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On the class of D_k-symmetrizable matrices[☆]

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Abstract

It is known that for every real square matrix A there exists a nonsingular real symmetric matrix S such that

$$SA = A'S,$$

where A' denotes the transpose of A. Using the notion of an M-matrix we derive a criterion for A to satisfy the above equality with a diagonal S of signature k. Such a matrix A will be called D_k-symmetrizable and the paper presents some results on this concept. In particular we show that a D_k-symmetrizable matrix shares many properties with a real symmetric matrix and that any real matrix A, up to an orthogonal similarity, is D_k-symmetrizable for some k.

1. Introduction

The spectrum localization problem is one of the most discussed problems in linear algebra. It is studied both in its numerical and algebraic aspects. In particular, in stability theory one wants to establish whether the spectrum of a matrix is included in a given region $\mathscr{W}$ of the complex plane without actually computing the eigenvalues of the matrix.

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Motivated by applications to linear and nonlinear stability for a general class of numerical methods for ordinary differential equations [3], as well as by applications to solving systems of linear equations with coefficient matrices that are symmetrizable by a diagonal matrix, we direct our attention to the case when $\mathscr{W}$ is the real axis. The paper proposes the concept of D_k-symmetrizability as a tool to estimate the number of real eigenvalues of a real matrix. For the background we refer the reader to [2–4,6,11–13]. The basic concept of D_k-symmetrizability is introduced in the following section. The main results of the paper are obtained in Section 3. In Section 4, we introduce a concept of S_k-symmetrizability and study how it relates to the concept of D_k-symmetrizability.

2. Notation and preliminaries

By $\mathbb{R}_{n\times n}$ we denote the set of all n-by-n real matrices. Throughout our considerations we use the symbol A for a matrix from $\mathbb{R}_{n\times n}$ and we assume that A is nonsymmetric. For a symmetric A our results hold trivially. We will use 0 to denote both the zero number and the zero vector—the context will make that clear.

For an n-by-r matrix U and index sets $\alpha \subseteq \{1,\dots,n\}$ and $\beta \subseteq \{1,\dots,r\}$ by $U[\alpha,\beta]$ we will denote the submatrix of U with row and column indices in α and β, respectively (if $n=r$ and $\alpha=\beta$ we will set $U[\alpha,\alpha]=U[\alpha]$).

For $X \in \mathbb{R}_{n\times n}$ by $\rho(X)$ and X' we will denote the spectral radius and the transpose of X, respetively. $\mathrm{Tr}(X)$ and X_2 will denote the trace of X and the sum of all 2-by-2 principal minors of X, respectively. As usual the symbols $\otimes$ and $\circ$ denote the Kronecker product and the Hadamard product of matrices, respectively.

Recall that Kronecker matrix multiplication does not require any restriction on the size of the matrices [7]. For our purposes if $X=(x_{ij}) \in \mathbb{R}_{n\times n}$ and $Y \in \mathbb{R}_{n\times n}$, then

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \cdots & x_{1n}Y \\ \vdots & & \vdots \\ x_{n1}Y & \cdots & x_{nn}Y \end{bmatrix}.$$

The Hadamard product of $X=(x_{ij}) \in \mathbb{R}_{n\times n}$ and $Y=(y_{ij}) \in \mathbb{R}_{n\times n}$ (with size restrictions, see [7]) is the matrix $X \circ Y = (x_{ij}y_{ij})$.

For $X=(x_{ij}) \in \mathbb{R}_{n\times n}$ we define:

- The matrix $\check{X} := (\check{x}_{ij}) = X - \mathrm{diag}(x_{11},\dots,x_{nn})$.
- The matrix $M(X) := (|X|E + I) \circ I - |X| = (m_{ij}) \in \mathbb{R}_{n\times n}$, where $E \in \mathbb{R}_{n\times n}$ is the matrix of all ones and $|X| := (|x_{ij}|)$.
- The matrix $\mathtt{M}(X) := (XX') \circ I - X \circ X' = (\widetilde{m}_{ij}) \in \mathbb{R}_{n\times n}$.
- The $(n-1)n/2$-by-n real matrix $L(X) := (l_{ij}) = (I \otimes X' - X' \otimes I)[\alpha,\beta]$, where

$$\alpha = \bigcup_{i=1}^{n} \big\{(i-1)(n+1)+1\big\}$$

and

$$\beta = \bigcup_{i=1}^{n-1} \left\{ i(n+1) - n + 1, i(n+1) - n + 2, \ldots, in \right\}.$$

