Bounds on the bases of irreducible generalized sign pattern matrices

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Abstract

Li et al. [Z. Li, F. Hall, C. Eschenbach, On the period and base of a sign pattern matrix, Linear Algebra Appl. 212/213 (1994) 101–120] extended the concept of the base (or “index of convergence”) and period from nonnegative matrices to powerful sign pattern matrices. In this paper we study the bases for non-powerful irreducible sign pattern matrices (and more generally, for generalized sign pattern matrices). We obtain sharp upper bounds, together with a complete characterization of the equality cases, of the bases for both primitive and irreducible sign pattern (and generalized sign pattern) matrices. We also show that there exist “gaps” in the base set of the classes of such matrices.

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1. Introduction

The sign of a real number \( a \), denoted by \( \text{sgn} \ a \), is defined to be 1, \(-1\) or 0, according to \( a > 0 \), \( a < 0 \) or \( a = 0 \). The sign pattern of a real matrix \( A \), denoted by \( \text{sgn} \ A \), is the \((0, 1, -1)\)-matrix obtained from \( A \) by replacing each entry by its sign.
The powers (especially the sign patterns of the powers) of a square sign pattern matrix $A$ have recently been studied to some extent (see [4,5,10]). Notice that in the computations of (the signs of) the entries of the power $A^k$, the ambiguous sign may arise when we add a positive sign to a negative sign. So a new symbol “#” has been introduced in [4] to denote the ambiguous sign. For convenience, we call the set $\Gamma = \{0, 1, -1, \#\}$ the generalized sign set and define the addition and multiplication involving the symbol # as follows (the addition and multiplication which do not involve # are obvious):

\[
(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \# \quad \text{(for all } a \in \Gamma),
\]
\[
0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \# \quad \text{(for all } b \in \Gamma \setminus \{0\}).
\]

It is straightforward to check that the addition and multiplication in $\Gamma$ defined in this way are commutative and associative, and the multiplication is distributive with respect to addition.

In [10], the matrices with entries in the set $\Gamma$ are called generalized sign pattern matrices. The addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product (including powers) of the generalized sign pattern matrices are still generalized sign pattern matrices.

From now on we assume that all the matrix operations considered in this paper are operations of the matrices over the set $\Gamma$.

For a generalized sign pattern matrix $A$, we use $|A|$ to denote the $(0, 1)$-matrix obtained from $A$ by replacing each nonzero entry by 1. Clearly $|A|$ completely determines the zero pattern of $A$. Notice that for the operations defined for the generalized sign set $\Gamma = \{0, 1, -1, \#\}$, we have $a + b = 0$ if and only if both $a$ and $b$ are zero (and $a \cdot b = 0$ if and only if one of $a$ and $b$ is zero).

So we have $|AB| = \|A\|\|B\|$ for generalized sign pattern matrices $A$ and $B$. In particular, we have $|A^k| = \|A\|^k$.

It is well known that graph theoretical methods are often useful in the study of the powers of square matrices, so we now introduce some graph theoretical concepts.

A signed digraph $S$ is a digraph where each arc of $S$ is assigned a sign 1 or $-1$. A walk $W$ in a digraph is a sequence of arcs: $e_1, e_2, \ldots, e_k$ such that the terminal vertex of $e_i$ is the same as the initial vertex of $e_{i+1}$ for $i = 1, \ldots, k - 1$. The number $k$ is called the length of the walk $W$, denoted by $l(W)$. The sign of the walk $W$ (in a signed digraph), denoted by $\text{sgn} W$, is defined to be $\prod_{i=1}^{k} \text{sgn}(e_i)$.

Two walks $W_1$ and $W_2$ in a signed digraph are called a pair of SSSD walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.

Let $A = (a_{ij})$ be a square sign pattern matrix of order $n$. The associated digraph $D(A)$ of $A$ (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j) | a_{ij} \neq 0\}$. The associated signed digraph $S(A)$ of $A$ is obtained from $D(A)$ by assigning the sign of $a_{ij}$ to each arc $(i, j)$ in $D(A)$.

We now use the associated signed digraph $S(A)$ to determine the (generalized) sign of the entries $(A^k)_{ij}$ of the power $A^k$ of a square sign pattern matrix $A$. Notice that we have the following formula for $(A^k)_{ij}$ (where $W_k(i, j)$ denotes the set of walks of length $k$ from vertex $i$ to vertex $j$ in $S(A)$):

\[
(A^k)_{ij} = \sum_{W \in W_k(i, j)} \text{sgn}(W).
\]
From this formula we have

(1) \((A^k)_{ij} = 0\) if and only if there is no walk of length \(k\) from \(i\) to \(j\) in \(S(A)\) (i.e., \(W_k(i, j) = \phi\)).

(2) \((A^k)_{ij} = 1\) (or \(-1\)) if and only if \(W_k(i, j) \neq \phi\) and all walks in \(W_k(i, j)\) have the same sign \(1\) (or \(-1\)).

(3) \((A^k)_{ij} = \#\) if and only if there is a pair of \(SSSD\) walks of length \(k\) from \(i\) to \(j\).

**Definition 1.1** [4]. A square generalized sign pattern matrix \(A\) is called powerful if each power of \(A\) contains no \# entry.

It is easy to see from the above relation between matrices and signed digraphs that a sign pattern matrix \(A\) is powerful if and only if the associated signed digraph \(S(A)\) contains no pairs of \(SSSD\) walks.

In [4], Li et al. introduced the concepts of base and period for (powerful) sign pattern matrices which are the generalizations of the concepts of “index of convergence” and period for square nonnegative matrices. Using similar definitions, these concepts can be extended from (powerful) sign pattern matrices to (square) generalized sign pattern matrices.

**Definition 1.2.** Let \(A\) be a square generalized sign pattern matrix of order \(n\) and \(A, A^2, A^3, \ldots\) be the sequence of powers of \(A\). (Since there are only \(4n^2\) different generalized sign patterns of order \(n\), there must be repetitions in the sequence.) Suppose \(A^l\) is the first power that is repeated in the sequence. Namely, suppose \(l\) is the least positive integer such that there is a positive integer \(p\) such that

\[A^l = A^{l+p}.\]  

(1.4)

Then \(l\) is called the generalized base (or simply base) of \(A\), and is denoted by \(l(A)\). The least positive integer \(p\) such that (1.4) holds for \(l = l(A)\) is called the generalized period (or simply period) of \(A\), and is denoted by \(p(A)\).

