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The pressure of the SU(N) lattice gauge theory at large N

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Abstract

We calculate bulk thermodynamic properties, such as the pressure, energy density, and entropy, in SU(4) and SU(8) lattice gauge theories, for the range of temperatures $T \leqslant 2.0T_c$ and $T \leqslant 1.6T_c$, respectively. We find that the $N = 4, 8$ results are very close to each other, and to what one finds in SU(3), and are far from the asymptotic free-gas value. We conclude that any explanation of the high-T pressure (or entropy) deficit must be such as to survive the $N \to \infty$ limit. We give some examples of this constraint in action and comment on what this implies for the relevance of gravity duals.

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1. Introduction

The thermodynamic properties of quantum chromodynamics (QCD), besides being of fundamental interest, are currently at the centre of intense experimental research. One of the most interesting phenomena has to do with the range of temperatures, T, above the phase transition (or crossover) at $T = T_c$, where the theory deconfines and chiral symmetry is restored. Traditionally, the description of this transition assumed that the hadronic phase gives way to a plasma, whose physical degrees of freedom are weakly interacting quarks and gluons. Recent experimental results have, however, challenged this 'simple' picture (for example, see [1] and references therein), and point to a picture of the 'plasma' as a very good fluid in the accessible range of T above T_c. In fact, numerical lattice results had already demonstrated the inadequacy of the simple quark–gluon plasma picture some time ago. Such lattice calculations, both for the pure gauge case [2] and with different kinds of fermions [3], found a large deficit in the pressure and entropy as compared to the Stephan–Boltzmann predictions for a free gluon gas (for pure glue), which remained at the level of more than 10% even at temperatures as high as $T \sim 4T_c$. Further evidence that points in the same direction is the survival of hadronic states above T_c, as seen in recent lattice simulations (for example, see [4] and references therein).

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These lattice calculations, and more recent experimental observations, have attracted considerable attention (see, e.g., [5] for a review). Approaches have ranged from modeling the system in terms of non-interacting quasi-particles with the quantum numbers of quarks and gluons but with temperature-dependent masses [6,7], to using higher-order perturbation theory (restricted by infrared divergences), sometimes including nonperturbative contributions on the dimensionally reduced 3D Euclidean lattice [8], large resummations (e.g., [9] and references therein), or, more recently, a description [10] in terms of a large number of loosely bound states that survive deconfinement and come in various representations of the gauge and flavor groups, and where one can use, for example, the lattice masses measured in [11].

In this Letter we ask whether this pressure (and entropy) deficit is a dynamical feature not just of SU(3) but of all SU(N) gauge theories—and, in particular, whether it survives the $N \to \infty$ limit. In this limit the theory becomes considerably simpler, although not (yet) analytically soluble, and so what happens there should strongly constrain the possible dynamics underlying the phenomenon. For example, in that limit supersymmetric SU(N) gauge theories become dual to weakly coupled gravity models, and in that context we recall the frequently mentioned prediction [12], that the pressure in the strong-coupling limit of the $\mathcal{N} = 4$ and $N = \infty$ supersymmetric gauge theory is 3/4 of its Stephan–Boltzmann value, which is similar to the deficit, referred to above, that one finds in the non-supersymmetric case.

To address this question we calculate the pressure for $T \leqslant 2T_c$ in SU(4), and SU(8) lattice gauge theories and compare the results to similar SU(3) calculations available in the literature (which we supplement where it is useful to do so). Recent calculations of various properties of SU(N) gauge theories [13] have demonstrated that SU(8) is in fact very close to SU(∞) for most purposes and have provided information on the location, β_c, of the deconfining transition for various L_t and N [14,15]. Thus our calculations should provide us with an accurate picture of what happens to the pressure at $N = \infty$.

In the next section we summarise the lattice setup, the relevant thermodynamics, and provide numerical checks that our system is large and homogeneous enough for our thermodynamic relations to be appropriate. We then present our results for the pressure, entropy and related quantities. We discuss the implications of our findings in the concluding section.

2. Lattice setup and methodology

The theory is defined on a discretised periodic Euclidean four-dimensional space–time with $L_s^3 \times L_t$ sites. Here $L_{s,t}$ is the lattice extent in the spatial and Euclidean time directions. The partition function

$$Z(T,V) = \sum_s \exp\left\{-\frac{E_s}{T}\right\} = \exp\left\{-\frac{F}{T}\right\} = \exp\left\{-\frac{fV}{T}\right\} \tag{2.1}$$

defines the free energy F and the free energy density, f, and can be expressed as a Euclidean path integral

$$Z(T,V) = \int DU \exp(-\beta S_W). \tag{2.2}$$

Here $T = (aL_t)^{-1}$ is the temperature and $V = (aL_s)^3$ is the spatial volume. When we change β, so as to change the lattice spacing $a(\beta)$, we change both T and V, if L_s and L_t are kept fixed. In the large-N limit, the 't Hooft coupling $\lambda = g^2 N$ is kept fixed, and so we must scale $\beta = 2N^2/\lambda \propto N^2$ in order to keep the lattice spacing fixed in that limit. We use the standard Wilson action S_W given by

$$S_W = \sum_P \left[1 - \frac{1}{N} \operatorname{Re}\operatorname{Tr} U_P\right]. \tag{2.3}$$

Here P is a lattice plaquette index, and U_P is the plaquette variable obtained by multiplying link variables along the circumference of a fundamental plaquette. We perform Monte Carlo simulations, using the Kennedy–Pendelton heat bath algorithm for the link updates, followed by five over-relaxations of all the SU(2) subgroups of SU(N).