Consequently,

$$\check{x}_{ij} = \begin{cases} 0 & \text{if } i = j, \\ x_{ij} & \text{if } i \neq j, \end{cases}$$

$$m_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} |x_{ik}| + 1 & \text{if } i = j, \\ -|x_{ij}| & \text{if } i \neq j, \end{cases}$$

$$\widetilde{m}_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} x_{i,k}^2 & \text{if } i = j, \\ -x_{ij}x_{ji} & \text{if } i \neq j \end{cases}$$

and

$$L(X) = \begin{bmatrix} x_{12} & -x_{21} & 0 & \cdots & \cdots & 0 \\ x_{13} & 0 & -x_{31} & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & 0 \\ x_{1n} & 0 & & & 0 & -x_{n1} \\ 0 & x_{23} & -x_{32} & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_{n-1,n} & -x_{n,n-1} \end{bmatrix}.$$

Definition 1. A matrix $X = (x_{ij}) \in \mathbb{R}_{n\times n}$ is said to be *sign-symmetric* if

$$\operatorname{sign}(x_{ij}) = \operatorname{sign}(x_{ji}) \quad \text{for all } 1 \leqslant i \neq j \leqslant n,$$

where

$$\operatorname{sign}(c) = \begin{cases} 1 & \text{if } c > 0, \\ 0 & \text{if } c = 0, \\ -1 & \text{if } c < 0. \end{cases}$$

(Compare with the definition of a strongly combinatorially symmetric matrix in [6].)

Definition 2 [9]. A matrix $X = (x_{ij}) \in \mathbb{R}_{n\times n}$ is said to be *combinatorially symmetric* if $x_{ij} \neq 0$ implies $x_{ji} \neq 0$.

By $\mathscr{T}$ we denote the set of all *diagonal matrices* $T \in \mathbb{R}_{n\times n}$ with diagonal entries ± 1.

By $\mathbb{S}$ we denote the set of all *symmetric matrices* $S \in \mathbb{R}_{n\times n}$ with entries from $\{-1, 0, 1\}$.

Definition 3. For a symmetric nonsingular matrix $X \in \mathbb{R}_{n\times n}$ we define its *signature* $s(X)$ as the absolute value of the difference of the number of positive eigenvalues and the number of negative eigenvalues.

We observe that for real symmetric nonsingular matrices the signatures come in steps of two: if for example one positive eigenvalue moves to the negatives, then the difference recorded in the signature changes by two. Moreover, observe that if $D_1, D_2 \in \mathbb{R}_{n\times n}$ are diagonal and $s(D_1) = n$, then $s(D_1 D_2) = s(D_2)$.

Definition 4. A matrix $X \in \mathbb{R}_{n\times n}$ is said to be *D_k-symmetrizable* if there exists a nonsingular diagonal matrix $D \in \mathbb{R}_{n\times n}$ with $s(D) = k$ and

$$DX = X'D.$$

Following [6,13] we observe that the D_k-symmetrizability of a matrix X defined via left diagonal multiplication can also be expressed via diagonal similarity

$$D^{1/2} X D^{-1/2} = D^{-1/2} X' D^{1/2}.$$

Comparing Definitions 3 and 4, it is clear that sign-symmetry of X is a necessary condition of D_k-symmetrizability.

Using [6] we observe that a reducible sign-symmetric matrix X necessarily has to be completely reducible, i.e., permutation similar to a direct sum. Then the question of D_k-symmetrizability becomes a question about D_k-symmetrizability for each of its summands. Hence the D_k-symmetrizability question for real sign-symmetric matrices reduces to the D_k-symmetrizability question for irreducible sign-symmetric matrices. We recall the notion of irreducibility.

