theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$,
(k) $\left(\frac{\alpha,0,0}{k}\right)_+$, $\alpha \in k^\times$,
(l) $\left(\frac{\alpha,0^2-\alpha \delta^2,0}{k}\right)_+$, $\alpha \in k^\times$ with $\alpha \not\in k^{\times^2}$, and $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$,
(m) $\left(\frac{0,\beta,0}{k}\right)_+$, $\beta \in k$ and $\gamma \in k^\times$,
(n) $\left(\frac{\alpha,0,0}{k}\right)_+$, $\alpha \in k^\times$,
(o) $\left(\frac{\alpha,0^2-\alpha \delta^2,0}{k}\right)_+$, $\alpha \in k^\times$ with $\alpha \not\in k^{\times^2}$, and $\theta, \delta \in k$ with $\theta \neq 0$ or $\delta \neq 0$.

Summarizing the foregoing results we obtain the classification of $D_4$-module algebra structures on $M_2(k)$.

Remark that the above $D_4$-module algebra structures on $M_2(k)$ except the cases (c) and (d) are $D_4$-Azumaya algebra structures when the corresponding parameters satisfy the conditions given in Theorem 3.6. The classification of 4-dimensional $D_4$-Azumaya algebras up to Brauer equivalence will not be included in this paper for the sake of length, and will appear elsewhere.

Finally, let us point out the elementary $D_4$-module algebra structures on the list in Theorem 4.10. This is of particular interest because the elementary $D_4$-module algebras represent the identity element in the Brauer group $BQ(k, H_4)$. Recall that a 4-dimensional $D_4$-module algebra $A$ is elementary if and only if $A$ is isomorphic to $\text{End}(M)$ for a 2-dimensional $D_4$-module $M$. From Section 1.4, we know that a 2-dimensional $D_4$-module $M$ must be of one of the following forms up to isomorphism:

(a) $2V(1, r)$, $r \in \mathbb{Z}_2$,
(b) $V(1,0) \oplus V(1, 1)$,
(c) $V(2, r)$, $r \in \mathbb{Z}_2$,
(d) $M_1(1, r, \eta)$, $r \in \mathbb{Z}_2$, where $\eta \in k \cup \{\infty\}$.

It is not hard to verify the following isomorphisms of $D_4$-module algebras.

- $\text{End}(2V(1, r)) \cong M_2(k)$ with the trivial $D_4$-action,
- $\text{End}(V(1, 0) \oplus V(1, 1)) \cong \left(\frac{1-1}{k}\right)\mathbb{Z}_2$,
- $\text{End}(V(2, r)) \cong \left(\frac{0,0,-1/2}{k}\right)_{\eta}$, and
- $\text{End}(M_1(1, r, \eta)) \cong \left(\frac{0,0,1}{k}\right)_{\eta}$, where $r \in \mathbb{Z}_2$, and where $\eta \in k \cup \{\infty\}$.

We end this section with all non-isomorphic elementary ($D_4$-Azumaya) algebra structures on $M_2(k)$.

**Proposition 4.11.** Let $A$ be a 4-dimensional $D_4$-module algebra. Then $A$ is elementary if and only if $A$ is isomorphic to one of the following:

(a) $M_2(k)$ with the trivial $D_4$-action,
(b) $\left(\frac{1-1}{k}\right)\mathbb{Z}_2$,
(c) $\left(\frac{0,0,1}{k}\right)_{\infty}$,
(d) $\left(\frac{0,0,1}{k}\right)_{\eta}$, where $\eta \in k$,
(e) $\left(\frac{0,0,-1/2}{k}\right)_{\eta}$.

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