For convenience, we will also define the corresponding concepts for signed digraphs. Let \(S\) be a signed digraph of order \(n\). Then there is a sign pattern matrix \(A\) of order \(n\) whose signed associated digraph \(S(A)\) is \(S\). We say that \(S\) is powerful if \(A\) is powerful (i.e., \(S\) contains no pair of \(SSSD\) walks). Also the base \(l(S)\) and period \(p(S)\) are defined to be those of \(A\). Namely we define \(l(S) = l(A)\) and \(p(S) = p(A)\).

In this paper we study the (generalized) base of the irreducible sign pattern (and generalized sign pattern) matrices. It was shown in [4, Theorem 4.3] that if an irreducible sign pattern matrix \(A\) is powerful, then \(l(A) = l(|A|)\). This means that the study of the base \(l(A)\) for powerful irreducible sign pattern matrices is essentially the study of the base (i.e., index of convergence) for nonnegative matrices. But if \(A\) is not powerful, then the situation is totally different. Therefore we will mainly consider the non-powerful cases in this paper.

In Section 3 we consider the primitive non-powerful cases, then in Section 4 we consider the imprimitive non-powerful cases and general cases. We obtain sharp upper bounds, together with complete characterization of the equality cases, of the bases for both primitive and irreducible sign pattern (and generalized sign pattern) matrices. We also show that there exist “gaps” in the base set of the classes of such matrices.
2. Some preliminaries

In this section, we introduce some definitions, notations and basic properties which we need to use in the presentations and proofs of our main results in Sections 3 and 4.

A square matrix $A$ (of order $n$) is reducible if there exists a permutation matrix $P$ (of order $n$) such that

$$P A P^T = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix},$$

where $B$ and $C$ are square non-vacuous matrices. $A$ is irreducible if it is not reducible. (For convenience, we say that $A$ and $P A P^T$ are permutation similar.)

A nonnegative square matrix $A$ is primitive if some power $A^k > 0$. The least such $k$ is called the primitive exponent (or simply exponent) of $A$, denoted by $\exp(A)$.

For convenience, a square generalized sign pattern matrix $A$ is called primitive if $|A|$ is primitive, and in this case we define $\exp(A) = \exp(|A|)$.

A digraph $D$ is called a primitive digraph, if there is a positive integer $k$ such that for each vertex $x$ and each vertex $y$ (not necessarily distinct) in $D$, there exists a walk of length $k$ from $x$ to $y$. The least such $k$ is called the primitive exponent (or simply exponent) of $D$, denoted by $\exp(D)$.

It is well known from the basic relations between matrices and digraphs that a square matrix $A$ is irreducible if and only if its associated digraph $D(A)$ is strongly connected, $A$ is primitive if and only if $D(A)$ is primitive, and in this case we have $\exp(A) = \exp(D(A))$.

It is also well known that a digraph $D$ is primitive if and only if $D$ is strongly connected and the greatest common divisor of the lengths of all the cycles of $D$ is 1 (see [1]).

There is an important characterization for powerful irreducible sign pattern matrices given in [4] which will be the starting point of our study on the bases of non-powerful irreducible sign pattern matrices. Theorem 2.1 is the graph theoretical version of this characterization.

Theorem 2.1 [4, Theorem 3.5]. Let $S$ be a strongly connected signed digraph and $h$ be the index of imprimitivity of $S$ (i.e., $h$ is the greatest common divisor of the lengths of all the cycles of $S$). Then $S$ is powerful if and only if $S$ satisfies the following two conditions:

(A1) All cycles in $S$ with lengths even multiples of $h$ (if any) are positive.

(A2) All cycles in $S$ with lengths odd multiples of $h$ have the same sign.

Notice that $S$ contains at least one cycle with length odd multiple of $h$ since $h$ is the greatest common divisor of the lengths of all the cycles of $S$.

Now suppose that $S$ is a primitive non-powerful signed digraph. Then the index of imprimitivity $h$ of $S$ is 1, and $S$ does not satisfy condition (A1) or (A2) in Theorem 2.1 since $S$ is non-powerful. Thus $S$ contains a pair of cycles $C_1$ and $C_2$ (say, with lengths $p_1$ and $p_2$, respectively) satisfying one of the following two conditions:

(B1) $p_i$ is odd and $p_j$ is even (where $\{i, j\} = \{1, 2\}$), and $\text{sgn} \, C_j = -1$.

(B2) Both $p_1$ and $p_2$ are odd and $\text{sgn} \, C_1 = -\text{sgn} \, C_2$.

For convenience, we call such a pair of cycles $C_1$ and $C_2$ (satisfying (B1) or (B2)) a “distin-
guished cycle pair”. Now it is easy to check that if $C_1$ and $C_2$ is a distinguished cycle pair with
lengths \( p_1 \) and \( p_2 \), respectively, then the (closed) walks \( W_1 = p_2 C_1 \) (use \( C_1 \) by \( p_2 \) times) and \( W_2 = p_1 C_2 \) (with the same length \( p_1 p_2 \)) have the different signs:

\[
(\text{sgn } C_1)^{p_2} = -(\text{sgn } C_2)^{p_1}.
\] (2.1)

Another important aspect in the study of the bases of primitive non-powerful sign pattern matrices (in Section 3) is the estimations and computations of the primitive exponents. One upper bound we will use in Section 3 is the following well-known Dulmage–Mendelsohn upper bound [2]:

\[
\exp(D) \leq n + s(n - 2),
\] (2.2)

where \( s \) is the length of the shortest cycle of the primitive digraph \( D \) of order \( n \). Also, a number of upper bounds for \( \exp(D) \) can be established by using the Frobenius numbers defined as below.

Let \( a_1, \ldots, a_k \) be positive integers. Define the Frobenius set \( S(a_1, \ldots, a_k) \) as

\[
S(a_1, \ldots, a_k) = \{r_1 a_1 + \cdots + r_k a_k | r_1, \ldots, r_k \text{ are nonnegative integers}\}.
\]

It is well known, by a lemma of Schur, that if \( \gcd(a_1, \ldots, a_k) = 1 \), then \( S(a_1, \ldots, a_k) \) contains all the sufficiently large positive integers. In this case we define the Frobenius number \( \phi(a_1, \ldots, a_k) \) to be the least integer \( \phi \) such that \( m \in S(a_1, \ldots, a_k) \) for all integers \( m \geq \phi \).

It follows from the above definition that \( \phi(a_1, \ldots, a_k) - 1 \) is not in \( S(a_1, \ldots, a_k) \).

It is also well known that if \( a, b \) are coprime positive integers, then \( \phi(a, b) = (a - 1)(b - 1) \).