Definition 5 [7]. A real matrix $X \in \mathbb{R}_{n\times n}$ is *reducible* if either

(i) $n = 1$ and $X = [0]$ or
(ii) $n \geqslant 2$ and there is a permutation matrix $P \in \mathbb{R}_{n\times n}$ and some integer r with $1 \leqslant r \leqslant n - 1$ such that
$$P'XP = \begin{bmatrix} Y & V \\ \Theta & Z \end{bmatrix}.$$
Here Y is an r-by-r matrix, Z is an $(n-r)$-by-$(n-r)$ matrix and Θ is the zero $(n-r)$-by-r matrix.

A square matrix X is *irreducible* if it is not reducible.

Following the discussion preceding Definition 5 we additionally specify our matrix A to be irreducible from now on.

Definition 6 [1]. A matrix $X \in \mathbb{R}_{n\times n}$ is an *M-matrix* if it can be expressed in the form

$$X = cI - Y$$

with a nonnegative Y and $c \geqslant \rho(Y)$.

If $c = \rho(Y)$, then X is a singular M-matrix, otherwise it is a nonsingular M-matrix.

For a nonsingular M-matrix X, its minimal real eigenvalue is positive [1,8]. It will be denoted by $q(X)$.

Definition 7. Let $X \in \mathbb{R}_{n\times n}$ be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

with square and nonsingular X_{11} and X_{22}. Then the matrices

$$[X/X_{11}] := X_{22} - X_{21}X_{11}^{-1}X_{12}$$

and

$$[X/X_{22}] = X_{11} - X_{12}X_{22}^{-1}X_{21}$$

are called the *Schur complement* in X of X_{11} or X_{22}, respectively.

We close this section by recalling some notions from graph theory.

The *digraph* $G = (V, H)$ is an ordered pair of two finite sets V and H, where the set H consists of some ordered pairs of elements of V, i.e., H is included in the cartesian product of V with itself. The elements of V are called *vertices* and the elements of H are called *arcs*. A graph $G_1 = (V_1, H_1)$ is called a *subgraph* of G if $V_1 \subseteq V$ and $H_1 \subseteq H$. A *path* π from vertex i to vertex j, $i \neq j$, is a sequence $i = i_0, i_1, \ldots, i_k = j$ of distinct vertices where $(i_0, i_1), \ldots, (i_{k-1}, i_k)$ are arcs. The length of the path π is k. A *p-cycle* in G is a sequence γ of vertices $i_1, \ldots, i_p, i_{p+1} = i_1$ where $p \geqslant 2$, in which $i_1, \ldots, i_p$ are distinct and $(i_1, i_2), \ldots, (i_{p-1}, i_k), (i_p, i_1)$ are arcs. A *connected* graph is defined as a graph that contains an undirected path between any two distinct vertices. A graph is called a *tree* if it is connected and does not contain any cycles. The subgraph F of the connected graph G is called a *spanning tree* of G if F is a tree and if F contains each vertex of G. The arcs of G not contained in F are called the *chords* of F.

With a given n-by-n matrix $A = (a_{ij})$ we associate a digraph $G(A)$ with n vertices in the following way: $G(A) = (N, H)$, where $N = \{1, \ldots, n\}$ and H is the set of such arcs (i, k) for which $a_{ik} \neq 0$. If $i_1, i_2, \ldots, i_p, i_1$ is a p-cycle in $G(A)$, $p \geqslant 2$, then the sequence $\{a_{i_s i_{s+1}}\}_{s=1}^{p}$, $i_{p+1} = i_1$, is a p-cycle in A.

If $\hat{a}_p = \{a_{i_s i_{s+1}}\}_{s=1}^{p}$, $i_{p+1} = i_1$, is a p-cycle in A, then we denote the transposed cycle $\{a_{i_{s+1} i_s}\}_{s=1}^{p}$, $i_{p+1} = i_1$, by $\hat{a}'_p$.

3. Results for D_k-symmetrizable matrices

We start by showing that D_k-symmetrizable matrices share many natural properties with the real symmetric matrices.