2.1. The method used

In lattice calculations of bulk thermodynamics, one can choose to use either the "integral" method (e.g., [2]) or the "differential" method (e.g., [16] or a new variant [17]) or one can attempt a direct evaluation of the density of states (e.g., [18]). We choose to use the first of these methods since the numerical price involved in using larger values of N drives us to smaller L_t, which means that the lattice spacing is too coarse (about 0.15 fm) for the differential method. We have performed preliminary checks for the applicability of the Wang–Landau algorithm [19] for the evaluation of the density of states in the SU(8) gauge theory, but found it numerically too costly for the present work.

The properties we will concentrate on are the pressure p, the energy density per unit volume ϵ, and the entropy S, as a function of temperature. These are given by

$$p = T\frac{\partial}{\partial V}\log Z(T,V) = \frac{T}{V}\log Z(T,V) = -f, \quad (2.4)$$

$$\epsilon = \frac{T^2}{V}\frac{\partial}{\partial T}\log Z(T,V), \quad (2.5)$$

$$\frac{S}{V} = \frac{\epsilon - f}{T} = \frac{\epsilon + p}{T}, \quad (2.6)$$

where the second equality in the first and last lines follows if the system is large and homogeneous, i.e., if V is large enough. In addition, it is useful to consider the quantity

$$\Delta \equiv \epsilon - 3p = T^5\frac{\partial}{\partial T}\frac{p}{T^4}, \quad (2.7)$$

which vanishes for an ideal gluon plasma. Again the second equality requires a large enough V. To calculate the pressure at temperature T in a volume V with lattice cut-off $a(\beta)$, we express $\log Z$ in the integral form

$$p(T) = \frac{T}{V}\log Z(T,V) = \frac{1}{a^4(\beta)L_s^3 L_t}\int_{\beta_0}^{\beta} d\beta' \frac{\partial \log Z}{\partial \beta'}. \quad (2.8)$$

(There is in general an integration constant, but it will disappear when we regularise the pressure later on in this section.) This integral form is useful because it is easy to see from Eqs. (2.2), (2.3) that

$$\frac{\partial \log Z}{\partial \beta} = -\langle S_W\rangle = N_p\langle u_p\rangle, \quad (2.9)$$

where $N_p = 6L_t L_s^3$ is the total number of plaquettes and $u_p \equiv \mathrm{Re}\,\mathrm{Tr}\,U_P/N$. So the pressure can be obtained by simply integrating the average plaquette over β. This pressure has been defined relative to that of the unphysical 'empty' vacuum and will therefore be ultraviolet-divergent in the continuum limit. To remove this divergence we need to define the pressure relative to that of a more physical system. We shall follow convention and subtract from $p(T)$ its value at $T = 0$, calculated with the same value of the cut-off $a(\beta)$. Thus our pressure will be defined with respect to its $T = 0$ value. Doing so we obtain from Eqs. (2.9), (2.8)

$$a^4\big[p(T) - p(0)\big] = 6\int_{\beta_0}^{\beta} d\beta'\big(\langle u_p\rangle_T - \langle u_p\rangle_0\big), \quad (2.10)$$

where $\langle u_p\rangle_0$ is calculated on some L^4 lattice which is large enough for it to be effectively at $T = 0$. We replace $p(T) - p(0) \to p(T)$, where from now on it is understood that $p(T)$ is defined relative to its value at $T = 0$, and we use $T = (aL_t)^{-1}$ to rewrite Eq. (2.10) as

$$\frac{p(T)}{T^4} = 6L_t^4\int_{\beta_0}^{\beta} d\beta'\big(\langle u_p\rangle_T - \langle u_p\rangle_0\big). \quad (2.11)$$

We remark that when our $L_s^3 L_t$ lattice is in the confining phase, then $\langle u_p\rangle$ is essentially independent of L_t and takes the same value as on a L_s^4 lattice (see below). This should become exact as $N \to \infty$ but is accurate enough even for SU(3). Thus as long as we choose β_0 in Eq. (2.11) such that $a(\beta_0)L_t > 1/T_c$ then the integration constant, referred to earlier, will cancel.

Finally, we evaluate Δ in Eq. (2.7) as follows:

$$\frac{\Delta}{T^4} = T\frac{\partial}{\partial T}\frac{p}{T^4} \quad (2.12)$$

$$= \frac{\partial \beta}{\partial \log T}\frac{\partial}{\partial \beta}\frac{p}{T^4} \quad (2.13)$$

$$= 6L_t^4\big(\langle u_p(\beta)\rangle_0 - \langle u_p(\beta)\rangle_T\big)\frac{\partial \beta}{\partial \log a(\beta)}. \quad (2.14)$$

To evaluate $\partial \log(a(\beta))/\partial\beta$ we can use calculations of the string tension, σ, in lattice units. For example, in [20] the calculated values of $a\sqrt{\sigma}$ are interpolated in β for various N and one can take the derivative of the interpolated form to use in Eq. (2.14). One could equally well use the calculated mass gap or the deconfining temperature. All these choices will give the same result up to modest $O(a^2)$ differences.

2.2. Average plaquette

We see from the above that what we need to do is to calculate average plaquettes closely enough in β so as to be able to perform the numerical integration in β. And we need the average plaquettes not only on the $L_t L_s^3$ lattice but also on a reference '$T=0$' L^4 lattice at each value of β. However, we mostly need values for $\beta \geqslant \beta_c$, where $a(\beta_c)L_t = 1/T_c$, since $p(T) - p(0) \simeq 0$ once $T < T_c$.

