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A combinatorial proof of Marstrand's theorem for products of regular Cantor sets

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ABSTRACT

In a paper from 1954 Marstrand proved that if $K \subset \mathbb{R}^2$ has a Hausdorff dimension greater than 1, then its one-dimensional projection has a positive Lebesgue measure for almost all directions. In this article, we give a combinatorial proof of this theorem when K is the product of regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, for which the sum of their Hausdorff dimension is greater than 1.

1. Introduction

If U is a subset of $\mathbb{R}^n$, the diameter of U is $|U| = \sup\{|x-y|; x, y \in U\}$ and, if $\mathcal{U}$ is a family of subsets of $\mathbb{R}^n$, the *diameter* of $\mathcal{U}$ is defined as

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $d > 0$, the *Hausdorff d-measure* of a set $K \subseteq \mathbb{R}^n$ is

$$m_d(K) = \lim_{\varepsilon \to 0} \left(\inf_{\substack{\mathcal{U} \text{ covers } K \\ \|\mathcal{U}\| < \varepsilon}} \sum_{U \in \mathcal{U}} |U|^d \right).$$

In particular, when $n = 1$, $m = m_1$ is the Lebesgue measure of Lebesgue measurable sets on $\mathbb{R}$. It is not difficult to show that there exists a unique $d_0 \geq 0$ for which $m_d(K) = +\infty$ if $d < d_0$ and $m_d(K) = 0$ if $d > d_0$. We define the Hausdorff dimension of K as $\mathrm{HD}(K) = d_0$. Also, for each $\theta \in \mathbb{R}$, let $v_\theta = (\cos\theta, \sin\theta)$, L_θ the line in $\mathbb{R}^2$ through the origin containing v_θ and $\mathrm{proj}_\theta : \mathbb{R}^2 \to L_\theta$ the orthogonal projection. From now on, we will restrict θ to the interval $[-\pi/2, \pi/2]$, because $L_\theta = L_{\theta+\pi}$.

In 1954, Marstrand [4] proved the following result on the fractal dimension of plane sets.

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Theorem. *If $K \subseteq \mathbb{R}^2$ is a Borel set such that* $\mathrm{HD}(K) > 1$, *then* $m(\mathrm{proj}_\theta(K)) > 0$ *for m-almost every* $\theta \in \mathbb{R}$.

The proof is based on a qualitative characterisation of the "bad" angles θ for which the result is not true. Specifically, Marstrand exhibits a Borel measurable function $f(x, \theta)$, $(x, \theta) \in \mathbb{R}^2 \times [-\pi/2, \pi/2]$, such that $f(x, \theta) = +\infty$ for m_d-almost every $x \in K$, for every "bad" angle. In particular,

$$\int_K f(x, \theta) \mathrm{d}m_d(x) = +\infty. \tag{1.1}$$

On the other hand, using a version of Fubini's Theorem, he proves that

$$\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_K f(x, \theta) \mathrm{d}m_d(x) = 0$$

which, in view of (1.1), implies that

$$m(\{\theta \in [-\pi/2, \pi/2]; m(\mathrm{proj}_\theta(K)) = 0\}) = 0.$$

These results are based on the analysis of rectangular densities of points.

Many generalisations and simpler proofs have appeared since. One of them came in 1968 by R. Kaufman who gave a very short proof of Marstrand's theorem using methods of potential theory. See [2] for his original proof and [5,9] for further discussion.

In this article, we prove a particular case of Marstrand's Theorem.

Theorem 1.1. *If K_1, K_2 are regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, such that $d = \mathrm{HD}(K_1)+\mathrm{HD}(K_2) > 1$, then $m(\mathrm{proj}_\theta(K_1 \times K_2)) > 0$ for m-almost every $\theta \in \mathbb{R}$.*

The argument also works to show that the push-forward measure of the restriction of m_d to $K_1 \times K_2$, defined as $\mu_\theta = (\mathrm{proj}_\theta)_*(m_d|_{K_1\times K_2})$, is absolutely continuous with respect to m, for m-almost every $\theta \in \mathbb{R}$. Denoting its Radon–Nykodim derivative by $\chi_\theta = d\mu_\theta/dm$, we also prove the following result.

