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Hessenberg matrices and the Pell and Perrin numbers

Fatih Yilmaz *, Durmus Bozkurt

Selcuk University, Science Faculty Department of Mathematics, 42250 Campus Konya, Turkey

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ABSTRACT

In this paper, we investigate the Pell sequence and the Perrin sequence and we derive some relationships between these sequences and permanents and determinants of one type of Hessenberg matrices.

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1. Introduction

The Pell and Perrin sequences [1] are defined by the following recurrence relations, respectively:

$$P_n = 2P_{n-1} + P_{n-2}, \quad \text{where } P_1 = 1,\ P_2 = 2,$$

$$R_n = R_{n-2} + R_{n-3}, \quad \text{where } R_0 = 3,\ R_1 = 0,\ R_2 = 2$$

for $n > 2$. The first few values of the sequences are

n	1	2	3	4	5	6	7	8	9
P_n	1	2	5	12	29	70	169	408	985
R_n	0	2	3	1	2	5	5	7	10

* Corresponding author.
E-mail addresses: fyilmaz@selcuk.edu.tr (F. Yilmaz), dbozkurt@selcuk.edu.tr (D. Bozkurt).

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The permanent of a matrix is similar to the determinant but all the signs used in the Laplace expansion of minors are positive. The permanent of an n-square matrix is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n [6].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A contractible on column k, if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk} r_i + a_{ik} r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}^T]^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a nonnegative matrix and B is a contraction of A [2], then

$$perA = perB. \tag{1}$$

It is known that there are a lot of relations between determinants or permanents of matrices and well-known number sequences. For example, in [2], the authors consider the relationships between the sums of Fibonacci and Lucas numbers by Hessenberg matrices.

In [4], Lee defined the matrix

$$\pounds_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$per(\pounds_n) = L_{n-1}$$

where L_n is the nth Lucas number.

In [5], the author investigated general tridiagonal matrix determinants and permanents. Also he showed that the permanent of the tridiagonal matrix based on $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}$, $\{b_i\}$, $\{c_i\}$.

In [3], the authors found $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an $n \times n$ $(1, -1)$ matrix S, such that $perA = \det(A \circ S)$, where $A \circ S$ denotes Hadamard product of A and S. Let S be a $(1, -1)$ matrix of order n, defined with

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}. \tag{2}$$

In the present paper, we consider the Pell and Perrin numbers as determinants and permanents of upper Hessenberg matrices, $A = (a_{ij})$ is an upper Hessenberg matrix if $a_{ij} = 0$ for $i > j + 1$.

2. Determinantal representations of the Pell and Perrin numbers

In this section, we define one type of upper Hessenberg matrix of odd order and show that the permanents of these type of matrices are the Pell numbers. Let $H_n = [h_{ij}]_{n\times n}$ be an n-square matrix with $h_{t,t+2} = 1$, $h_{s,s+2} = -1$ for $t = 2, 4, \ldots, \frac{n-3}{2}$ and $s = 1, 3, \ldots, \frac{n-1}{2}$ and $h_{i,j} = 1$ for $|i - j| \leqslant 1$ and otherwise 0. Namely:

$$H_n = \begin{bmatrix} 1 & 1 & -1 & & & & & \\ 1 & 1 & 1 & 1 & & & 0 & \\ & 1 & 1 & 1 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 1 & 1 & 1 & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}. \tag{3}$$

Theorem 1. *Let H_n be an n-square matrix as in* (3)*, then*

$$perH_n = perH_n^{(n-2)} = P_n$$

where P_n is the nth Pell number.

Proof. By definition of the matrix H_n, it can be contracted on column 1. Let H_n^r be the rth contraction of H_n. If $r = 1$, then

$$H_n^1 = \begin{bmatrix} 2 & 0 & 1 & & & & 0 \\ 1 & 1 & 1 & -1 & & & \\ 0 & 1 & 1 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 1 & -1 \\ & & & & 1 & 1 & 1 \\ 0 & & & & & 1 & 1 \end{bmatrix}.$$

Since H_n^1 also can be contracted according to the first column,

$$H_n^2 = \begin{bmatrix} 2 & 3 & -2 & 0 & 0 & & & \\ 1 & 1 & 1 & 1 & 0 & & 0 & \\ & 1 & 1 & 1 & -1 & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}.$$