We performed calculations in SU(4) on $16^3 \times 5$ lattices and in SU(8) on $8^3 \times 5$ lattices for a range of β values corresponding to $T/T_c \in [0.89, 1.98]$ for SU(4), and to $T/T_c \in [0.97, 1.57]$ for SU(8). Since we use $L_t = 5$, while the data for SU(3) in [2] is for $L_t = 4, 6, 8$, we also performed simulations for SU(3) on $20^3 \times 5$ lattices with $T/T_c \in [1, 2]$. The results are presented in Tables 1–3.

In addition to the finite T calculations we have performed '$T=0$' calculations on 20^4 lattices for SU(3), and on 16^4 lattices for SU(4). These have the advantage of being on the same spatial volumes as the corresponding finite T calculations, and we know from previous calculations [21,22] that, for the range of $a(\beta)$ involved, these volumes are large enough to be, effectively, at zero T. For SU(8), however, using 8^4 lattices would not be adequate for the largest β-values, as we

Table 1
Statistics and results of the Monte Carlo simulations for SU(4)

β	$T>0$		$T=0$	
	s_T	(lattice sweeps) $\times 10^{-3}$	s_0	(lattice sweeps) $\times 10^{-3}$
10.55	0.537478(84)	10	0.537487(81)	5
10.60	0.543862(58)	20	0.543797(25)	15
10.62	0.546212(64)	10	0.546068(33)	10
10.64	0.550279(70)	10	0.548208(16)	20
10.68	0.554213(32)	20	0.552177(16)	20
10.72	0.557649(30)	20	0.555861(14)	20
10.75	0.560051(27)	20	0.558462(13)	20
10.80	0.563923(32)	20	0.562587(16)	20
10.85	0.567592(24)	20	0.566453(17)	20
10.90	0.571107(17)	20	0.570118(16)	20
11.00	0.577707(17)	20	0.576981(11)	20
11.02	0.578985(18)	20	0.578279(11)	20
11.10	0.583911(20)	20	0.583352(12)	20
11.30	0.595398(13)	20	0.595039(10)	20

Table 2
Statistics and results of the Monte Carlo simulations for SU(8)

$T>0$				$T=0$		
β	s_T	L_s	(lattice sweeps) $\times 10^{-3}$	β	s_0	(lattice sweeps) $\times 10^{-3}$
43.90	0.525330(80)	14	5	43.85	0.523819(37)	>20
43.93	0.526873(79)	8	19.5	44	0.528788(18)	>20
44.00	0.531307(50)	10	>20	44.35	0.538491(13)	>20
44.10	0.534164(34)	12	7	44.85	0.549794(9)	>20
44.20	0.536650(70)	14	5	45.7	0.565708(4)	>20
44.30	0.539181(30)	8	20			
44.45	0.542629(38)	8	30			
44.60	0.545812(35)	8	20			
44.80	0.549968(37)	8	30			
45.00	0.553926(38)	8	20			
45.50	0.562992(28)	12	10			

Table 3
Statistics and results of the Monte Carlo simulations for SU(3)

β	$T > 0$		$T = 0$	
	s_T	(lattice sweeps) $\times 10^{-3}$	s_0	(lattice sweeps) $\times 10^{-3}$
5.800	0.568664(100)	10	0.567667(29)	11
5.805	0.569688(153)	20	0.568438(23)	11
5.810	0.570624(55)	10	0.569218(18)	11
5.815	0.571297(81)	10	0.569996(26)	11
5.820	0.572205(78)	10	0.570788(16)	11
5.900	0.583058(38)	10	0.581854(20)	11
6.150	0.609377(27)	10	0.608971(8)	11
6.200	0.613966(31)	10	0.613628(13)	11

will see below. (The same is not true for the finite T calculation on $8^3 \times 5$ lattices where it is $1/aT$ that sets the scale for finite volume corrections.) We, therefore, take instead the SU(8) calculations on larger lattices in [22], and interpolate between the values of β used there, to obtain average plaquettes at the values of β we require. To perform this interpolation we fit with the ansatz

$$\langle u_p\rangle_0(\beta) = \langle u_p\rangle_0^{\text{P.T.}}(\beta) + \frac{\pi^2}{12}\frac{G_2}{N\sigma^2}\big(a\sqrt{\sigma}\big)^4 + c_4 g^8 + c_5 g^{10}, \tag{2.15}$$

where $\langle u_p\rangle_0^{\text{P.T.}}(\beta)$ is the lattice perturbative result to $\mathcal{O}(g^6)$ from [23] and $N = 8$. Our best fit has $\chi^2/\text{dof} = 0.93$ with $\text{dof} = 2$, and the best fit parameters are $c_4 = -6.92$, $c_5 = 26.15$, and a gluon condensate of $\frac{G_2}{N\sigma^2} = 0.72$.