Also, by using the formula for the Frobenius numbers of the arithmetical progressions [7], we have

\[
\phi(n - 2, n - 1, n) = \left\lfloor \frac{n - 2}{2} \right\rfloor (n - 2).
\] (2.3)

Let \( v \) be a vertex of a primitive digraph \( D \). The vertex exponent of \( v \), denoted by \( \exp_D(v) \), is defined to be the least positive integer \( k \) such that for each vertex \( u \) in \( D \), there is a walk of length \( k \) from \( v \) to \( u \).

Let \( R = \{l_1, \ldots, l_r\} \) be a set of cycle lengths in a primitive digraph \( D \) such that \( \gcd(l_1, \ldots, l_r) = 1 \). For each vertex \( x \) and vertex \( y \) in \( D \), let \( d(x, y) \) be the distance from \( x \) to \( y \) and let \( d_R(x, y) \) (called “the relative distance from \( x \) to \( y \) with respect to \( R \)”\) be the length of the shortest walk from \( x \) to \( y \) which meets at least one cycle of length \( l_i \) for each \( i = 1, \ldots, r \). Let \( \phi_R = \phi(l_1, \ldots, l_r) \) be the Frobenius number. Then we have the following upper bounds for the primitive exponent and vertex exponent (see [8]):

\[
\exp(D) \leq \phi_R + \max_{x,y \in V(D)} d_R(x, y)
\] (2.4)

and

\[
\exp_D(v) \leq \phi_R + \max_{u \in V(D)} d_R(v, u).
\] (2.5)

These upper bounds will be used in Section 3.

3. The primitive non-powerful cases

We have mentioned above (in Section 1) that the main situation we need to consider in the study of the bases of irreducible sign pattern matrices is the non-powerful situation. In this section we consider the primitive non-powerful cases, while in the next section we consider the imprimitive non-powerful cases and the general cases.
Definition 3.1. Let $S$ be a non-powerful signed digraph. Then the “ambiguous index” of $S$, denoted by $r(S)$, is defined to be the least integer $r$ such that there is a pair of SSSD walks of length $r$ in $S$.

An $m \times n$ matrix with all entries equal to 1 is denoted by $J_{m \times n}$. An $m \times n$ generalized sign pattern matrix $A$ with all entries equal to # is denoted by $#J_{m \times n}$, or simply $#J$ in case the size of the matrix need not be indicated explicitly.

Proposition 3.1. Let $S$ be a primitive, non-powerful signed digraph. Then we have

1. There is an integer $k$ such that there exists a pair of SSSD walks of length $k$ from each vertex $x$ to each vertex $y$ in $S$.
2. If there exists a pair of SSSD walks of length $k$ from each vertex $x$ to each vertex $y$, then there also exists a pair of SSSD walks of length $k+1$ from each vertex $u$ to each vertex $v$ in $S$.
3. The minimal such $k$ (as in (1)) is just $l(S)$, the base of $S$.

Proof. (1) Let $k = \exp(S) + d(S) + r(S)$, where $d(S)$ is the diameter of the digraph $S$. Since $S$ is non-powerful, there exists a pair of SSSD walks $W_1$ and $W_2$ of length $r(S)$, say from vertex $u$ to vertex $v$, in $S$. Now take each vertex $x$ and each vertex $y$ in $S$. Let $P$ be a shortest path in $S$ from $x$ to $y$ with length $d(x, u)$. Then $d(x, u) \leq d(S)$. Let $W$ be a walk of length $\exp(S) + d(S) - d(x, u)$ from $v$ to $y$ (since $S$ is primitive, such a walk exists). Then it is easy to see that $P + W_1 + W$ and $P + W_2 + W$ are a pair of SSSD walks of length $k$ from $x$ to $y$ in $S$.

(2) It is obvious since a primitive digraph must be strongly connected.

(3) Let $A$ be the sign pattern matrix whose associated signed digraph is $S$. Then $l(A) = l(S)$. Notice that in the definition of the base $l(A)$, each repeated power of $A$ must be repeated infinitely many times in the power sequence $A, A^2, A^3, \ldots$. So by the matrix version of (1) and (2), we have $l(A) = \min\{k | A^k = #J\}$ for primitive, non-powerful $S$. Thus (3) follows by the relation $l(A) = l(S)$. □

Lemma 3.1. Let $S$ be a primitive non-powerful signed digraph, $W_1$ and $W_2$ be a pair of SSSD walks of length $r$ from vertex $u$ to vertex $v$. Then we have

1. $l(S) \leq d(S) + r + \exp_S(v)$,
2. $l(S) \leq d(S) + r(S) + \exp(S)$.

Proof. (1) Let $x$ and $y$ be any two (not necessarily distinct) vertices of $S$. Let $P$ be a shortest path in $S$ from $x$ to $u$ with length $d(x, u)$. Then clearly $d(x, u) \leq d(S)$. So $\exp_S(v) + (d(S) - d(x, u)) \geq \exp_S(v)$ and thus there exists a walk $Q$ from $v$ to $y$ of length $\exp_S(v) + d(S) - d(x, u)$. Therefore $P + W_1 + Q$ and $P + W_2 + Q$ is a pair of SSSD walks of length $d(S) + r + \exp_S(v)$ from $x$ to $y$, and so (1) follows from Proposition 3.1.

(2) It follows directly from (1) by taking $r = r(S)$ and the fact $\exp_S(v) \leq \exp(S)$. □

In the remainder of this paper, let $D_1$ and $D_2$ be the primitive digraphs of order $n$ as given in Figs. 3.1 and 3.2, respectively.

Then it is well known from the theory of nonnegative primitive matrices [6] that

$$\exp(D_1) = (n - 1)^2 + 1, \quad \exp(D_2) = (n - 1)^2$$  \hspace{1cm} (3.1)
Lemma 3.2. Let $S_1$ be a non-powerful signed digraph of order $n$ with $D_1$ as its underlying digraph (see Fig. 3.1). Then we have
\begin{equation}
\exp(S_1) = 2(n - 1)^2 + n.
\end{equation}

Proof. First we show that there is a pair of $SSSD$ walks of length $(n - 1)^2 + 1$ from vertex $n - 1$ to vertex 1. For this purpose, let $Q_1$ and $Q_2$ be the paths of lengths 1 and 2 from the vertex $n - 1$ to vertex 1, let $C_{n-1}$ and $C_n$ be the cycles of lengths $n - 1$ and $n$ in $S_1$ (see Fig. 3.1). Take
\begin{align*}
W_1 &= Q_1 + (n - 1)C_{n-1}, \\
W_2 &= Q_2 + (n - 2)C_n.
\end{align*}

Let $P$ be the unique path from vertex 1 to vertex $n - 1$. Then
\begin{align*}
W_1 + P &= nC_{n-1}, \\
W_2 + P &= (n - 1)C_n.
\end{align*}

Since $S_1$ is non-powerful, and $C_{n-1}$ and $C_n$ are the only two cycles of $S_1$, $C_{n-1}$ and $C_n$ must be a distinguished cycle pair by Theorem 2.1. So $nC_{n-1}$ and $(n - 1)C_n$ have different signs by (2.1). Hence $W_1$ and $W_2$ also have different signs, and so are a pair of $SSSD$ walks of length $(n - 1)^2 + 1$.