Theorem 1.2. *χ_θ is an L^2 function for m-almost every $\theta \in \mathbb{R}$.*

Remark 1.3. Theorem 1.2, as in this work, follows from most proofs of Marstrand's theorem and, in particular, is not new as well.

Our proof makes a study on the fibers ${\mathrm{proj}_\theta}^{-1}(v) \cap (K_1 \times K_2)$, $(\theta, v) \in \mathbb{R} \times L_\theta$, and relies on two facts:

(I) A regular Cantor set of Hausdorff dimension d is regular in the sense that the m_d-measure of small portions of it has the same exponential behaviour.

(II) This enables us to conclude that, except for a small set of angles $\theta \in \mathbb{R}$, the fibers ${\mathrm{proj}_\theta}^{-1}(v) \cap (K_1 \times K_2)$ are not concentrated in a thin region. As a consequence, $K_1 \times K_2$ projects into a set of positive Lebesgue measure.

The idea of (II) is based on the work [6] of the second author. He proves that, if K_1 and K_2 are regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and at least one of them is non-essentially affine (a technical condition), then the arithmetic sum $K_1 + K_2 = \{x_1 + x_2; x_1 \in K_1, x_2 \in K_2\}$ has the expected Hausdorff dimension:

$$\mathrm{HD}(K_1 + K_2) = \min\{1, \mathrm{HD}(K_1) + \mathrm{HD}(K_2)\}.$$

Marstrand's Theorem for products of Cantor sets has many useful applications in dynamical systems. It is fundamental in certain results of dynamical bifurcations, namely homoclinic bifurcations in surfaces. For instance, in [10] it is used to show that hyperbolicity is not prevalent in homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one; in [7] it is used to prove that stable intersections of regular Cantor sets are dense in the region where the sum of their Hausdorff dimensions is larger than one; in [8] to show that, for homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one, typically there are open sets of parameters with positive Lebesgue density at the initial bifurcation parameter corresponding to persistent homoclinic tangencies.

2. Regular Cantor sets of class $C^{1+\alpha}$

We say that $K \subset \mathbb{R}$ is a *regular Cantor set of class* $C^{1+\alpha}$, $\alpha > 0$, if:

(i) there are disjoint compact intervals $I_1, I_2, \ldots, I_r \subseteq [0, 1]$ such that $K \subset I_1 \cup \cdots \cup I_r$ and the boundary of each I_i is contained in K;

(ii) there is a $C^{1+\alpha}$ expanding map ψ defined in a neighbourhood of $I_1 \cup I_2 \cup \cdots \cup I_r$ such that $\psi(I_i)$ is the convex hull of a finite union of some intervals I_j, satisfying:

  (ii.1) for each $i \in \{1, 2, \ldots, r\}$ and n sufficiently big, $\psi^n(K \cap I_i) = K$;

  (ii.2) $K = \bigcap_{n\in\mathbb{N}} \psi^{-n}(I_1 \cup I_2 \cup \cdots \cup I_r)$.

The set $\{I_1, \ldots, I_r\}$ is called a Markov partition of K. It defines an $r \times r$ matrix $B = (b_{ij})$ by

$$\begin{aligned} b_{ij} &= 1, \quad \text{if } \psi(I_i) \supseteq I_j \\ &= 0, \quad \text{if } \psi(I_i) \cap I_j = \emptyset, \end{aligned}$$

which encodes the combinatorial properties of K. Given such a matrix, consider the set $\Sigma_B = \{\underline{\theta} = (\theta_1, \theta_2, \ldots) \in \{1, \ldots, r\}^{\mathbb{N}}; b_{\theta_i\theta_{i+1}} = 1, \forall\, i \geq 1\}$ and the shift transformation $\sigma : \Sigma_B \to \Sigma_B$ given by $\sigma(\theta_1, \theta_2, \ldots) = (\theta_2, \theta_3, \ldots)$.