For the scaling of the lattice spacing with β, needed in Eq. (2.15) and Eq. (2.14) and in the temperature scale, we used the interpolation of $a\sqrt{\sigma}$ as a function of β, as given in [20].[1] For the temperature scale we need in addition to locate the value of β that corresponds to $T = T_c$ for the relevant value of L_t, and for this we have used the values in [15,20]. In the case of SU(3) we compared the resulting $T/T_c(\beta)$ with that of [2] where the physical scale was set by T_c. We find that the two functions lie on top of each other for $L_t = 6$. This is consistent with the fact that the SU(3) value of $T_c/\sqrt{\sigma}$ for $L_t = 5, 6$ are the same within one sigma [15]. This is true for SU(8) as well, where the value of $T_c/\sqrt{\sigma}$ for $a = 1/(5T_c)$, and $a = 1/(8T_c)$, are the same within one sigma [20], and we find no point to perform similar comparisons there. For SU(4) the value of $T_c/\sqrt{\sigma}$ at $a = 1/(5T_c), 1/(6T_c)$ is ~ 5, and ~ 3.7 sigma away from the value at $a = 1/(8T_c)$ [20], which may suggest that in this case $T/T_c(\beta)$ at values of β that correspond to $T \simeq 8/(5T_c)$ will be smaller when fixing the physical scale with T_c rather than with the string tension. Nevertheless, the shift between the two is at the level of $\sim 2\%$, and will not change the results presented here. In addition, to fix $T/T_c(\beta)$ by fixing T_c, requires a larger scale calculation of $\beta_c(L_t, L_s)$ that will include evaluation of finite volume corrections, similar to what was done for $L_t = 5$ in [15]. In view of the small shifts and the high calculational price, we shall ignore this potential ambiguity in this Letter.

2.3. Finite volume effects

For $N = 4, 8$, one is able to use lattice volumes much smaller than what one needs for SU(3) [2]. That this is so for the deconfinement transition, has been explicitly demonstrated in [15,20], and is theoretically expected, much more generally, as $N \to \infty$. The main remaining concern has to do with tunnelling between confined and deconfined phases near T_c. When $V \to \infty$ tunnelling occurs only at $\beta = \beta_c$ (in a calculation of sufficient statistics) and the system is in the appropriate pure phase for $T < T_c$ and for $T > T_c$. On a finite volume, where this is no longer true, one minimises finite-V corrections by calculating the average plaque-

[1] This is excluding the first three β values in the case of SU(4), which are outside the interpolation regime of [20]. In that case we have performed a new interpolation fit to include these points as well. This gave the string tensions $a\sqrt{\sigma} = 0.3739(15), 0.3440(10), 0.3336(10)$ and the derivatives $-d\log a/d\beta = 1.83(7), 1.55(7), 1.48(5)$ for $\beta = 10.55, 10.60, 10.62$.

Table 4
Finite volume effects for plaquette average in the deconfined phase on a $L_t = 5$ lattice, for SU(8)

β	$L_s = 8$	$L_s = 10$	$L_s = 12$	$L_s = 14$
43.95	–	0.529788(100)	0.529944(65)	–
44.00	0.531343(45)	0.531307(50)	–	–
44.10	0.534219(54)	–	0.534164(34)	–
44.20	0.536714(33)	–	0.536689(54)	0.536650(70)
44.25	–	–	0.537954(60)	0.537850(100)
44.30	0.539181(29)	–	–	0.539220(100)
45.50	0.563093(41)	–	0.562992(28)	–

ttes only in field configurations that are confining, for $T < T_c$, or deconfining, for $T > T_c$. This ensures that the system is as close as possible to being 'large and homogeneous' as is required in the derivations of this section. Because the latent heat grows $\propto N^2$ [20] the region δT around T_c in which there is significant tunnelling shrinks as $\delta T \propto 1/N^2$ for a given V. Hence, we can reduce V as N increases without increasing the ambiguity of the calculation. For SU(3), where the phase transition is only weakly first order, frequent tunneling occurs in the vicinity of T_c in the volume we use, and it is not practical to attempt to separate phases. This will smear the apparent variation of the pressure across T_c in the case of SU(3).

We now turn to a more detailed discussion of finite volume effects. If ξ is the longest correlation length, in lattice units, in a volume of length L, then finite volume effects will be negligible if $\xi \ll L$. In addition finite volume corrections will be suppressed as $N \to \infty$. In our particular context, ξ is given by the inverse mass of the lightest (non-vacuum) state that couples to the loop that winds around the temporal torus. In both the confined and deconfined phases, these masses decrease as $T \to T_c$. Therefore, the largest length scale is set by the masses at $T = T_c$. As N increases these masses increase towards their limits, with $1/N^2$ corrections that are quite large [20].

2.3.1. *The deconfined phase*

In the deconfined phase, on an $L_s^3 \times 5$ lattice at $T = T_c^+$, the value of ξ is about 12.5 lattice spacings for SU(3), while it is about 5.2, and 2.4 lattice spacings for SU(4), and SU(8), respectively [20]. This suggests that our choice of $L_s = 16$ for SU(4) and $L_s = 8$ for SU(8) should be adequate. In addition, it is known from calculations of T_c [14,15,20] that on such lattices the tunnelling is sufficiently rare that even at $T = T_c$ one can reliably categorise field configurations as confined or deconfined and hence calculate the average plaquette in just the deconfined phase if one so wishes. For our supplementary SU(3) calculations we use $L_s = 20$ which is much smaller in units of ξ. In practice this means that in this case we are unable to separate phases at $T \simeq T_c$.

To explicitly confirm our control of finite volume effects, we have compared the SU(8) value of $\langle u_p(\beta)\rangle$ as measured in the deconfined phase of the our $8^3 \times 5$ lattice with other $L_s^3 \times 5$ results from other studies [24]. As summarised in Table 4, the results are consistent at the 2σ level.

2.3.2. *The confined phase*

As we remarked above (see below for explicit evidence) we have $\langle u_p\rangle_T \simeq \langle u_p\rangle_0$ in the confined phase and so the contribution in Eq. (2.11) of the range of β where the finite T system is confined is very small. Nonetheless, we include an integration over that range for completeness and so we need to discuss possible finite V corrections for this case as well.

