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A comparison between the variational iteration method and Adomian decomposition method

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Abstract

In this paper, we present a comparative study between the variational iteration method and Adomian decomposition method. The study outlines the significant features of the two methods. The analysis will be illustrated by investigating the homogeneous and the nonhomogeneous advection problems.

MSC: 47H05; 47H10

Keywords: Variational iteration method; Adomian decomposition method; Advection problems

1. Introduction

This paper outlines a reliable comparison between two powerful methods that were recently developed. The first is the variational iteration method (VIM) developed by He in [5–10,13,14] and used in [1,2,15,16] among many others. The second is Adomian decomposition method (ADM) developed by Adomian in [3,4], and used heavily in the literature in [17–24] and the references therein. The two methods give rapidly convergent series with specific significant features for each scheme. The homogeneous and the nonhomogeneous advection problem

$$u_t + uu_x = f(x,t), \tag{1}$$

where $u = u(x,t)$, will be used as a vehicle for this study. For $f(x,t) = 0$, Eq. (1) reduces to the homogeneous advection model. The nonlinear advection equation (1) arises in the description of various physical processes. The existence of nontrivial exact solutions is the question of physical interest. Such exact solutions are important because numerical solutions may not identify the scientific phenomenon under investigation.

A substantial amount of research work has been directed for the study of the nonlinear problems, and on advection problem in particular. In this paper, our work stems mainly on two of the most recently developed methods, the VIM and ADM. The two methods, which accurately compute the solutions in a series form or in an exact form, are of great interest to applied sciences.

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The main advantage of the two methods is that it can be applied directly for all types of differential and integral equations, homogeneous or inhomogeneous. Another important advantage is that the methods are capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution. The effectiveness and the usefulness of both methods are demonstrated by finding exact solutions to the models that will be investigated. However, each method has its own characteristic and significance that will be examined.

2. The methods

In what follows we will highlight briefly the main points of each of the two methods, where details can be found in [3–14].

2.1. He's variational iteration method

Consider the differential equation

$$Lu + Nu = g(t), \tag{2}$$

where L and N are linear and nonlinear operators, respectively, and $g(t)$ is the source inhomogeneous term.

He [5–14] introduced the variational iteration method where a correction functional for Eq. (2) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\,\mathrm{d}\xi, \tag{3}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

It is obvious now that the main steps of He's variational iteration method require first the determination of the Lagrangian multiplier λ that will be identified optimally. Having determined the Lagrangian multiplier, the successive approximations $u_{n+1}, n \geqslant 0$, of the solution u will be readily obtained upon using any selective function u_0. Consequently, the solution

$$u = \lim_{n\to\infty} u_n. \tag{4}$$

In other words, the correction functional (3) will give several approximations, and therefore the exact solution is obtained at the limit of the resulting successive approximations.

2.2. Adomian decomposition method

Adomian decomposition method [3,4] defines the unknown function $u(x)$ by an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{5}$$

where the components $u_n(x)$ are usually determined recurrently. The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where A_n are the so-called Adomian polynomials of $u_0, u_1, \ldots, u_n$ defined by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots, \tag{7}$$

or equivalently

$$\begin{aligned}
&A_0 = F(u_0),\\
&A_1 = u_1 F'(u_0),\\
&A_2 = u_2 F'(u_0) + \tfrac{1}{2}u_1^2 F''(u_0),\\
&A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \tfrac{1}{3}u_1^3 F'''(u_0),\\
&A_4 = u_4 F'(u_0) + (u_1 u_3 + \tfrac{1}{2}u_2^2) F''(u_0) + \tfrac{1}{2}u_1^2 u_2 F'''(u_0) + \tfrac{1}{24}u_1^4 F^{(iv)}(u_0).\\
&\cdots .
\end{aligned} \tag{8}$$

It is now well known that these polynomials can be generated for all classes of nonlinearity according to specific algorithms defined by (7). Recently, an alternative algorithm for constructing Adomian polynomials has been developed by Wazwaz [20].

The variational iteration method gives several successive approximation through using the iteration of the correction functional. This powerful technique handles both linear and nonlinear equations in a unified manner without any need for the so-called Adomian polynomials. However, Adomian decomposition method provides the components of the exact solution, where these components should follow the summation given in (5). Moreover, the VIM requires the evaluation of the Lagrangian multiplier λ, whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic work.

In what follows, a homogeneous and a nonhomogeneous advection problem will be examined by using the two schemes presented above. The two physical models will be used for illustrative purposes regarding the comparison goal.