Proposition 1. *Let A be D_k-symmetrizable, $\beta \subset \{1, \ldots, n\}$, and let $s(D[\beta]) = k_\beta$. Then $A[\beta]$ is $\widetilde{D}_{k_\beta}$-symmetrizable, where $\widetilde{D} = D[\beta]$.*

The proof is obvious.

Proposition 2. *Let A be D_k-symmetrizable and nonsingular. Then A^{-1} is D_k-symmetrizable.*

Proof. From Definition 4 we have

$$DA^{-1} = D(DA)^{-1}D = D(A'D)^{-1}D = (A^{-1})'D. \qquad \square$$

Proposition 3. *Let A and $B \in \mathbb{R}_{n\times n}$ be D_k-symmetrizable and let $AB = BA$. Then AB is D_k-symmetrizable.*

In particular, for every positive integer m, A^m is D_k-symmetrizable if A is.

The proof is similar to the proof of Proposition 2 and is thus omitted.

From Proposition 3 and our comment on D_n-symmetrizability in Section 2 we immediately get the following corollary.

Corollary 1. *Let A and $B \in \mathbb{R}_{n\times n}$ be D_n-symmetrizable and let $AB = BA$. Then AB is sign-symmetric.*

Proposition 4. *Let $B \in \mathbb{R}_{n\times n}$ and let A and B be $D^{(1)}_{k_1}$- and $D^{(2)}_{k_2}$-symmetrizable, respectively, and let $s(D^{(1)} \otimes D^{(2)}) = k_{12}$. Then $A \otimes B$ is $\widetilde{D}_{k_{12}}$-symmetrizable with $\widetilde{D} = D^{(1)} \otimes D^{(2)}$.*

Proof. By Definition 4 and the properties of the Kronecker product [8] we see that

$$\begin{aligned}(D^{(1)} \otimes D^{(2)})(A \otimes B) &= (D^{(1)}A) \otimes (D^{(2)}B)\\ &= (A'D^{(1)}) \otimes (B'D^{(2)})\\ &= (A' \otimes B')(D^{(1)} \otimes D^{(2)})\\ &= (A \otimes B)'(D^{(1)} \otimes D^{(2)}). \qquad \square\end{aligned}$$

Corollary 2. *Let A and $B \in \mathbb{R}_{n\times n}$ be $D^{(1)}_n$- and $D^{(2)}_n$-symmetrizable, respectively. Then $A \otimes B$ is sign-symmetric.*

Proof. Since $s(D^{(1)} \otimes D^{(2)}) = n^2$, the Kronecker product $A \otimes B$ is $\widetilde{D}_{n^2}$-symmetrizable by Proposition 4. To complete the proof we refer to our comment on symmetrizability with "full" signature n in Section 2. □

Proposition 5. *Let $B \in \mathbb{R}_{n\times n}$ be irreducible, let A and B be $D^{(1)}_{k_1}$- and $D^{(2)}_{k_2}$- symmetrizable, respectively, and let $s(D^{(1)} \circ D^{(2)}) = k_{12}$. Then $A \circ B$ is $\widetilde{D}_{k_{12}}$-symmetrizable with $\widetilde{D} = D^{(1)} \circ D^{(2)}$.*

In particular, if B is symmetric, then $A \circ B$ is $D^{(1)}_{k}$-symmetrizable.

Proof. Set $A = (a_{ij}),\ B = (b_{ij}),\ D^{(1)} = \operatorname{diag}(d^{(1)}_1, \dots, d^{(1)}_n)$, and $D^{(2)} = \operatorname{diag}(d^{(2)}_1, \dots, d^{(2)}_n)$. Then a direct calculation yields

$$a_{ij}d^{(1)}_i = a_{ji}d^{(1)}_j \ \text{ and } \ b_{ij}d^{(2)}_i = b_{ji}d^{(2)}_j, \quad 1 \leqslant i, j \leqslant n,$$

from which we obtain

$$a_{ij}b_{ij}d^{(1)}_i d^{(2)}_i = a_{ji}b_{ji}d^{(1)}_j d^{(2)}_j.$$

Therefore

$$(D^{(1)} \circ D^{(2)})(A \circ B) = (A \circ B)'(D^{(1)} \circ D^{(2)}).$$

To complete the proof observe that a symmetric matrix is always D_n-symmetrizable with $D = I$. □