There is a natural homeomorphism between the pairs (K, ψ) and (Σ_B, σ). For each finite word $\underline{a} = (a_1, \ldots, a_n)$ such that $b_{a_ia_{i+1}} = 1, i = 1, \ldots, n-1$, the intersection

$$I_{\underline{a}} = I_{a_1} \cap \psi^{-1}(I_{a_2}) \cap \cdots \cap \psi^{-(n-1)}(I_{a_n})$$

is a non-empty interval with diameter $|I_{\underline{a}}| = |I_{a_n}|/|(\psi^{n-1})'(x)|$ for some $x \in I_{\underline{a}}$, which is exponentially small if n is large. Then, $\{h(\underline{\theta})\} = \bigcap_{n\geq 1} I_{(\theta_1,\ldots,\theta_n)}$ defines a homeomorphism $h : \Sigma_B \to K$ that commutes the diagram

$$\begin{array}{ccc} \Sigma_B & \xrightarrow{\ \sigma\ } & \Sigma_B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ K & \xrightarrow[\ \psi\ ]{} & K \end{array}$$

If $\lambda = \sup\{|\psi'(x)|; x \in I_1 \cup \cdots \cup I_r\} \in (1, +\infty)$, then $|I_{(\theta_1,\ldots,\theta_{n+1})}| \geq \lambda^{-1} \cdot |I_{(\theta_1,\ldots,\theta_n)}|$ and so, for $\rho > 0$ small and $\underline{\theta} \in \Sigma_B$, there is a positive integer $n = n(\rho, \underline{\theta})$ such that

$$\rho \leq |I_{(\theta_1,\ldots,\theta_n)}| \leq \lambda\rho.$$

Definition 2.1. A ρ-decomposition of K is any finite set $(K)_\rho = \{I_1, I_2, \ldots, I_r\}$ of disjoint closed intervals of $\mathbb{R}$, each one of them intersecting K, whose union covers K and such that

$$\rho \leq |I_i| \leq \lambda\rho, \quad i = 1, 2, \ldots, r.$$

Remark 2.2. Although ρ-decompositions are not unique, we use, for simplicity, the notation $(K)_\rho$ to denote any of them. We also use the same notation $(K)_\rho$ to denote the set $\cup_{I\in(K)_\rho} I \subset \mathbb{R}$ and the distinction between these two situations will be clear throughout the text.

Every regular Cantor set of class $C^{1+\alpha}$ has a ρ-decomposition for $\rho > 0$ small: by the compactness of K, the family $\{I_{(\theta_1,\ldots,\theta_{n(\rho,\underline{\theta})})}\}_{\underline{\theta}\in\Sigma_B}$ has a finite cover (in fact, it is only necessary for ψ to be of class C^1). Also, one can define ρ-decomposition for the product of two Cantor sets K_1 and K_2, denoted by $(K_1 \times K_2)_\rho$. Given $\rho \neq \rho'$ and two decompositions $(K_1 \times K_2)_{\rho'}$ and $(K_1 \times K_2)_\rho$, consider the partial order

$$(K_1 \times K_2)_{\rho'} \prec (K_1 \times K_2)_\rho \iff \rho' < \rho \quad \text{and} \quad \bigcup_{Q'\in(K_1\times K_2)_{\rho'}} Q' \subseteq \bigcup_{Q\in(K_1\times K_2)_\rho} Q.$$

In this case, $\text{proj}_\theta((K_1 \times K_2)_{\rho'}) \subseteq \text{proj}_\theta((K_1 \times K_2)_\rho)$ for any θ.

A remarkable property of regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, is bounded distortion.