Now by (2.5) we have (since the vertex 1 is both on the cycle $C_{n-1}$ and the cycle $C_n$)
\begin{equation}
\exp(S_1) = \exp(D_1) \leq \phi(n, n - 1) + d(D_1) \leq (n - 1)(n - 2) + (n - 1) = (n - 1)^2.
\end{equation}

So by the result (1) of Lemma 3.1 (where $S = S_1$, $r = (n - 1)^2 + 1$ and $v = 1$), we have
\begin{equation}
l(S_1) \leq d(S_1) + [(n - 1)^2 + 1] + (n - 1)^2 \leq 2(n - 1)^2 + n.
\end{equation}
Next we show that there is no pair of $SSSD$ walks of length $k = 2(n - 1)^2 + n - 1$ from vertex $n$ to vertex $n$. Let $W_1$ and $W_2$ be any two walks of length $k$ from $n$ to $n$. Then each (closed walk) $W_i$ is a “union” of several cycles $C_{n-1}$ and several (at least one) cycles $C_n$. Thus we have

$$k = l(W_i) = a_in + b_i(n - 1) \quad (a_i \geq 1, b_i \geq 0) \quad (i = 1, 2).$$

So $(a_2 - a_1)n = (b_1 - b_2)(n - 1)$. Write $b_1 - b_2 = nx$; then $a_2 - a_1 = (n - 1)x$. We claim that $x = 0$.

If $x \geq 1$, then $a_2 \geq n$ (since $a_1 \geq 1$), so $k = (a_2 - n)n + b_2(n - 1) + n^2$ which implies $\phi(n, n - 1) - 1 = n^2 - 3n + 1 = k - n^2 = (a_2 - n)n + b_2(n - 1) \in S(n, n - 1)$, contradicting the definition of the Frobenius number $\phi(n, n - 1)$. Similarly we can get a contradiction if $x \leq -1$.

Thus we have $x = 0$. So $a_1 = a_2$, $b_1 = b_2$ and thus $\text{sgn}(W_1) = \text{sgn}(W_2)$. This argument shows that

$$l(S_1) \geq 2(n - 1)^2 + n.$$

Combining the above two inequalities we obtain $l(S_1) = 2(n - 1)^2 + n$. □

**Lemma 3.3.** Let $S_2$ be a non-powerful signed digraph of order $n \geq 3$ with $D_2$ as its underlying digraph (see Fig. 3.2). Then we have

1. If the (only) two cycles of length $n - 1$ of $S_2$ have different signs, then
   $$l(S_2) \leq n^2 - n + 2.$$

2. If the two cycles of length $n - 1$ of $S_2$ have the same sign, then
   $$l(S_2) = 2(n - 1)^2 + (n - 1).$$

**Proof.** (1) In Fig. 3.2, let $Q_1 = (n - 1, 1) + (1, 2)$ and $Q_2 = (n - 1, n) + (n, 2)$ be two paths of length 2 from vertex $n - 1$ to vertex 2. If the two cycles of length $n - 1$ of $S_2$ have different signs, then we must have $\text{sgn} Q_1 = -\text{sgn} Q_2$, so clearly $r(S_2) \leq 2$. Thus we have $l(S_2) \leq d(S_2) + r(S_2) + \exp(S_2) \leq n - 1 + 2 + (n - 1)^2 = n^2 - n + 2$.

(2) If the two cycles of length $n - 1$ in $S_2$ have the same sign, then $\text{sgn} Q_1 = \text{sgn} Q_2$. Also each cycle of length $n - 1$ and the cycle of length $n$ will form a distinguished cycle pair by Theorem 2.1, since $S_2$ is non-powerful and the only three cycles of $S_2$ are the two cycles of length $n - 1$ and one cycle of length $n$. So $nC_{n-1}$ and $(n - 1)C_n$ will have different signs by (2.1).

Now let $P_1 = (n, 1)$ be the (unique) path of length 1 from $n$ to 1, and $P_2 = (n, 2) + Q + (n - 1, 1)$ be the path of length $n - 1$ from $n$ to 1, where $Q$ is the unique path from 2 to $n - 1$. Let $P = (1, 2) + Q + (n - 1, n)$ be the unique path (of length $n - 1$) from 1 to $n$ and let

$$W_1 = P_1 + (n - 2)C_n, \quad W_2 = P_2 + (n - 2)C_{n-1}.$$ 

Then $W_1 + P = (n - 1)C_n$ and $W_2 + P = nC_{n-1}$. So $W_1$ and $W_2$ have different signs and thus are a pair of $SSSD$ walks of length $(n - 1)^2$ in $S_2$. So we have $r(S_2) \leq (n - 1)^2$. Thus by Lemma 3.1 and $\exp(D_2) = (n - 1)^2$ we have

$$l(S_2) \leq d(S_2) + r(S_2) + \exp(S_2) \leq 2(n - 1)^2 + (n - 1).$$

Next we show that there is no pair of $SSSD$ walks of length $k = 2(n - 1)^2 + n - 2$ from vertex 1 to vertex $n$ in $S_2$. Let $W_1$ and $W_2$ be any two walks of length $k$ from 1 to $n$. Then each $W_i$ is a “union” of the unique path $P$ from 1 to $n$ (of length $n - 1$) and several cycles of length $n - 1$ and several cycles of length $n$. Thus we have
\[ k = l(W_i) = a_in + b_1(n - 1) + n - 1 \quad (a_i, b_i \geq 0) \quad (i = 1, 2). \]

So \((a_2 - a_1)n = (b_1 - b_2)(n - 1)\). Write \(b_1 - b_2 = nx\); then \(a_2 - a_1 = (n - 1)x\). We claim that \(x = 0\).