In the confined phase, on an $L_s^3 \times 5$ lattice at $T = T_c^-$, the value of ξ is about 9.5 lattice spacings for SU(3), but drops to about 5 and 3.5 for SU(4) and SU(8), respectively [20]. This leaves our choice of L_s still reasonable for SU(4) but somewhat worse for SU(8). In Table 5 we provide a finite volume check for the latter case that proves reassuring.

Finally we return to our earlier comment that for the '$T = 0$' L^4 lattice calculations, a size $L = 8$ in SU(8) would not be large enough. This is demonstrated, for our largest β-value, in Table 6, where we also present the value of $L_t \times T/T_c(\beta)$ (in our $L_t = 5$ calculations). In the confined L_s^4 lattice, finite volume effects will be suppressed when the latter is much smaller

Table 5
Finite volume effects for plaquette average in the confined phase on a $L_t = 5$ lattice, for SU(8)

β	$L_s = 8$	$L_s = 10$	$L_s = 12$	$L_s = 14$
43.90	0.525750(87)	–	0.525613(54)	0.525425(90)
43.95	–	0.527240(34)	0.527275(48)	0.527280(50)
44.00	–	–	0.528867(33)	0.528810(50)
44.10	–	–	0.531880(45)	0.531900(60)

Table 6
Finite volume effects for plaquette average in the confined phase on a L^4 lattice, for SU(8). The last column is for $L_t = 5$

β	$L_s = 8$	$L_s = 10$	$L_s = 16$	$L_t \times T/T_c$
44.00	0.528876(39)	0.528788(18)	–	5.05
45.70	0.566089(23)	–	0.565708(4)	8.20

than L_s. Clearly, for $\beta = 45.70$ and $L_s = 8$, this is not the case.

By contrast, for SU(4) the finite volume effects seems not to be large on the 16^4 lattice as we checked for our largest value of $\beta = 11.30$. There the value of the plaquette on a 20^4 lattice is 0.595014(4) [21], which is consistent within $\sim 2.3\sigma$ with the value presented in Table 1. This is in spite of the fact that for this coupling $L_t \times T/T_c = 10$, and is not so much smaller than $L_s = 16$.

3. Results

To obtain the pressure from the values of the average plaquette presented in Tables 1–3 we need to perform the integration in Eq. (2.11), which we do by numerical trapezoids. We have already remarked that the contribution to the pressure from the confined phase is negligible. In Table 7 we provide some accurate evidence for this. We show the values of the average plaquette on L^4 lattices, corresponding to $T \simeq 0$, as well as the values on $L_s^3 \times 5$ lattices at $T \simeq T_c$, with the latter obtained separately in the confined and deconfined phases. (These volumes are large enough for there to be no tunnelling, or even attempted tunnelling, within our available statistics.) We see that for both SU(4) and SU(8) there is no visible difference between the plaquette at $T = 0$ and $T = T_c$ in the confined phase at, say, the 2σ level. Any difference (and there obviously must be some difference) is clearly negligible when compared to the difference between the confined and deconfined phases at (and above) T_c.

In presenting our results for the pressure, we shall normalize to the lattice Stephan–Boltzmann result given by

$$\left(p/T^4\right)_{\text{free-gas}} = \left(N_c^2 - 1\right)\frac{\pi^2}{45} R_{\mathrm{I}}(L_t). \tag{3.1}$$

Here R_{I} includes the effects of discretization errors in the integral method [25,26]. For large values of L_t, and an infinite volume, it is given by

$$R_{\mathrm{I}}(L_t) = 1 + \frac{8}{21}\left(\frac{\pi}{L_t}\right)^2 + \frac{5}{21}\left(\frac{\pi}{L_t}\right)^4 + \mathcal{O}\left(\frac{1}{L_t}\right)^6. \tag{3.2}$$

Since some values of L_t discussed in this Letter are not very large, we shall use the full correction, which includes higher orders in $1/L_t$, instead of Eq. (3.2). This was calculated numerically for the infinite volume limit in [25] for $L_t = 4, 6, 8$, and we supplement this calculation, with the same numerical routines [26], for other values of L_t. A summary of $R_{\mathrm{I}}(L_t)$ in the infinite volume limit is given in Table 8.

We find that the full correction for $L_t = 5$ is a $\sim 21\%$ effect, which, without this normalisation, might obscure the physical effects that we are interested in. This is an appropriate normalisation because we expect Eq. (3.1) to provide the $T \to \infty$ limit of p/T^4. The same applies to the internal energy density, since $\epsilon \to 3p$ as $T \to \infty$, and so when presenting our results for ϵ/T^4 we normalise it with the expression in Eq. (3.1). For similar reasons we shall use the same normalisation when presenting our results for the entropy. For Δ/T^4 it is less clear what normalisation one should use since $\Delta = \epsilon - 3p \to 0$ as $T \to \infty$, but for

Table 7
The plaquette average in the confined phase, C, at $T \simeq T_c$ compared to the $T = 0$ value and to the value in the deconfined phase, D, for SU(4) and SU(8)

β	N	Lattice	$\langle u_p \rangle$	Phase	T
10.635	4	$32^3 \times 5$	0.549563(33)	D	T_c
		$32^3 \times 5$	0.547689(11)	C	T_c
		10^4	0.547640(27)	C	0
43.965	8	$12^3 \times 5$	0.530352(23)	D	T_c
		$12^3 \times 5$	0.527725(27)	C	T_c
		10^4	0.527648(24)	C	0

Table 8
The lattice discretisation errors correction factor $R_I(L_t)$ in the infinite volume limit