Lemma 2.3. *Let (K, ψ) be a regular Cantor set of class $C^{1+\alpha}$, $\alpha > 0$, and $\{I_1, \ldots, I_r\}$ a Markov partition. Given $\delta > 0$, there exists a constant $C(\delta) > 0$, decreasing on δ, with the following property: if $x, y \in K$ satisfy*

(i) $|\psi^n(x) - \psi^n(y)| < \delta$;
(ii) *The interval $[\psi^i(x), \psi^i(y)]$ is contained in $I_1 \cup \cdots \cup I_r$, for $i = 0, \ldots, n-1$,*

then

$$e^{-C(\delta)} \leq \frac{|(\psi^n)'(x)|}{|(\psi^n)'(y)|} \leq e^{C(\delta)}.$$

In addition, $C(\delta) \to 0$ as $\delta \to 0$.

A direct consequence of bounded distortion is the required regularity of K, contained in the next result.

Lemma 2.4. *Let K be a regular Cantor set of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K)$. Then $0 < m_d(K) < +\infty$. Moreover, there is $c > 0$ such that, for any $x \in K$ and $0 \leq r \leq 1$,*

$$c^{-1} \cdot r^d \leq m_d(K \cap B_r(x)) \leq c \cdot r^d.$$

The same happens for products $K_1 \times K_2$ of Cantor sets (without loss of generality, considered with the box norm).

Lemma 2.5. *Let K_1, K_2 be regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K_1) + \mathrm{HD}(K_2)$. Then $0 < m_d(K_1 \times K_2) < +\infty$. Moreover, there is $c_1 > 0$ such that, for any $x \in K_1 \times K_2$ and $0 \leq r \leq 1$,*

$${c_1}^{-1} \cdot r^d \leq m_d((K_1 \times K_2) \cap B_r(x)) \leq c_1 \cdot r^d.$$

See chapter 4 of [9] for the proofs of these lemmas. In particular, if $Q \in (K_1 \times K_2)_\rho$, there is $x \in (K_1 \cup K_2) \cap Q$ such that $B_{\lambda^{-1}\rho}(x) \subseteq Q \subseteq B_{\lambda\rho}(x)$ and so

$$(c_1\lambda^d)^{-1} \cdot \rho^d \leq m_d((K_1 \times K_2) \cap Q) \leq c_1\lambda^d \cdot \rho^d.$$

Changing c_1 by $c_1\lambda^d$, we may also assume that

$${c_1}^{-1} \cdot \rho^d \leq m_d((K_1 \times K_2) \cap Q) \leq c_1 \cdot \rho^d,$$

which allows us to obtain estimates on the cardinality of ρ-decompositions.

Lemma 2.6. *Let K_1, K_2 be regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K_1) + \mathrm{HD}(K_2)$. Then there is $c_2 > 0$ such that, for any ρ-decomposition $(K_1 \times K_2)_\rho$, $x \in K_1 \times K_2$ and $0 \leq r \leq 1$,*

$$\#\{Q \in (K_1 \times K_2)_\rho;\ Q \subseteq B_r(x)\} \leq c_2 \cdot \left(\frac{r}{\rho}\right)^d.$$

In addition, ${c_2}^{-1} \cdot \rho^{-d} \leq \#(K_1 \times K_2)_\rho \leq c_2 \cdot \rho^{-d}$.

Proof. We have

$$\begin{aligned} c_1 \cdot r^d &\geq m_d((K_1 \times K_2) \cap B_r(x)) \\ &\geq \sum_{Q \subseteq B_r(x)} m_d((K_1 \times K_2) \cap Q) \\ &\geq \sum_{Q \subseteq B_r(x)} {c_1}^{-1} \cdot \rho^d \\ &= \#\{Q \in (K_1 \times K_2)_\rho;\ Q \subseteq B_r(x)\} \cdot {c_1}^{-1} \cdot \rho^d \end{aligned}$$

and then

$$\#\{Q \in (K_1 \times K_2)_\rho;\, Q \subseteq B_r(x)\} \le {c_1}^2 \cdot \left(\frac{r}{\rho}\right)^d.$$