If \(x \geq 1\), then \(b_1 \geq n\), so \(k = a_in + (b_1 - n)(n - 1) + (n - 1)(n + 1)\), which implies \(\phi(n, n - 1) - 1 = n^2 - 3n + 1 = k - (n - 1)(n + 1) = a_in + (b_1 - n)(n - 1) \in S(n, n - 1)\), contradicting the definition of \(\phi(n, n - 1)\). A similar contradiction can be obtained if \(x \leq -1\). Thus we have \(x = 0\). So \(a_1 = a_2, b_1 = b_2\) and thus \(\text{sgn}(W_1) = \text{sgn}(W_2)\) (because the two cycles of length \(n - 1\) have the same sign). This shows that

\[ l(S_1) \geq 2(n - 1)^2 + (n - 1). \]

So we obtain \(l(S_1) = 2(n - 1)^2 + n - 1\). \(\square\)

**Lemma 3.4.** Let \(S\) be a primitive non-powerful signed digraph of order \(n \geq 5\) with \(D\) as its underlying digraph where \(D\) is not isomorphic to \(D_1\) or \(D_2\). Then we have

\[ l(S) \leq 2n^2 - 4n + 5. \]

**Proof.** Let \(s\) be the length of the shortest cycle of \(D\). Then \(s \leq n - 2\) since \(D\) is not isomorphic to \(D_1\) or \(D_2\). So by the Dulmage–Mendelsohn upper bound (2.2) for the exponents of primitive digraphs we have

\[ \exp(S) = \exp(D) \leq n + s(n - 2) \leq n + (n - 2)^2 = n^2 - 3n + 4. \] (3.3)

Since \(S\) is primitive non-powerful, there is a distinguished cycle pair \(C_1\) and \(C_2\) (with lengths, say, \(p_1\) and \(p_2\), respectively) by Theorem 2.1, where \(p_1C_2\) and \(p_2C_1\) have different signs by (2.1).

**Case 1.** \(C_1\) and \(C_2\) have no common vertices.

Then \(p_1 + p_2 \leq n\). Let \(Q\) be a shortest path from \(C_1\) to \(C_2\) with length \(q\). Then \(q \leq n - p_1 - p_2 + 1\), and \(p_2C_1 + Q\) and \(Q + p_1C_2\) is a pair of SSSD walks with length \(p_1p_2 + q\). So we have

\[ r(S) \leq p_1p_2 + q \leq p_1p_2 + n - p_1 - p_2 + 1 = (p_1 - 1)(p_2 - 1) + n \]
\[ \leq \left[ \frac{1}{2}(p_1 + p_2 - 2) \right]^2 + n \leq \left[ \frac{1}{2}(n - 2) \right]^2 + n = \frac{n^2}{4} + 1. \]

Thus by Lemma 3.1 we have

\[ l(S) \leq d(S) + r(S) + \exp(S) \leq (n - 1) + \frac{n^2}{4} + 1 + n^2 - 3n + 4 \]
\[ = \frac{5}{4}n^2 - 2n + 4 < 2n^2 - 4n + 5. \]

**Case 2.** \(C_1\) and \(C_2\) have some common vertices.

**Subcase 2.1.** \(p_1 = p_2\). Then \(C_1\) and \(C_2\) is also a pair of SSSD walks (since \(C_1\) and \(C_2\) have common vertices) of length \(p_1\). Thus we have \(r(S) \leq p_1 \leq n\). So we have
Subcase 2.2. \( \min(p_1, p_2) \leq n - 2 \).

Now \( p_2 C_1 \) and \( p_1 C_2 \) is a pair of SSSD walks (since \( C_1 \) and \( C_2 \) have common vertices) of length \( p_1 p_2 \). So we have

\[
r(S) \leq p_1 p_2 \leq (n - 2)n
\]

and thus

\[
l(S) \leq d(S) + r(S) + \exp(S) \leq n - 1 + (n - 2)n + n^2 - 3n + 4
\]

\[
= 2n^2 - 4n + 3 < 2n^2 - 4n + 5.
\]

Subcase 2.3. \( \{p_1, p_2\} = \{n - 1, n\} \).

Then similar to Subcase 2.2 we have

\[
r(S) \leq p_1 p_2 \leq (n - 1)n.
\]

In this case if \( s \leq n - 3 \), then \( \exp(S) \leq n + s(n - 2) \leq n^2 - 4n + 6 \). So we have

\[
l(S) \leq d(S) + r(S) + \exp(S) \leq n - 1 + (n - 1)n + n^2 - 4n + 6 = 2n^2 - 4n + 5.
\]

Now if \( s = n - 2 \), then \( n, n - 1, n - 2 \) are all the cycle lengths of \( S \). By (2.3) we have

\[
\phi(n - 2, n - 1, n) = \left\lfloor \frac{n - 2}{2} \right\rfloor (n - 2) \leq \frac{1}{2} (n - 2)^2.
\]

Also, for each vertex \( x \) and each vertex \( y \) in \( D \), we have either

\[
d_{[n-2, n-1, n]}(x, y) = d(x, y) \quad \text{(if } d(x, y) \geq 2)\]

or (by adding a cycle of length \( n \) to the shortest path from \( x \) to \( y \))

\[
d_{[n-2, n-1, n]}(x, y) \leq d(x, y) + n \quad \text{(if } d(x, y) \leq 1)\]

In both cases we will have

\[
d_{[n-2, n-1, n]}(x, y) \leq n + 1.
\]

Thus by (2.4) we have

\[
\exp(D) \leq \phi(n - 2, n - 1, n) + \max_{x,y \in V(D)} d_{[n-2, n-1, n]}(x, y) \leq \frac{1}{2} (n - 2)^2 + n + 1.
\]

So by Lemma 3.1 we have

\[
l(S) \leq d(S) + r(S) + \exp(S) \leq n - 1 + (n - 1)n + \frac{1}{2} (n - 2)^2 + n + 1
\]

\[
= \frac{3}{2} n^2 - n + 2 < 2n^2 - 4n + 5. \quad \square
\]

Combining the above lemmas, we have the following result.

**Theorem 3.1.** Let \( S \) be a primitive non-powerful signed digraph of order \( n \geq 5 \). Then we have

1. \( l(S) \leq 2(n - 1)^2 + n = 2n^2 - 3n + 2 \). \hspace{1cm} (3.4)
2. Equality holds in (3.4) if and only if the underlying digraph of \( S \) is isomorphic to \( D_1 \) (in Fig. 3.1).
3. \( l(S) = 2(n - 1)^2 + n - 1 \) if and only if the underlying digraph of \( S \) is isomorphic to \( D_2 \) (in Fig. 3.2) whose two cycles of length \( n - 1 \) have the same sign in \( S \).
(4) For each integer \( k \) with \( 2n^2 - 4n + 5 < k < 2n^2 - 3n + 1 \), there is no primitive non-powerful signed digraph \( S \) of order \( n \) with \( l(S) = k \).