$L_t = 2$	$L_t = 3$	$L_t = 4$	$L_t = 5$	$L_t = 6$	$L_t = 8$
2.04526(4)	1.6913(2)	1.3778(1)	1.2129(6)	1.1323(1)	1.0659(1)

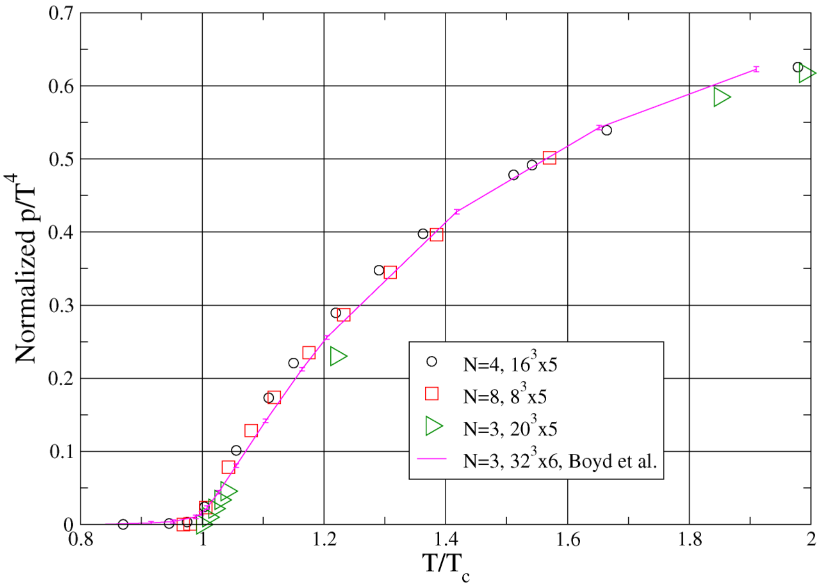

Fig. 1. The pressure, normalized to the lattice Stephan–Boltzmann pressure, including the full discretization errors. The symbol's vertical sizes are representing the largest error bars (which are received for the highest temperature). The solid line is for SU(3) and $L_t = 6$ from [2].

ease of comparison we shall once again normalise using Eq. (3.1).

To facilitate the comparison of our results with earlier work on SU(3) [2], which was done for $L_t = 4, 6, 8$, we have performed SU(3) simulations with $L_t = 5$. The spatial size is $L_s = 20$ which should be sufficiently large in the light of our above discussion of finite volume effects (and we note that it satisfies an empirical rule that one needs $L_s/L_t \geqslant 4$ [27]).

We present our $N = 4$ and $N = 8$ results for p/T^4 in Fig. 1. We also show there our calculations of the SU(3) pressure for $L_t = 5$, as well as the $L_t = 6$ calculations from [2]. Although our errors on the SU(3) pressure are probably underestimated, since the mesh in β is quite coarse, nonetheless one can clearly infer that the pressure in the SU(4) and SU(8) cases is remarkably close to that in SU(3) and hence that the well-known pressure deficit observed in

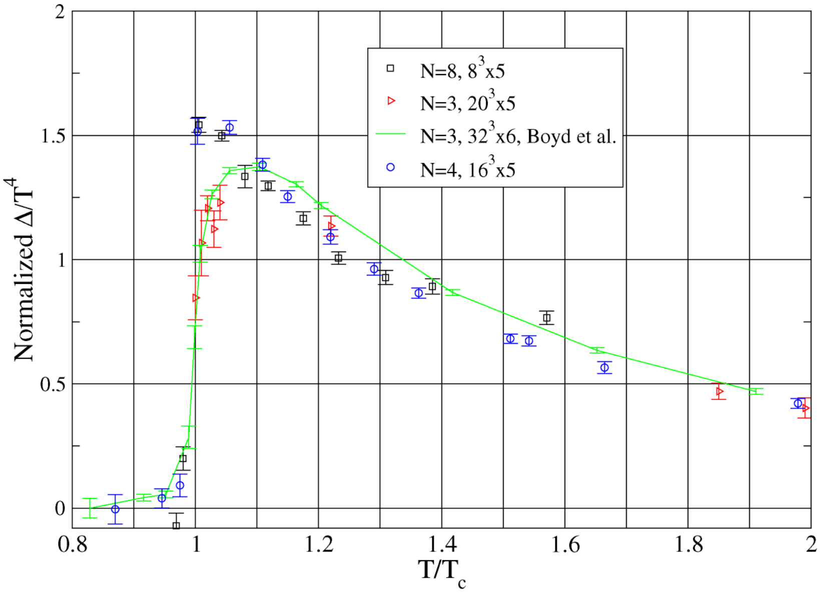

Fig. 2. Results for $\Delta(T)/T^4 = T\frac{\partial p/T^4}{\partial T}$, normalized by the same coefficient as we normalize the pressure. The solid line is for SU(3) and $L_t = 6$ from [2].

Table 9
Plaquette average in the deconfined phase for lattice with fixed coupling, at different values of L_t, and with β that corresponds to roughly the deconfining temperature at $L_t = 5$: $\beta = 5.800, 10.635, 44.00$ for $N = 3, 4, 8$. The data for $L_t = 5$ are obtained for $L = 64, 32, 10$ for $N = 3, 4, 8$ (for $N = 3$, $\delta\langle u_p\rangle$ is the difference between the plaquette as calculated within separate confined and deconfined sequences of field configurations)

N	$L^3 \times 5$	$8^3 \times 4$	$8^3 \times 3$	$8^3 \times 2$	10^4	$-d\log a/d\beta$
3	$\delta\langle u_p\rangle = 0.00080(5)$	0.570987(37)	0.573311(34)	0.578121(27)	0.567642(29)	2.075(17)
4	0.549563(33)	0.551604(33)	0.554047(27)	0.559163(24)	0.547640(27)	1.440(23)
8	0.531202(92)	0.533066(25)	0.535991(24)	0.541518(17)	0.528788(18)	0.384(20)

SU(3) is in fact a property of the large-N planar theory.