3. The homogeneous advection problem

We first consider the homogeneous advection problem [23]

$$\begin{aligned}
&u_t + u u_x = 0,\\
&u(x,0) = -x.
\end{aligned} \tag{9}$$

3.1. Using He's variational iteration method

The correction functional for (9) reads as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right)\mathrm{d}\xi. \tag{10}$$

This yields the stationary conditions

$$\begin{aligned}
&\lambda'(\xi) = 0,\\
&1 + \lambda(\xi) = 0.
\end{aligned} \tag{11}$$

This in turn gives

$$\lambda = -1. \tag{12}$$

Substituting this value of the Lagrangian multiplier into functional (10) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right)\mathrm{d}\xi, \quad n \geqslant 0. \tag{13}$$

As stated before, we can use any selective function for u_0; preferably we use the initial condition $u_0 = -x$. Consequently, using (13) yields the following successive approximations:

$$
\begin{aligned}
&u_0(x,t) = -x,\\
&u_1(x,t) = -x - xt,\\
&u_2(x,t) = -x - xt - xt^2 - \tfrac{1}{3}xt^3,\\
&u_3(x,t) = -x - xt - xt^2 - xt^3 - \tfrac{2}{3}xt^4 - \text{small terms},\\
&u_4(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - \tfrac{14}{15}xt^5 - \text{small terms},\\
&u_5(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - xt^5 - \text{small terms},\\
&\vdots\\
&u_n(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - xt^5 - \cdots - xt^n - \text{small terms}.
\end{aligned}
\tag{14}
$$

Recall that

$$u = \lim_{n\to\infty} u_n, \tag{15}$$

which gives

$$u(x,t) = -x(1 + t + t^2 + t^3 + t^4 + \cdots), \tag{16}$$

which leads to the closed form solution

$$u(x,t) = \frac{x}{t-1}. \tag{17}$$

3.2. Adomian decomposition method

We first rewrite Eq. (9) in an operator form

$$
\begin{aligned}
&Lu = -uu_x,\\
&u(x,0) = -x,
\end{aligned}
\tag{18}
$$

where the differential operator L is

$$L = \frac{\partial}{\partial t}. \tag{19}$$

The inverse L^{-1} is assumed as an integral operator given by

$$L^{-1}(.) = \int_0^t (.)\,\mathrm{d}t. \tag{20}$$

Applying the inverse operator L^{-1} on both sides of (18) and using the initial condition we find

$$u(x,t) = -x - L^{-1}(uu_x). \tag{21}$$

Substituting (5) and (6) into the functional equation (18) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = -x - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{22}$$

where A_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(x,t)$ by $-x$, the remaining components $u_n(x,t),\ n \geqslant 1$, can be determined by using the recurrence relation

$$
\begin{aligned}
&u_0(x,t) = -x,\\
&u_{k+1}(x) = -L^{-1}(A_k), \quad k \geqslant 0,
\end{aligned}
\tag{23}
$$

where A_k are Adomian polynomials that represent the nonlinear term uu_x and given by

$$\begin{aligned} &A_0 = u_0 u_{0_x}, \\ &A_1 = u_0 u_{1_x} + u_1 u_{0_x}, \\ &A_2 = u_0 u_{2_x} + u_1 u_{1_x} + u_2 u_{0_x}, \\ &\cdots . \end{aligned} \tag{24}$$

Other polynomials can be generated in a similar way to enhance the accuracy of approximation.

Combining (23) and (24) yields

$$\begin{aligned} &u_0(x,t) = -x, \\ &u_1(x,t) = -xt, \\ &u_2(x,t) = -xt^2, \\ &u_3(x,t) = -xt^3, \\ &u_4(x,t) = -xt^4, \\ &\cdots . \end{aligned} \tag{25}$$

In view of (25), the solution $u(x,t)$ is readily obtained in a series form by

$$u(x) = -x(1 + t + t^2 + t^3 + t^4 + \cdots), \tag{26}$$

or in a closed form by

$$u(x,t) = \frac{x}{t-1}. \tag{27}$$

4. The nonhomogeneous advection problem

We first consider the nonhomogeneous advection problem [23]

$$\begin{aligned} &u_t + uu_x = 2t + x + t^3 + xt^2, \\ &u(x,0) = 0. \end{aligned} \tag{28}$$

4.1. Using He's variational iteration method

The correction functional for (28) reads as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi) \frac{\partial u_n(x,\xi)}{\partial x} - (2\xi + x + \xi^3 + x\xi^2) \right) \mathrm{d}\xi. \tag{29}$$

Proceeding as before, we find the stationary conditions

$$\begin{aligned} &\lambda'(\xi) = 0, \\ &1 + \lambda(\xi) = 0. \end{aligned} \tag{30}$$