**Proof.** Combining Lemmas 3.2–3.4. □

The result (4) of Theorem 3.1 means that there exist “gaps” in the base set of the class of primitive non-powerful signed digraphs of order \( n \).

**Remark.** It is easy to check directly that the results of Theorem 3.1 are also true for \( n \leq 4 \).

### 4. The imprimitive non-powerful cases and general cases

In this section we first consider the imprimitive non-powerful cases, and then consider the general cases by combining the (primitive and imprimitive) non-powerful cases with the powerful cases studied in [4].

A square irreducible matrix \( A \) is called imprimitive if it is not primitive. It is well known that if \( A \) is imprimitive with index of imprimitivity \( p \) (i.e., \( p = p(|A|) \)), then \( A \) is permutation similar to a matrix of the following block partitioned form (also called the “imprimitive normal form” of \( A \)):

\[
\begin{pmatrix}
0 & A_1 & 0 & \cdots & 0 \\
0 & 0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & A_{p-1} \\
A_p & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(4.1)

where the zero blocks along the diagonal are square. If in (4.1) the block \( A_i \) is of the size \( n_i \times n_{i+1} \) (i = 1, \ldots, p, where the subscripts are read \( \text{mod } p \)), then we denote the matrix (4.1) as \((n_1, A_1, n_2, \ldots, n_p, A_p, n_1)\), or simply \((A_1, \ldots, A_p)\) in case the sizes of the blocks need not be indicated explicitly (see [9]). For convenience, we also define \( A_{j+p} = A_j \) for all \( j \) and define

\[
A_i(m) = A_iA_{i+1}\cdots A_{i+m-1}
\]

(4.2)
to be the product of \( m \) successive matrices (where \( A_i(0) \) is defined to be the identity matrix of order \( n_i \)).

In order to give a formula for the powers of the matrix (4.1), we introduce another notation. Let \( m \) be a nonnegative integer and \( Z_i \) be a matrix of the size \( n_i \times n_{i+m} \) (\( i = 1, \ldots, p \, \text{mod } p \)). We define \((Z_1, Z_2, \ldots, Z_p)_m\) to be the block partitioned matrix \((A_{ij}) \) (\( i, j = 1, \ldots, p \)) with the blocks

\[
A_{ij} = \begin{cases} 
Z_i & \text{if } j - i \equiv m \, \text{mod } p, \\
0 & \text{otherwise}.
\end{cases}
\]

It is easy to see from this definition that \((A_1, \ldots, A_p)_1 = (A_1, \ldots, A_p)\).

Using these notations together with the recursive computations, we have the following formula for the power \( A^m \) of \( A = (A_1, \ldots, A_p) \) (also see [9]):

\[
A^m = (A_1(m), \ldots, A_p(m))_m.
\]

(4.3)
The following necessary and sufficient condition for a nonnegative matrix $A$ of the form (4.1) to be irreducible with period $p$ is an important fact which is needed in this section.

**Lemma 4.1** [1]. Let $A = (A_1, \ldots, A_p)$ be a matrix of the form (4.1). Then the following two conditions are equivalent:

1. $A$ is irreducible with index of imprimitivity $p$ (i.e., $p(|A|) = p$).
2. Each block $A_i$ ($1 \leq i \leq p$) contains no zero row and no zero column, and each $A_i(p)$ ($1 \leq i \leq p$) is a primitive matrix.

Lemma 4.4 considers when an irreducible sign pattern matrix of the form (4.1) with index of imprimitivity $p$ is not powerful. Before proving Lemma 4.4, we give some basic facts in Lemmas 4.2 and 4.3.

**Lemma 4.2.** Let $B_1, \ldots, B_k$ be generalized sign pattern matrices without zero rows or zero columns such that the number of columns of $B_i$ is equal to the number of rows of $B_{i+1}$ ($i = 1, \ldots, k-1$). Then

1. If some $B_i$ contains a # entry, then the product $B_1 \cdots B_k$ also contains a # entry.
2. If some $B_i = \#J$, then also $B_1 \cdots B_k = \#J$.

The proof of Lemma 4.2 is straightforward, and so is omitted.

**Lemma 4.3.** Let $X$ and $Y$ be $m \times n$ and $n \times m$ generalized sign pattern matrices without zero rows or zero columns. Then

1. $|l(XY) - l(YX)| \leq 1$.
2. $XY$ is powerful if and only if $YX$ is powerful.
3. $XY$ is primitive if and only if $YX$ is primitive.

**Proof.** By Lemma 4.2 and the fact that $(YX)^{k+1} = Y(XY)^kX$. □

**Lemma 4.4.** Let $A = (A_1, \ldots, A_p)$ be an irreducible sign pattern matrix of the form (4.1) with index of imprimitivity $p$ (i.e., $p(|A|) = p$). Then the following conditions are equivalent:

1. $A$ is not powerful.
2. There exists some $i$ such that $A_i(p)$ is not powerful.
3. For each $j = 1, \ldots, p$, $A_j(p)$ is not powerful.

**Proof.** By Lemma 4.1, each $A_i$ contains no zero row or zero column. It follows that each $A_i(m)$ will contain no zero row or zero column.

(1) $\implies$ (2). Since $A$ is not powerful, some power $A^m$ contains a # entry. By the formula (4.3) it follows that some $A_i(m)$ contains a # entry. Now take an integer $k$ such that $pk \geq m$. Then we have

$$A_i(p)^k = A_i(pk) = A_i(m)A_{i+m}(pk - m).$$

So by Lemma 4.2 $A_i(p)^k$ also contains a # entry, thus $A_i(p)$ is not powerful.
(2) $\implies$ (1). If $A_i(p)$ is not powerful, then some power $A_i(p)^k$ contains a # entry. But $A_i(p)^k = A_i(p^k)$, so by (4.3) $A^{pk}$ contains a # entry, thus $A$ is not powerful.

(3) $\implies$ (2). Obvious.