In Fig. 2 we present our results for Δ/T^4 as calculated from Eq. (2.14). This quantity can be considered as a measure of the interaction and non-conformality of the theory, since it is identically zero both for the noninteracting Stephan–Boltzmann case, and for the $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory. As remarked above, we normalise with the expression in Eq. (3.1). We also note that in this case there are no errors from a numerical integration, and this enables a fair comparison with the SU(3) data of [2]. Comparing the results for different N we see that, just as for the pressure, the results for all these gauge theories are very similar.

To see what is the behaviour of Δ/T^4 at even higher temperatures, we use the plaquette averages on lattices with $L_t = 2, 3, 4, 5$, that have been calculated at fixed couplings which correspond to $T \simeq T_c$ for $L_t = 5$ [20]. We present the results in Table 9. For the evaluation of Δ one needs $d\log a/d\beta$ which we present in the table as well.

In such calculations where one varies T by varying L_t, the lattice spacing varies as $a = 1/L_t \times 1/T$ when expressed in units of the relevant temperature scale, and so lattice spacing corrections will vary with T.

The resulting values of Δ in the case of SU(3) are plotted in Fig. 3 where they are compared to the results obtained from calculations where one varies T

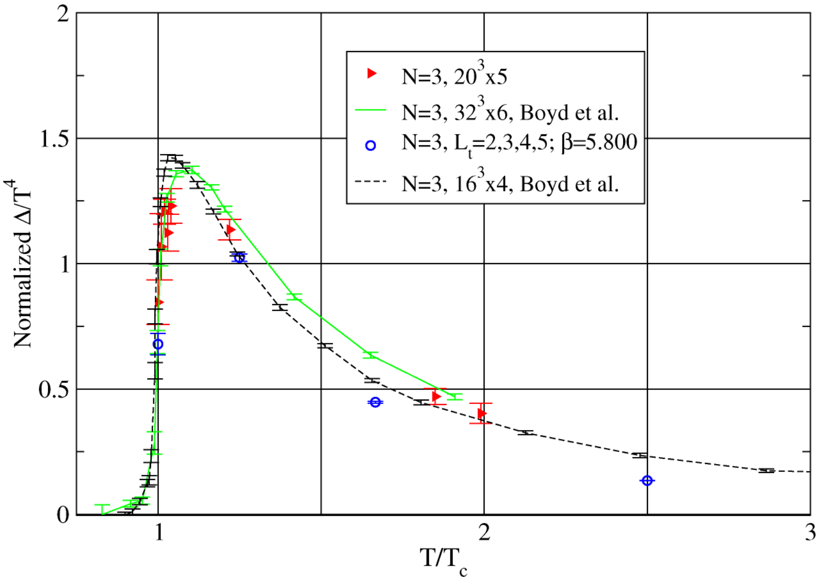

Fig. 3. Results for $\Delta(T)/T^4 = T\frac{\partial p/T^4}{\partial T}$, normalized to the free-gas result. The lines are for SU(3) and $L_t = 4, 6$ from [2]. Triangles correspond to $L_t = 5$, and changing β, while circles correspond to changing L_t and keeping a fixed $\beta = 5.800$.

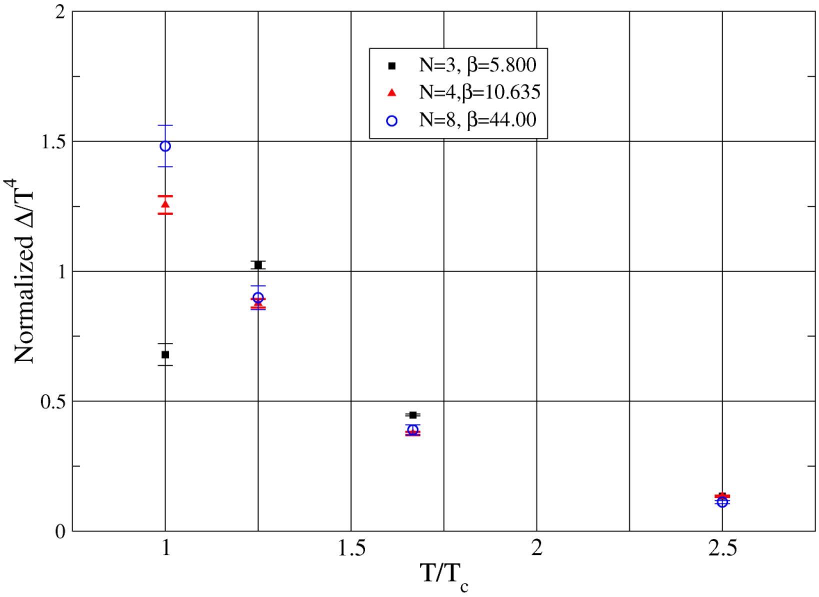

Fig. 4. Results for $\Delta(T)/T^4 = T\frac{\partial p/T^4}{\partial T}$ for $N = 3, 4, 8$, by fixing $\beta = \beta_c(L_t = 5)$, while changing $L_t = 2, 3, 4, 5$.

by varying β at fixed L_t. These calculations include ours for $L_t = 5$ and those of [2] for $L_t = 4, 6$.