This in turn gives

$$\lambda = -1. \tag{31}$$

Substituting this value of the Lagrangian multiplier into functional (29) gives the iteration formula

$$u_{n+1}(x,t) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3 + u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi} + \tilde{u}_n(x,\xi) \frac{\partial u_n(x,\xi)}{\partial x} \right) \mathrm{d}\xi, \tag{32}$$

obtained upon integrating the source nonhomogeneous term, $n \geqslant 0$. As stated before, we can use any selective function for u_0; preferably we use the initial condition $u(x,0)=0$. Using (32) yields the following successive approximations:

$$\begin{aligned}
u_0(x,t) &= t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3,\\
u_1(x,t) &= t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3,\\
u_2(x,t) &= t^2 + xt - \tfrac{2}{15}xt^5 - \tfrac{7}{72}t^6,\\
u_3(x,t) &= t^2 + xt - \text{small terms},\\
u_4(x,t) &= t^2 + xt - \text{small terms},\\
u_5(x,t) &= t^2 + xt - \text{small terms},\\
&\vdots\\
u_n(x,t) &= t^2 + xt - \text{small terms}.
\end{aligned} \tag{33}$$

Recall that

$$u = \lim_{n\to\infty} u_n, \tag{34}$$

which gives the exact solution

$$u(x,t) = t^2 + xt. \tag{35}$$

4.2. Adomian decomposition method

We first rewrite Eq. (28) in an operator form

$$\begin{aligned}
Lu &= 2t + x + t^3 + xt^2 - uu_x,\\
u(x,0) &= 0,
\end{aligned} \tag{36}$$

where the differential operator L is

$$L = \frac{\partial}{\partial t}. \tag{37}$$

The inverse L^{-1} is assumed as an integral operator given by

$$L^{-1}(.) = \int_0^t (.)\,\mathrm{d}t. \tag{38}$$

Applying the inverse operator L^{-1} on both sides of (36) and using the initial condition we find

$$u(x,t) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3 - L^{-1}(uu_x). \tag{39}$$

Substituting (5) and (6) into the functional equation (36) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \tag{40}$$

where A_n are the so-called Adomian polynomials. Identifying the zeroth component $u_0(x,t)$ by $t^2 + xt + \frac{1}{4}t^4 + \frac{1}{3}xt^3$, the remaining components $u_n(x,t),\ n \geqslant 1$, can be determined by using the recurrence relation

$$\begin{aligned}
u_0(x,t) &= t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3,\\
u_{k+1}(x) &= -L^{-1}(A_k), \quad k \geqslant 0,
\end{aligned} \tag{41}$$

where A_k are Adomian polynomials that were evaluated before in the homogeneous case. This in turn gives the components

$$\begin{aligned} &u_0(x,t) = t^2 + xt + \tfrac{1}{4}t^4 + \tfrac{1}{3}xt^3, \\ &u_1(x,t) = -\tfrac{1}{4}t^4 - \tfrac{1}{3}xt^3 - \tfrac{2}{15}xt^5 - \tfrac{7}{72}t^6 - \tfrac{1}{63}xt^7 - \tfrac{1}{96}t^8, \\ &\cdots. \end{aligned} \tag{42}$$

It is important to recall here that the noise terms appear between the two components u_0 and u_1. The noise terms are identified as the identical terms with opposite signs. We then cancel the noise terms $\pm\frac{1}{4}t^4 \pm \frac{1}{3}xt^3$ between the components u_0 and u_1, and justify that the remaining terms of u_0 satisfy the equation. Consequently, the exact solution is

$$u(x,t) = t^2 + xt. \tag{43}$$

5. Discussions

The main goal of this work is to conduct a comparative study between He's variational iteration method and the Adomian decomposition method. The two methods are powerful and efficient methods that both give approximations of higher accuracy and closed form solutions if existing.

An important conclusion can made here. He's variational iteration method gives several successive approximations through using the iteration of the correction functional. However, Adomian decomposition method provides the components of the exact solution, where these components should follow the summation given in (5). Moreover, the VIM requires the evaluation of the Lagrangian multiplier λ, whereas ADM requires the evaluation of the Adomian polynomials that mostly require tedious algebraic calculations. It is interesting to point out that unlike the successive approximations obtained by the VIM, the ADM provides the solution in successive components that will be added to get the series solution.

More importantly, the VIM reduces the volume of calculations by not requiring the Adomian polynomials, hence the iteration is direct and straightforward. However, ADM requires the use of Adomian polynomials for nonlinear terms, and this needs more work. For nonlinear equations that arise frequently to express nonlinear phenomenon, He's variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.

For nonhomogeneous equations, the appearance of noise terms, if the criterion set in [17] for this appearance exists, will facilitate the calculations. However, the existence of noise terms requires necessary conditions that are not always available.

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