On the other hand,

$$m_d(K_1 \times K_2) = \sum_{Q \in (K_1 \times K_2)_\rho} m_d((K_1 \times K_2) \cap Q) \le \sum_{Q \in (K_1 \times K_2)_\rho} c_1 \cdot \rho^d,$$

implying that

$$\#(K_1 \times K_2)_\rho \ge {c_1}^{-1} \cdot m_d(K_1 \times K_2) \cdot \rho^{-d}.$$

Taking $c_2 = \max\{{c_1}^2\,,\ c_1/m_d(K_1 \times K_2)\}$, we conclude the proof. □

3. Proof of Theorem 1.1

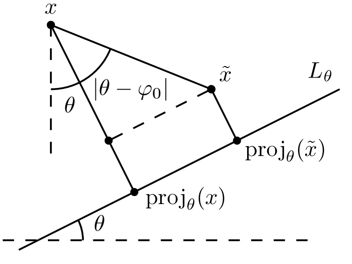
Given rectangles Q and $\tilde{Q}$, let

$$\Theta_{Q,\tilde{Q}} = \{\theta \in [-\pi/2, \pi/2];\, \mathrm{proj}_\theta(Q) \cap \mathrm{proj}_\theta(\tilde{Q}) \ne \emptyset\}.$$

Lemma 3.1. *If $Q, \tilde{Q} \in (K_1 \times K_2)_\rho$ and $x \in (K_1 \times K_2) \cap Q$, $\tilde{x} \in (K_1 \times K_2) \cap \tilde{Q}$, then*

$$m(\Theta_{Q,\tilde{Q}}) \le 2\pi\lambda \cdot \frac{\rho}{d(x,\tilde{x})}.$$

Proof. Consider the figure.



Since $\mathrm{proj}_\theta(Q)$ has diameter at most $\lambda\rho$, $d(\mathrm{proj}_\theta(x), \mathrm{proj}_\theta(\tilde{x})) \le 2\lambda\rho$ and then, by elementary geometry,

$$\begin{aligned} \sin(|\theta - \varphi_0|) &= \frac{d(\mathrm{proj}_\theta(x), \mathrm{proj}_\theta(\tilde{x}))}{d(x,\tilde{x})} \\ &\le 2\lambda \cdot \frac{\rho}{d(x,\tilde{x})} \\ \Longrightarrow |\theta - \varphi_0| &\le \pi\lambda \cdot \frac{\rho}{d(x,\tilde{x})}, \end{aligned}$$

because $\sin^{-1} y \le \pi y/2$. As φ_0 is fixed, the lemma is proved. □

We point out that, although ingenuous, Lemma 3.1 expresses the crucial property of transversality that makes the proof work, and all results related to Marstrand's theorem use a similar idea in one way or another. See [11] where this transversality condition is also exploited.

Fixed a ρ-decomposition $(K_1 \times K_2)_\rho$, let

$$N_{(K_1 \times K_2)_\rho}(\theta) = \#\{(Q,\tilde{Q}) \in (K_1 \times K_2)_\rho \times (K_1 \times K_2)_\rho;\, \mathrm{proj}_\theta(Q) \cap \mathrm{proj}_\theta(\tilde{Q}) \ne \emptyset\}$$

for each $\theta \in [-\pi/2, \pi/2]$ and

$$E((K_1 \times K_2)_\rho) = \int_{-\pi/2}^{\pi/2} N_{(K_1\times K_2)_\rho}(\theta)\mathrm{d}\theta.$$

Proposition 3.2. *Let K_1, K_2 be regular Cantor sets of class $C^{1+\alpha}$, $\alpha > 0$, and let $d = \mathrm{HD}(K_1) + \mathrm{HD}(K_2)$. Then there is $c_3 > 0$ such that, for any ρ-decomposition $(K_1 \times K_2)_\rho$,*

$$E((K_1 \times K_2)_\rho) \le c_3 \cdot \rho^{1-2d}.$$

Proof. Let $s_0 = \lceil \log_2 \rho^{-1} \rceil$ and choose, for each $Q \in (K_1 \times K_2)_\rho$, a point $x \in (K_1 \times K_2) \cap Q$. By a double counting and using Lemmas 2.6 and 3.1, we have