Corollary 3. *Let A be D_k-symmetrizable. Then $A \circ A$ is D_n-symmetrizable. Moreover, if A is nonsingular, then $A \circ A^{-1}$ is D_n-symmetrizable.*

Proof. As $s(D \circ D) = n$ and A^{-1} is D_k-symmetrizable by Proposition 2, the assertion follows directly from Proposition 5. □

Keeping in mind that a D_n-symmetrizable matrix is sign-symmetric (see Section 2), we get the following result as an immediate consequence of Corollary 3.

Observation 1. *Let A be nonsingular and D_k-symmetrizable. Then $A \circ A^{-1}$ is sign-symmetric.*

Proposition 6. *Let $D \in \mathbb{R}_{n\times n}$ be diagonal and nonsingular. Partition A as*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1}$$

for a square nonsingular block A_{11}. Partition D conformally as

$$\begin{bmatrix} D^{(1)} & \Theta \\ \Theta & D^{(2)} \end{bmatrix}. \tag{2}$$

If A is D_k-symmetrizable, then $[A/A_{11}]$ is $D^{(2)}_{k_2}$-symmetrizable, where $k_2 = s(D^{(2)})$.

Proof. Definition 4 together with (1) and (2) yields

$$DA = \begin{bmatrix} D^{(1)}A_{11} & D^{(1)}A_{12} \\ D^{(2)}A_{21} & D^{(2)}A_{22} \end{bmatrix} = \begin{bmatrix} A'_{11}D^{(1)} & A'_{21}D^{(2)} \\ A'_{12}D^{(1)} & A'_{22}D^{(2)} \end{bmatrix} = A'D. \tag{3}$$

Using the Schur complement, (3) yields

$$[DA/D^{(1)}A_{11}] = D^{(2)}A_{22} - D^{(2)}A_{21}(D^{(1)}A_{11})^{-1}D^{(1)}A_{12} = D^{(2)}[A/A_{11}]$$

and

$$\begin{aligned}[A'D/A'_{11}D^{(1)}] &= A'_{22}D^{(2)} - A'_{12}D^{(1)}(A'_{11}D^{(1)})^{-1}A'_{21}D^{(2)} \\ &= (A_{22} - A_{21}A_{11}^{-1}A_{12})'D^{(2)} \\ &= [A/A_{11}]'D^{(2)}.\end{aligned}$$

Since $[DA/D^{(1)}A_{11}] = [A'D/A'_{11}D^{(1)}]$, the assertion follows. □

Proposition 7. *Let $D \in \mathbb{R}_{n\times n}$ be diagonal and nonsingular. Partition A as in* (1) *with a square nonsingular A_{22} and D conformally as in* (2).

If A is D_k-symmetrizable, then $[A/A_{22}]$ is $D^{(1)}_{k_1}$-symmetrizable, where $k_1 = s(D^{(1)})$.

The proof is similar to the proof of Proposition 6 and is thus omitted.

Before we state and prove a characterization of D_k-symmetrizability we note that this matrix property is independent of the diagonal entries of a matrix (see also [6]). Therefore we may prescribe these as is convenient. In particular, we can transform a given singular matrix to a nonsingular one.