(2) $\implies$ (3). Assume $j < i$ (the case $j \geq i$ can be proved similarly). Let $X = A_j(i - j)$, $Y = A_i(p - i + j)$. Then we have $A_j(p) = XY$ and $A_i(p) = YX$. So the result follows from Lemma 4.3. □

Now we consider the generalized base of the (irreducible) imprimitive matrices. We will basically adopt the approach as in [9] (for studying the upper bound of the indices of convergence of the nonnegative imprimitive matrices). Without loss of generality we may assume that $A$ is of the form (4.1). Since it is proved in [4] that $l(A) = l(|A|)$ if $A$ is powerful, we will only consider the case where $A$ is not powerful.

Lemma 4.5. Let $A = (A_1, \ldots, A_p)$ be an irreducible sign pattern matrix of the form (4.1) with index of imprimitivity $p$ (i.e., $p(|A|) = p$). Suppose $A$ is not powerful. Then we have

1. There exists some positive integer $k$ such that $A_i(k) = \#J$ for all $i = 1, \ldots, p$.
2. If $A_i(k) = \#J$ for all $i = 1, \ldots, p$, then $A_i(k + 1) = \#J$ for all $i = 1, \ldots, p$.
3. $p(A) = p$.
4. Let $l = \min\{k | A_i(k) = \#J \text{ for all } i = 1, \ldots, p\}$. Then $l(A) = l$.

Proof. (1) By Lemmas 4.1 and 4.4, $A_i(p)$ is primitive non-powerful. It follows from Proposition 3.1 that there exists some integer $r$ such that $A_i(p)^r = \#J$. Take $X = A_1(i - 1)$ and $Y = A_i(p - i + 1)$. Then we have $A_1(p) = XY$ and $A_i(p) = YX$. So by Lemma 4.2 we have

$$A_i(p)^{r+1} = (XY)^{r+1} = Y(XY)^r X = Y(#J)X = #J \quad (i = 1, \ldots, p).$$

Take $k = p(r + 1)$. Then we have $A_i(k) = A_i(p(r + 1)) = A_i(p)^{r+1} = \#J$ for all $i = 1, \ldots, p$.

(2) If $A_i(k) = \#J$, then $A_i(k + 1) = A_i(k)A_{i+k} = \#J$ by Lemma 4.2.

(3) If $l \not\equiv r \pmod{p}$, then $A' \neq A''$ since the nonzero blocks of $A'$ and $A''$ are in the different positions. So we have $p(A) \geq p$. On the other hand, suppose $k$ is the smallest integer such that $A_i(k) = \#J$ for all $i = 1, \ldots, p$. Then we have $A_i(k) = A_i(k + p)$ and thus

$$A^k = (A_1(k), \ldots, A_p(k))_k = (A_1(k + p), \ldots, A_p(k + p))_{k+p} = A^{k+p}.$$

So $p(A) \leq p$ (and $l(A) \leq k$) and hence $p(A) = p$.

(4) From the proof of (3) we already know that $l(A) \leq l$. Also, by the definition of $l$, there exists some $i \in \{1, \ldots, p\}$ such that $A_i(l - 1) \neq \#J$. But $A_i(l - 1 + p) = \#J$ since $l - 1 + p \geq l$. So $A_i(l - 1) \neq A_i(l - 1 + p)$. It follows that $A_i^{l-1} \neq A_i^{l-1+p}$. Since $p = p(A)$ by (3), we conclude that $l(A) > l - 1$. Combining this with $l(A) \leq l$ we have $l(A) = l$. □

The next lemma gives an upper bound of $l(A)$ in terms of the generalized base of the primitive non-powerful matrices $A_i(p)(i = 1, \ldots, p)$.

Lemma 4.6. Let $A = (A_1, \ldots, A_p)$ be an irreducible non-powerful sign pattern matrix of the form (4.1) with index of imprimitivity $p \geq 2$. Suppose $1 \leq i_1 < i_2 < \cdots < i_t \leq p$ and $l_{ij} = l(A_{ij}(p))$ is the generalized base of the primitive non-powerful matrix $A_{ij}(p)$ ($1 \leq j \leq t$). Then

$$l(A) \leq p \max(l_{i_1}, \ldots, l_{i_t}) + p - t.$$  

(4.4)
Proof. Let $h = \max(l_{i_1}, \ldots, l_{i_t})$ and $l = ph + p - t$. Since $l_{i_j} = l(A_{i_j}(p))$ and $h \geq l_{i_j}$, we have $A_{i_j}(ph) = (A_{i_j}(p))^h = #J$. Now we consider $A_k(l)$ for $1 \leq k \leq p$. Note that $$|(i_1, \ldots, i_t)| + |[k, \ldots, k + p - t]| = p + 1 > p,$$ so there exist $j$ and $q$ ($1 \leq j \leq t$, $0 \leq q \leq p - t$) such that $i_j \equiv k + q(\mod p)$. Thus $$A_{ij} = A_{k+q}$$ and $$A_k(l) = A_k(q)A_{k+q}(l - q) = A_k(q)A_{k+q}(ph)A_{k+q+ph}(l - ph - q) = A_k(q)A_{i_j}(ph)A_{k+q+ph}(p - t - q) = #J$$ for all $k = 1, \ldots, p$. By Lemma 4.5 we have $$l(A) \leq l = ph + p - t = p \max(l_{i_1}, \ldots, l_{i_t}) + p - t. \quad \Box$$

Corollary 4.1. Let $A = (n_1, A_1, n_2, \ldots, n_p, A_p, n_1)$ be an irreducible non-powerful sign pattern matrix of the form (4.1) with index of imprimitivity $p$, and let $m = \min(n_1, n_2, \ldots, n_p)$. Then $$l(A) \leq p(2m - 1)^2 + m + 1 - 1. \quad (4.5)$$

Proof. Assume $m = n_i$ for some $i$. Note that $A_i(p)$ is an $n_i \times n_i$ primitive non-powerful matrix with the generalized base $l_i = l(A_i(p)) \leq 2(n_i - 1)^2 + n_i = 2(m - 1)^2 + m$ (by Lemma 4.4 and Theorem 3.1). So by taking $t = 1$ and $i_1 = i$ in Lemma 4.6 we have $l(A) \leq pl_i + p - 1 \leq p(2(m - 1)^2 + m + 1) - 1. \quad \Box$

Now we are ready to prove the upper bound of $l(A)$ for irreducible non-powerful sign pattern matrix $A$.