$$\begin{aligned} E((K_1 \times K_2)_\rho) &= \sum_{Q,\tilde{Q}\in(K_1\times K_2)_\rho} m(\Theta_{Q,\tilde{Q}}) \\ &= \sum_{s=1}^{s_0} \sum_{\substack{Q,\tilde{Q}\in(K_1\times K_2)_\rho \\ 2^{-s}<d(x,\tilde{x})\le 2^{-s+1}}} m(\Theta_{Q,\tilde{Q}}) \\ &\le \sum_{s=1}^{s_0} c_2 \cdot \rho^{-d} \left[c_2 \cdot \left(\frac{2^{-s+1}}{\rho} \right)^d \right] \cdot \left(2\pi\lambda \cdot \frac{\rho}{2^{-s}} \right) \\ &= 2^{d+1}\pi\lambda {c_2}^2 \cdot \left(\sum_{s=1}^{s_0} 2^{s(1-d)} \right) \cdot \rho^{1-2d}. \end{aligned}$$

Because $d > 1$, $c_3 = 2^{d+1}\pi\lambda {c_2}^2 \cdot \sum_{s\ge1} 2^{s(1-d)} < +\infty$ satisfies the required inequality. □

This implies that, for each $\varepsilon > 0$, the upper bound

$$N_{(K_1\times K_2)_\rho}(\theta) \le \frac{c_3 \cdot \rho^{1-2d}}{\varepsilon} \tag{3.1}$$

holds for every θ except for a set of measure at most ε. Letting $c_4 = {c_2}^{-2} \cdot {c_3}^{-1}$, we will show that

$$m(\mathrm{proj}_\theta((K_1 \times K_2)_\rho)) \ge c_4 \cdot \varepsilon \tag{3.2}$$

for every θ satisfying (3.1). For this, divide $[-2, 2] \subseteq L_\theta$ in $\lfloor 4/\rho \rfloor$ intervals $J_1^\rho, \ldots, J_{\lfloor 4/\rho\rfloor}^\rho$ of equal length (at least ρ) and define

$$s_{\rho,i} = \#\{Q \in (K_1 \times K_2)_\rho;\ \mathrm{proj}_\theta(x) \in J_i^\rho\}, \quad i = 1, \ldots, \lfloor 4/\rho \rfloor.$$

Then $\sum_{i=1}^{\lfloor 4/\rho\rfloor} s_{\rho,i} = \#(K_1 \times K_2)_\rho$ and

$$\sum_{i=1}^{\lfloor 4/\rho\rfloor} {s_{\rho,i}}^2 \le N_{(K_1\times K_2)_\rho}(\theta) \le c_3 \cdot \rho^{1-2d} \cdot \varepsilon^{-1}.$$

Let $S_\rho = \{1 \le i \le \lfloor 4/\rho \rfloor;\ s_{\rho,i} > 0\}$. By the Cauchy–Schwarz inequality,

$$\#S_\rho \ge \frac{\left(\sum_{i\in S_\rho} s_{\rho,i} \right)^2}{\sum_{i\in S_\rho} {s_{\rho,i}}^2} \ge \frac{{c_2}^{-2} \cdot \rho^{-2d}}{c_3 \cdot \rho^{1-2d} \cdot \varepsilon^{-1}} = \frac{c_4 \cdot \varepsilon}{\rho}.$$

For each $i \in S_\rho$, the interval J_i^ρ is contained in $\mathrm{proj}_\theta((K_1 \times K_2)_\rho)$ and then

$$m(\mathrm{proj}_\theta((K_1 \times K_2)_\rho)) \ge c_4 \cdot \varepsilon,$$

which proves (3.2).