Theorem 1. *The following are equivalent.*

(i) *A is D_k-symmetrizable.*
(ii) *There exists $T \in \mathscr{T}$ such that $s(T) = k$, TA is sign-symmetric, and $\mathtt{M}(TA)$ is a singular M-matrix.*
(iii) *There exists $T \in \mathscr{T}$ such that $s(T) = k$, TA is sign-symmetric, $M(A)$ is a nonsingular M-matrix, and $q((M(A))^{-1} \circ M(A)) = 1$.*
(iv) *There are a diagonal $T \in \mathscr{T}$ and a spanning tree $\top$ of $G(A)$ such that*
   (a) *$s(T) = k$ and TA is sign-symmetric,*
   (b) *if for $r > 2$ $\hat{a}_r$ is an r-cycle of TA corresponding to a chord of $\top$ and $\hat{a}'_r$ is the transposed r-cycle, then $\hat{a}_r = \hat{a}'_r$.*

The proof relies on four results from [5,6,9]:

Result 1 [5, Theorem 5.4]. *Let $A = (a_{ij}) \in \mathbb{R}_{n\times n}$, $a_{ij} \leqslant 0$ for $i \neq j$, and suppose that there exists a vector $x > 0$ such that $Ax \geqslant 0$. Then A is a singular M-matrix.*

Result 2 [5, Theorem 5.6]. *Let $A \in \mathbb{R}_{n\times n}$ be a singular irreducible M-matrix. Then A has rank $n-1$ and there exists a vector $y > 0$ such that $Ay = 0$.*

Result 3 [6, Theorem 4]. *Let $A \in \mathbb{R}_{n\times n}$ be an M-matrix. Then $q(A^{-1} \circ A) \leqslant 1$. For irreducible A equality occurs if and only if A is D_n-symmetrizable for D with a positive diagonal.*

Result 4 [9, Theorem 3]. *Let $A = (a_{ij}) \in \mathbb{R}_{n\times n}$ be a real combinatorially symmetric matrix. Then there exists a real diagonal matrix D such that $D^{-1}\breve{A}D$ is symmetric if and only if*

(i) *there is a spanning tree $\top$ of $G(A)$ such that the 2-cycles of A corresponding to the edges of $\top$ are all positive, and*

(ii) *if for $r > 2$ $\hat{a}_r$ is an r-cycle of A corresponding to a chord of $\top$ and $\hat{a}'_r$ is the transposed r-cycle, then $\hat{a}_r = \hat{a}'_r$.*

Proof (*Theorem* 1). (i) $\Rightarrow$ (ii): Let $D = \operatorname{diag}(d_1, \ldots, d_n)$ and define $\widetilde{D} = \operatorname{diag}(\widetilde{d}_1, \ldots, \widetilde{d}_n)$ by setting $\widetilde{d}_i = |d_i|$, $i = 1, \ldots, n$. Thus for some $T \in \mathscr{T}$, D can be expressed as

$$D = \widetilde{D}T \tag{4}$$

from which we immediately get $s(T) = k$.

Using (4) we obtain from Definition 4 that

$$\widetilde{D}(TA) = (TA)'\widetilde{D}, \tag{5}$$

which implies sign-symmetry of TA.

Using the definition of $L(TA)$ we can write (5) as

$$L(TA)\widetilde{d} = 0, \tag{6}$$

with $\widetilde{d} = [\widetilde{d}_1, \ldots, \widetilde{d}_n]^{\mathrm{T}}$.

Then the solvability in $\widetilde{d}$ of linear system (6) is equivalent to the solvability in $\widetilde{d}$ of the linear system:

$$(L(TA))'L(TA)\widetilde{d} = 0. \tag{7}$$

A direct calculation shows that

$$(L(TA))'L(TA) = \mathtt{M}(TA) \tag{8}$$

and therefore (7) becomes

$$\mathtt{M}(TA)\widetilde{d} = 0.$$

By sign-symmetry of TA the off-diagonal entries of $\mathtt{M}(TA)$ are nonpositive and, as $\widetilde{d}$ is positive, the implication in question follows from Theorem 5.4 in [5].