Going with this process, we have

$$H_n^3 = \begin{bmatrix} 5 & 0 & 2 & 0 & 0 & & & \\ 1 & 1 & 1 & 1 & 0 & & 0 & \\ & 1 & 1 & 1 & -1 & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}$$

and contracting H_n^3 according to the first column

$$H_n^4 = \begin{bmatrix} 5 & 7 & -5 & 0 & 0 & & & \\ 1 & 1 & 1 & 1 & 0 & & 0 & \\ & 1 & 1 & 1 & -1 & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}.$$

Continuing this method, we obtain the rth contraction

$$H_n^r = \begin{bmatrix} P_{r+1} & 0 & P_r & 0 & 0 & & & \\ 1 & 1 & 1 & -1 & 0 & & 0 & \\ & 1 & 1 & 1 & 1 & & & \\ & & 1 & 1 & 1 & -1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}, \quad \text{if } r \text{ is odd}$$

$$H_n^r = \begin{bmatrix} P_r & P_{r-1}+P_r & -P_r & 0 & 0 & & & \\ 1 & 1 & 1 & 1 & 0 & & 0 & \\ & 1 & 1 & 1 & -1 & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & \ddots & \ddots & \ddots & \ddots & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}, \quad \text{if } r \text{ is even}$$

where $2 \leqslant r \leqslant n-4$. Hence

$$H_n^{n-3} = \begin{bmatrix} P_k & P_{k-1}+P_k & -P_k \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which, by contraction of H_n^{n-3} on column 1,

$$H_n^{n-2} = \begin{bmatrix} P_n & 0 \\ 1 & 1 \end{bmatrix}.$$

By (1), we have $perH_n = perH_n^{(n-2)} = P_n$. $\square$

Let $K(n) = [k_{ij}]$ be $n \times n$ matrix with $k_{11} = 1$, $k_{12} = 2$, $k_{13} = 3$, $k_{21} = 1$, $k_{23} = 1$ and $k_{m,m+1} = k_{m+1,m} = 1$ for $m = 3, 4, 5, \ldots, n-1$ and $k_{p,p+2} = 1$ for $p = 3, 4, \ldots, n-2$. Clearly:

$$K_n = \begin{bmatrix} 1 & 2 & 3 & 0 & & & \\ 1 & 0 & 0 & 0 & & 0 & \\ & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 & 1 \\ & 0 & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}. \tag{4}$$

Theorem 2. *Let K_n be an n-square matrix as in* (3), *then*

$$perK_n = perK_n^{(n-2)} = R_n$$

where R_n is the nth Perrin number.

Proof. By definition of the matrix K_n, it can be contracted on column 1. Namely,

$$K_n^1 = \begin{bmatrix} 2 & 3 & 0 & 0 & & & 0 \\ 1 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{bmatrix}.$$

K_n^1 also can be contracted on the first column,

$$K_n^2 = \begin{bmatrix} 3 & 2 & 2 & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{bmatrix}.$$

Continuing this process, we have

$$K_n^r = \begin{bmatrix} R_{r+1} & R_{r+2} & R_r & 0 & 0 & & 0 \\ 1 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 1 \\ 0 & & & & & 1 & 0 \end{bmatrix}$$

for $1 \leqslant r \leqslant n-4$. Hence

$$K_n^{n-3} = \begin{bmatrix} R_{n-2} & R_{n-1} & R_{n-3} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which by contraction of K_n^{n-3} on column 1, gives

$$K_n^{n-2} = \begin{bmatrix} R_{n-1} & R_n \\ 1 & 0 \end{bmatrix}.$$

By applying (1) we have $perK_n = perK_n^{(n-2)} = R_n$, which is desired. $\square$

Let S be a matrix as in (2) and denote the matrices $H_n \circ S$ and $K_n \circ S$ by A_n and B_n, respectively. Thus

$$A_n = \begin{bmatrix} 1 & 1 & -1 & & & & & \\ -1 & 1 & 1 & 1 & & & 0 & \\ & -1 & 1 & 1 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & -1 & 1 & 1 & 1 & \\ & & & & -1 & 1 & 1 & -1 \\ & 0 & & & & -1 & 1 & 1 \\ & & & & & & -1 & 1 \end{bmatrix}$$

and

$$B_n = \begin{bmatrix} 1 & 2 & 3 & 0 & & & \\ -1 & 0 & 0 & 0 & & 0 & \\ & -1 & 0 & 1 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 & 1 \\ & 0 & & & -1 & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}.$$

Then, we have

$$det(A_n) = perH_n = P_n$$

and

$$det(B_n) = perK_n = R_n.$$

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