Proof of Theorem 1.1. Fix a decreasing sequence

$$(K_1 \times K_2)_{\rho_1} \succ (K_1 \times K_2)_{\rho_2} \succ \cdots \tag{3.3}$$

of decompositions such that $\rho_n \to 0$ and, for each $\varepsilon > 0$, consider the sets

$$G_\varepsilon^n = \{\theta \in [-\pi/2, \pi/2];\, N_{(K_1\times K_2)_{\rho n}}(\theta) \le c_3 \cdot {\rho_n}^{1-2d} \cdot \varepsilon^{-1}\}, \quad n \ge 1.$$

Then $m([-\pi/2, \pi/2] \setminus G_\varepsilon^n) \le \varepsilon$, and the same holds for the set

$$G_\varepsilon = \bigcap_{n\ge 1} \bigcup_{l=n}^{\infty} G_\varepsilon^l.$$

If $\theta \in G_\varepsilon$, then

$$m(\mathrm{proj}_\theta((K_1 \times K_2)_{\rho n})) \ge c_4 \cdot \varepsilon, \quad \text{for infinitely many } n,$$

which implies that $m(\mathrm{proj}_\theta(K_1 \times K_2)) \ge c_4 \cdot \varepsilon$. Finally, the set $G = \cup_{n\ge 1} G_{1/n}$ satisfies $m([-\pi/2, \pi/2] \setminus G) = 0$ and $m(\mathrm{proj}_\theta(K_1 \times K_2)) > 0$, for any $\theta \in G$. □

4. Proof of Theorem 1.2

Given any $X \subset K_1 \times K_2$, let $(X)_\rho$ be the restriction of the ρ-decomposition $(K_1 \times K_2)_\rho$ to those rectangles which intersect X. As done in Section 3, we will obtain estimates on the cardinality of $(X)_\rho$. Being a subset of $K_1 \times K_2$, the upper estimates from Lemma 2.6 also hold for X. The lower estimate is given by

Lemma 4.1. *Let X be a subset of $K_1 \times K_2$ such that $m_d(X) > 0$. Then there is $c_6 = c_6(X) > 0$ such that, for any ρ-decomposition $(K_1 \times K_2)_\rho$ and $0 \le r \le 1$,*

$$c_6 \cdot \rho^{-d} \le \#(X)_\rho \le c_2 \cdot \rho^{-d}.$$

Proof. As $m_d(X) < +\infty$, there exists $c_5 = c_5(X) > 0$ (see Theorem 5.6 of [1]) such that

$$m_d(X \cap B_r(x)) \le c_5 \cdot r^d, \quad \text{for all } x \in X \text{ and } 0 \le r \le 1,$$

and then

$$m_d(X) = \sum_{Q\in(X)_\rho} m_d(X \cap Q) \le \sum_{Q\in(X)_\rho} c_5 \cdot (\lambda\rho)^d = (c_5 \cdot \lambda^d) \cdot \rho^d \cdot \#(X)_\rho.$$

Just take $c_6 = {c_5}^{-1} \cdot \lambda^{-d} \cdot m_d(X)$. □

Proposition 4.2. *The measure $\mu_\theta = (\mathrm{proj}_\theta)_*(m_d|_{K_1\times K_2})$ is absolutely continuous with respect to m, for m-almost every $\theta \in \mathbb{R}$.*

Proof. Note that the implication

$$X \subset K_1 \times K_2, \qquad m_d(X) > 0 \implies m(\mathrm{proj}_\theta(X)) > 0 \tag{4.1}$$

is sufficient for the required absolute continuity. In fact, if $Y \subset L_\theta$ satisfies $m(Y) = 0$, then

$$\mu_\theta(Y) = m_d(X) = 0,$$

where $X = {\mathrm{proj}_\theta}^{-1}(Y)$. Otherwise, by (4.1) we would have $m(Y) = m(\mathrm{proj}_\theta(X)) > 0$, contradicting the assumption.

We prove that (4.1) holds for every $\theta \in G$, where G is the set defined in the proof of Theorem 1.1. The argument is the same made after Proposition 3.2: as, by the previous lemma, $\#(X)_\rho$ has lower and upper estimates depending only on X and ρ, we obtain that

$$m(\mathrm{proj}_\theta((X)_{\rho n})) \ge {c_3}^{-1} \cdot {c_6}^2 \cdot \varepsilon, \quad \text{for infinitely many } n,$$

and then $m(\mathrm{proj}_\theta(X)) > 0$. □

Let $\chi_\theta = d\mu_\theta/dm$. In principle, this is an L^1 function. We prove that it is an L^2 function, for every $\theta \in G$.