As we see from Fig. 3 our $L_t = 5$ SU(3) results do in fact lie between the $L_t = 4, 6$ results of [2] as one would expect. We observe that the T dependence is very similar in all cases, and that the remaining L_t dependence appears to be much the same for the different kinds of calculation. This gives us confidence that performing calculations where we vary T by varying L_t at fixed β does not intro-

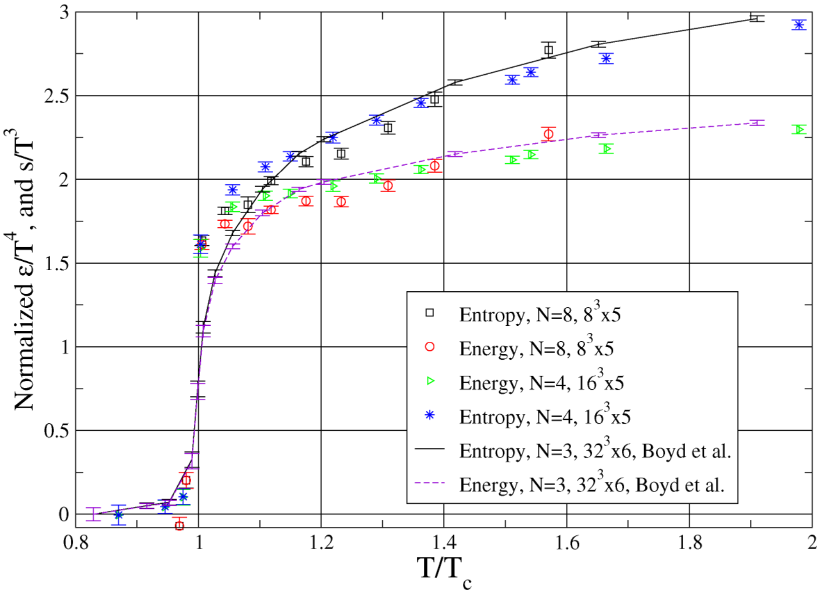

Fig. 5. Results for energy density and entropy, normalized to the lattice Stephan–Boltzmann result, including the full discretization errors. The solid line is for SU(3) and $L_t = 6$ from [2].

duce any unanticipated and important systematic errors.

Having performed this check, we compare in Fig. 4 our results for Δ in the range $T_c \leqslant T \leqslant 2.5T_c$ that corresponds to $5 \geqslant L_t \geqslant 2$. This comparison confirms what we observed in Fig. 2 over a smaller range of T: Δ is very similar for all the values of N (except very close to T_c), implying that this is also a property of the $N = \infty$ planar limit.

Finally, we present in Fig. 5 our results for the normalized energy density $\epsilon = \Delta + 3p$, and the entropy per unit volume $s = (\epsilon + p)/T$. The lines are the SU(3) result of [2] with $L_t = 6$. Again we see very little dependence on the gauge group, implying very similar curves for $N = \infty$.

4. Summary and discussion

In this Letter we have analyzed numerically the bulk thermodynamics of SU(4) and SU(8) gauge theories. We found that the pressure, when normalized to the Stephan–Boltzmann lattice pressure, is practically the same as for SU(3), in the range $T_c \leqslant T \leqslant 1.6T_c$ that we analyze. We found the same to be the case for the internal energy and entropy, as well as for the quantity $\Delta = \epsilon - 3p$ (where we were able to explore temperatures up to $T \simeq 2.5T_c$). All this implies that the dynamics that drives the deconfined system far from its noninteracting gluon plasma limit, must remain equally important in the $N = \infty$ planar theory. This is encouraging since that limit is simpler to approach analytically, in particular using gravity duals.

Our results have been (mostly) obtained for lattice spacings $a = 1/(5T)$ and it would be useful to perform a larger scale calculation that allows us to perform an explicit continuum extrapolation. However, past SU(3) calculations of the pressure, and calculations in SU(N) of various physical quantities, strongly suggest that our choice of a already provides us with a reliable preview of what such a more complete calculation would produce.

Our results imply that any explanation of the QCD pressure deficit must survive the large-N limit, and so should not be driven by special features particular to SU(3). This can provide a strong constraint on such explanations. For example, in approaches based on higher-order perturbation theory, it tells us that the important contributions must be planar. In models focussing on resonances and bound states, it must be that the dominant states are coloured, since the contribution of colour singlets will vanish as $N \to \infty$. Models using 'quasi-particles' should place these in colour representations that do not exclude their presence at $N = \infty$, and in fact give them T-dependent properties which depend weakly on N. Also, topological fluctuations should play no role in this deficit since the evidence is that there are no topological fluctuations of any size in the deconfined phase at large N [28,29].

Finally, we emphasize that our conclusion that the SU(3) pressure and entropy deficits are features of the large-N gauge theory, means that these 'observable' phenomena can, in principle, be addressed using AdS/CFT gravity duals. Indeed, it is precisely where the deficit is large that the coupling must be strong and this is also precisely where, at large N, such dualities can be established. As has been frequently emphasized (see, for example, [16,17]) the deficit in the normalized entropy is not far from the value of $s/s_{\text{free-gas}} = 3/4$ given by the AdS/CFT prediction. In this Letter we have found that large-N gauge theories show the same behaviour, as we see in Fig. 5, where, for the entropy, the horizontal line $s_{\text{normalized}}/T^3 = 3$ would correspond to $s/s_{\text{free-gas}} = 3/4$. Our results can therefore serve as a bridge between the AdS/CFT approach to large-N and the observable world of QCD.

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