(ii) $\Rightarrow$ (i): Observe that by the irreducibility of A we have that $\mathtt{M}(TA)$ is an irreducible singular M-matrix. Then the system

$$\mathtt{M}(TA)x = 0 \tag{9}$$

has a positive solution $x = [x_1, \dots, x_n]'$ according to Theorem 5.6 in [5]. Using (8), system (9) becomes

$$(L(TA))' L(TA)x = 0$$

from which we get

$$(L(TA)x)'(L(TA)x) = 0. \tag{10}$$

From (10) it follows that

$$L(TA)x = 0.$$

Using the definition of $L(PTA)$, we have

$$\hat{D}(TA) = (TA)'\hat{D} \tag{11}$$

with $\hat{D} = \operatorname{diag}(x_1, \dots, x_n)$. Setting $D = \hat{D}T$, formula (11) becomes $DA = A'D$ with $s(D) = s(T) = k$. So, (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii): The existence of $T \in \mathscr{T}$ such that $s(T) = k$ and TA is sign-symmetric follows from the argument used in the corresponding part of the proof of (i) $\Rightarrow$ (ii). The remaining part of condition (iii) follows from Theorem 4 in [6].

(iii) $\Rightarrow$ (i): As TA is sign-symmetric and $M(TA) = M(A)$ we conclude that TA is $\hat{D}_n$-symmetrizable from Theorem 4 in [6]. Hence we have

$$\hat{D}(TA) = (TA)'\hat{D} \tag{12}$$

with $s(\hat{D}) = n$. Setting $D = \hat{D}T$, formula (12) becomes

$$DA = A'D,$$

where $s(D) = s(T) = k$. So, (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iv): As A is D_k-symmetrizable with a diagonal matrix D, by the reasoning used in the proof of (i) $\Rightarrow$ (ii), there is a diagonal $T \in \mathscr{T}$ such that (iv)(a) holds. Moreover, we can write D as $D = |D|T$ and since $s(|D|) = n$, TA is D_n-symmetrizable by Definition 4. So, again by Definition 4, there is a positive diagonal $\hat{D}$ (the square root of $|D|$) such that $\hat{D}TA\hat{D}^{-1}$ is symmetric. Hence $\hat{D}(\check{TA})\hat{D}^{-1}$ is also symmetric and, as TA is combinatorially symmetric, to show (iv)(b) it suffices to apply Theorem 3 from [9].

(iv) $\Rightarrow$ (i): By Theorem 3 from [9] there is a positive diagonal matrix $\hat{D}$ such that $\hat{D}TA\hat{D}^{-1}$ is symmetric. Then by Definition 4, TA is D_n-symmetrizable with $D = \hat{D}^2$. Setting $\dot{D} = DT$ it is easy to see that $s(\dot{D}) = s(T) = k$ and that A satisfies $\dot{D}A = A'\dot{D}$. Thus A is D_k-symmetrizable and the proof is complete. $\square$

Remark 1. Observe that for a symmetric matrix $C \in \mathbb{R}_{n\times n}$, which is obviously D_n-symmetrizable with $D = I$, $\mathtt{M}(C)$ is a singular M-matrix by Theorem 2. Thus our theorem provides a way to construct singular M-matrices.

Proposition 8. *Let $S \in \mathtt{S}$ and let A be D_k-symmetrizable. Then $S \circ A$ is D_k-symmetrizable as well.*

As S is D_n-symmetrizable with $D = I$, the proof is a direct consequence of Proposition 5.

As one application of D_k-symmetrizability we now evaluate the number of real eigenvalues of a real matrix.

Theorem 2. *Let A be D_k-symmetrizable.*

(i) *All the eigenvalues of A are real and $s(A) = s(DA)$ if $k = n$.*
(ii) *A has at least k real eigenvalues if $k < n$.*

Proof
(i) See Theorem 7.6.3 in [7].
(ii) Observe that $A = D^{-1}(DA)$ for the symmetric matrix DA and use Corollary 2 in [11]. □

For D_n-symmetrizable matrices we can modify Theorem 2 as follows:

Theorem 3. *Let A be D_n-symmetrizable and let $S \in \mathbb{S}$. Then all the eigenvalues of $S \circ A$ are real. Moreover, all the eigenvalues of any principal submatrix of A are real.*

Proof. The first part of the assertion is a direct consequence of Proposition 8 and Theorem 7.6.3 from [7]. Let α be a proper subset of $N = \{1, \ldots, n\}$ and let $S \in \mathscr{S}$ be such that $S[N\backslash\alpha]$ is the zero matrix.