Proof of Theorem 1.2. Let $\theta \in G_{1/m}$, for some $m \in \mathbb{N}$. Then

$$N_{(K_1\times K_2)_{\rho_n}}(\theta) \le c_3 \cdot {\rho_n}^{1-2d} \cdot m, \quad \text{for infinitely many } n. \tag{4.2}$$

For each of these n, consider the partition $\mathcal{P}_n = \{J_1^{\rho_n}, \ldots, J_{\lfloor 4/\rho_n \rfloor}^{\rho_n}\}$ of $[-2, 2] \subset L_\theta$ into intervals of equal length and let $\chi_{\theta,n}$ be the expectation of χ_θ with respect to $\mathcal{P}_n$. As $\rho_n \to 0$, the sequence of functions $(\chi_{\theta,n})_{n\in\mathbb{N}}$ converges pointwise to χ_θ. By Fatou's Lemma, we are done if we prove that each $\chi_{\theta,n}$ is L^2 and its L^2-norm $\|\chi_{\theta,n}\|_2$ is bounded above by a constant independent of n.

By definition,

$$\mu_\theta(J_i^{\rho_n}) = m_d((\mathrm{proj}_\theta)^{-1}(J_i^{\rho_n})) \le s_{\rho_n,i} \cdot c_1 \cdot {\rho_n}^d, \quad i = 1, 2, \ldots, \lfloor 4/\rho_n \rfloor,$$

and then

$$\chi_{\theta,n}(x) = \frac{\mu_\theta(J_i^{\rho_n})}{|J_i^{\rho_n}|} \le \frac{c_1 \cdot s_{\rho_n,i} \cdot {\rho_n}^d}{|J_i^{\rho_n}|}, \quad \forall\, x \in J_i^{\rho_n},$$

implying that

$$\begin{aligned}
\|\chi_{\theta,n}\|_2^2 &= \int_{L_\theta} |\chi_{\theta,n}|^2 \mathrm{d}m \\
&= \sum_{i=1}^{\lfloor 4/\rho_n \rfloor} \int_{J_i^{\rho_n}} |\chi_{\theta,n}|^2 \mathrm{d}m \\
&\le \sum_{i=1}^{\lfloor 4/\rho_n \rfloor} |J_i^{\rho_n}| \cdot \left(\frac{c_1 \cdot s_{\rho_n,i} \cdot {\rho_n}^d}{|J_i^{\rho_n}|} \right)^2 \\
&\le {c_1}^2 \cdot {\rho_n}^{2d-1} \cdot \sum_{i=1}^{\lfloor 4/\rho_n \rfloor} {s_{\rho_n,i}}^2 \\
&\le {c_1}^2 \cdot {\rho_n}^{2d-1} \cdot N_{(K_1\times K_2)_{\rho_n}}(\theta).
\end{aligned}$$

In view of (4.2), this last expression is bounded above by

$$({c_1}^2 \cdot {\rho_n}^{2d-1}) \cdot (c_3 \cdot {\rho_n}^{1-2d} \cdot m) = {c_1}^2 \cdot c_3 \cdot m,$$

which is a constant independent of n. □

5. Concluding remarks

The proofs of Theorems 1.1 and 1.2 work not just for the case of products of regular Cantor sets, but in greater generality, whenever $K \subset \mathbb{R}^2$ is a Borel set for which there is a constant $c > 0$ such that, for any $x \in K$ and $0 \le r \le 1$,

$$c^{-1} \cdot r^d \le m_d(K \cap B_r(x)) \le c \cdot r^d,$$

since this alone implies the existence of ρ-decompositions for K.

The good feature of the proof is that the discretisation idea may be applied to other contexts. For example, we prove in [3] a Marstrand type theorem in an arithmetical context.

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