Theorem 4.1. Let $A$ be an irreducible non-powerful sign pattern matrix of order $n$ with index of imprimitivity $p \geq 2$, and let $n = pr + s$, where $r = \left\lfloor \frac{n}{p} \right\rfloor$ and $0 \leq s \leq p - 1 ([x]$ is the largest integer not exceeding $x$). Then $$l(A) \leq p(2r - 1)^2 + r + s. \quad (4.6)$$

Proof. Without loss of generality, we may assume that $A$ is in the imprimitive normal form $A = (n_1, A_1, n_2, \ldots, n_p, A_p, n_1)$, where $n_1 + n_2 + \cdots + n_p = n$. Let $m = \min(n_1, n_2, \ldots, n_p)$, then $m \leq r$.

Case 1. $m \leq r - 1$. By Corollary 4.1 we have $$l(A) \leq p(2m - 1)^2 + m + 1 - 1 \leq p(2r - 2)^2 + (r - 1) + 1 - 1 < p(2(r - 1)^2 + r) + s.$$ The last inequality follows from the fact that $r \geq m + 1 \geq 2$.

Case 2. $m = r$. By the fact that $n_1 + n_2 + \cdots + n_p = n = pr + s$ and $0 \leq s \leq p - 1$, it is easy to see that there exist $p - s$ indices $i_1, \ldots, i_{p-s}$ with $1 \leq i_1 < i_2 < \cdots < i_{p-s} \leq p$ such that
Let \( i_{1} = i_{2} = \cdots = i_{p-r} = r \). For \( j = 1, \ldots, p-s \), \( A_{ij}(p) \) is an \( r \times r \) primitive non-powerful matrix with the generalized base \( l_{ij} = l(A_{ij}(p)) \leq 2(r-1)^{2} + r \) by Theorem 3.1. Taking \( t = p-s \) in Lemma 4.6, we then have \( l(A) \leq p \max(l_{i_{1}}, \ldots, l_{i_{p-r}}) + p - (p-s) \leq p(2(r-1)^{2} + r) + s \). This completes the proof. \( \square \)

**Corollary 4.2.** Let \( A, n, p, r, s \) as in Theorem 4.1. Then we have

\[
l(A) < 2n^2 - 5n + 5. \tag{4.7}
\]

**Proof.** If \( r = 1 \), then by (4.6) we have \( l(A) \leq pr + s = n < 2n^2 - 5n + 5 \). So we may assume that \( r \geq 2 \). Notice that \( p \geq 2 \) and \( r \geq 2 \) will imply

\[
(n-1)^2 - p(r-1)^2 \geq (pr-1)^2 - p(r-1)^2 = (p-1)(pr^2 - 1) \\
\geq pr^2 - 1 \geq pr + 2p - 1 \geq n + p \geq n + 2.
\]

So \( p(r-1)^2 \leq (n-1)^2 - (n+2) = n^2 - 3n - 1 \). Thus by (4.6) we have

\[
l(A) \leq p(2(r-1)^2 + r) + s = 2p(r-1)^2 + n \leq 2(n^2 - 3n - 1) + n \\
= 2n^2 - 5n - 2 < 2n^2 - 5n + 5. \quad \square
\]

Now by combining our above results for the non-powerful cases with the results in [4] for the powerful cases on the estimations of the generalized bases of irreducible (including primitive and imprimitive) sign pattern matrices, we obtain the following theorem.

**Theorem 4.2.** Let \( A \) be an irreducible sign pattern matrix of order \( n \geq 3 \). Then we have

\[
(1) \quad l(A) \leq 2(n-1)^2 + n. \tag{4.8}
\]

(2) Equality holds in (4.8) if and only if \( A \) is non-powerful and the associated digraph \( D(A) \) of \( A \) is isomorphic to \( D_1 \) (in Fig. 3.1).

(3) \( l(A) = 2(n-1)^2 + n - 1 \) if and only if \( A \) is non-powerful and the associated digraph \( D(A) \) of \( A \) is isomorphic to \( D_2 \) (in Fig. 3.2) whose two cycles of length \( n-1 \) have the same sign in \( S(A) \).

(4) For each integer \( k \) with \( 2n^2 - 4n + 5 < k < 2n^2 - 3n + 1 \), there is no irreducible sign pattern matrix \( A \) of order \( n \) with \( l(A) = k \).

**Proof.** We consider three cases.

**Case 1.** \( A \) is powerful.

Then \( l(A) = l(|A|) \) by [4, Theorem 4.3]. So by the results on the indices of convergence of nonnegative irreducible matrices [3] we have

\[
l(A) = l(|A|) \leq (n-1)^2 + 1 < 2n^2 - 4n + 5.
\]

**Case 2.** \( A \) is non-powerful and imprimitive.

Then \( p = p(A) = p(|A|) \geq 2 \) (by Lemma 4.5). So by Corollary 4.2 we have \( l(A) < 2n^2 - 5n + 5 < 2n^2 - 4n + 5 \).
Case 3. $A$ is non-powerful and primitive.

Then the results follow directly from Theorem 3.1. □

The result (4) of Theorem 4.2 actually means that there exist “gaps” in the base set of the class of irreducible sign pattern matrices of order $n$.

Finally, we would like to point out that if $A$ itself contains a # entry, then also $l(A) \leq 2n^2 - 4n + 5$. To see this, we only need to consider Case 3 of the proof of Theorem 4.2 (Case 1 does not occur and the estimations in Case 2 still hold if $A$ contains a # entry). Now in this case, the “ambiguous index” $r(S) = 1$, since $A$ itself contains a # entry (where $S$ is the associated signed digraph of $A$). Thus by Lemma 3.1 we have (for $n \geq 3$)

$$l(A) = l(S) \leq d(S) + r(S) + \exp(S) \leq n - 1 + 1 + (n - 1)^2 + 1 < 2n^2 - 4n + 5.$$ 

This comment suggests that the results of Theorem 4.2 can be extended to generalized sign pattern matrices as follows.

**Theorem 4.3.** Let $A$ be an irreducible generalized sign pattern matrix of order $n \geq 3$. Then

1. $l(A) \leq 2(n - 1)^2 + n$. \hfill (4.9)
2. Equality holds in (4.9) if and only if $A$ is a non-powerful sign pattern matrix and the associated digraph $D(A)$ of $A$ is isomorphic to $D_1$ (in Fig. 3.1).
3. $l(A) = 2(n - 1)^2 + n - 1$ if and only if $A$ is a non-powerful sign pattern matrix and the associated digraph $D(A)$ of $A$ is isomorphic to $D_2$ (in Fig. 3.2) whose two cycles of length $n - 1$ have the same sign in $S(A)$.
4. For each integer $k$ with $2n^2 - 4n + 5 < k < 2n^2 - 3n + 1$, there is no irreducible generalized sign pattern matrix $A$ of order $n$ with $l(A) = k$.

**References**