Then, as $(S \circ A)[N\backslash\alpha]$ is the zero matrix and all the eigenvalues of $S \circ A$ are real, the remaining part of the assertion follows. □

With some further assumptions on A we can improve Theorem 2 for $k < n$. In particular, we can extract a set $\mathscr{M} \subset \mathbb{R}_{n\times n}$ consisting of matrices with exactly $n - 2$ real eigenvalues.

Theorem 4. *Let A be D_{n-2}-symmetrizable and let*

$$(\mathrm{Tr}(A))^2 < \frac{2n}{n-1} A_2. \tag{13}$$

Then A has exactly $n - 2$ real eigenvalues.

The assertion follows from Corollary 2 in [12] and Criterion 1 in [10].

Example. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then A is D_0-symmetrizable with $D = \mathrm{diag}(1, -1)$ and satisfies (13). So, by Theorem 4, A has no real eigenvalues.

4. Results on S_k-symmetrizable matrices

Both the theory and applications of D_k-symmetrizability suggest to consider more general symmetrizers, namely to introduce the notion of S_k-symmetrizability.

Definition 8. A matrix $X \in \mathbb{R}_{n\times n}$ is S_k-*symmetrizable* if there exists a nonsingular and symmetric matrix $S \in \mathbb{R}_{n\times n}$ with $s(S) = k$ and

$$SX = X'S. \tag{14}$$

Remark 2. It is known [11–13] that for every matrix $X \in \mathbb{R}_{n\times n}$ there is a symmetric and nonsingular matrix $S \in \mathbb{R}_{n\times n}$ with

$$SX = X'S.$$

Hence, following Definition 8, every real square matrix is S_k-symmetrizable for some k.

The following theorem shows how S_k-symmetrizability relates to D_k-symmetrizability.

Theorem 5. *The following are equivalent*:

(i) *A is S_k-symmetrizable.*
(ii) *There is an orthogonal $Q \in \mathbb{R}_{n\times n}$ such that $Q'AQ$ is D_k-symmetrizable.*

Proof. (i) $\Rightarrow$ (ii): By Schur's unitary triangularization theorem there is an orthogonal $Q \in \mathbb{R}_{n\times n}$ such that QSQ' is diagonal. Since A satisfies (14) we have

$$QSQ'QAQ' = QA'Q'QSQ$$

which, by setting $D = QSQ'$, becomes

$$DQAQ' = (QAQ')'D.$$

As D is nonsingular and $s(D) = s(QSQ') = s(S)$ we see that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): If QAQ' is D_k-symmetrizable, we have

$$DQAQ' = (QAQ')'D \tag{15}$$

with $s(D) = k$. After a slight manipulation (15) becomes

$$SA = A'S$$

with the symmetric and nonsingular matrix $S = Q'DQ$. Since obviously $s(S) = s(D)$, (ii) $\Rightarrow$ (i) holds. □

Corollary 4. *Every $B \in \mathbb{R}_{n\times n}$ is D_k-symmetrizable up to an orthogonal similarity.*

The proof is a direct consequence of Theorem 5 and Remark 2.

We conclude the paper with a result which, in some sense, generalizes the singular value decomposition [7] for the case of real square matrices.

Theorem 6. *For every $B \in \mathbb{R}_{n\times n}$ there exist orthogonal P and Q such that PBQ is sign-symmetric and $PQ \in \mathscr{T}$.*

Proof. From Corollary 4 there is an orthogonal $Q \in \mathbb{R}_{n\times n}$ such that

$$DQ'BQ = (Q'BQ)'D, \tag{16}$$

where D is nonsingular and $S(D) = k$. Hence for some $T \in \mathscr{T}$ we can set $D = |D|T$ and $P = TQ'$ so that (16) becomes $|D|PBQ = (PBQ)'|D|$. Hence, as $s(|D|) = n$, PBQ is D_n-symmetrizable and the orthogonality of P is obvious. It is easy to see that $PQ = TQ'Q = T$ and therefore $PQ \in \mathscr{T}$. $\square$

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