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On limiting values of stochastic differential equations with small noise intensity tending to zero

R. Buckdahn [a,*], Y. Ouknine [b], M. Quincampoix [a]

[a] Laboratoire de Mathématiques, UMR CNRS 6205, Université de Bretagne Occidentale 6 avenue Le Gorgeu, 29200 Brest, France

[b] Faculté des Sciences, Université Cadi Ayadd, Marrakech, Morocco

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Abstract

When the right-hand side of an ordinary differential equation (ODE in short) is not Lipschitz, neither existence nor uniqueness of solutions remain valid. Nevertheless, adding to the differential equation a noise with nondegenerate intensity, we obtain a stochastic differential equation which has pathwise existence and uniqueness property. The goal of this short paper is to compare the limit of solutions to stochastic differential equation obtained by adding a noise of intensity ε to the generalized Filippov notion of solutions to the ODE. It is worth pointing out that our result does not depend on the dimension of the space while several related works in the literature are concerned with the one dimensional case.

Introduction

Let us consider a function $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ to which we associate the following ODE

$$x'(t) = f\big(x(t)\big), \quad t \geqslant 0, \quad x(0) = x. \tag{1}$$

Without regularity assumptions on f (for instance Lipschitz continuity), it is well known that neither existence, nor uniqueness hold true in general.

* Corresponding author.
E-mail addresses: rainer.buckdahn@univ-brest.fr (R. Buckdahn), ouknine@ucam.ac.ma (Y. Ouknine), marc.quincampoix@univ-brest.fr (M. Quincampoix).

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Now consider the following stochastic differential equation (SDE in short) obtained by adding to the right-hand side of (1) a noise with small intensity:

$$dX_\varepsilon(t) = f\big(X_\varepsilon(t)\big)\,dt + \varepsilon\,dW_t, \quad t \geqslant 0, \qquad x(0) = x, \tag{2}$$

where $\varepsilon > 0$ is small. Here $(W(t), t \geqslant 0)$ denotes an d-dimensional standard Brownian motion on some complete probability space $(\Omega, \mathcal{F}, P)$ and $(\Omega, \mathcal{F}, P; W)$ the corresponding reference probability system. We denote by $(\mathcal{F}_t)_{t\geqslant 0}$ the natural filtration generated by W and augmented by the P-null sets of $\mathcal{F}$.

Eqs. (2) possess the very crucial property that a unique strong solution exists with the only assumption that f is bounded and measurable (cf. [12]). It is possible to prove by a tightness argument of Prokhorov's type that the laws of solutions to (2) are in a relatively (weakly sequentially) compact set of probabilities.

We address the question of the properties of the limits of solutions to (2) when $\varepsilon \to 0^+$. More precisely we want to compare the limit in law of such solutions with the solutions—in a generalized sense—to (1).

In the literature, this problem has been extensively studied when f is continuous and consequently the ODE (1) has at least a solution. In the one dimensional case, the articles [2,3,11] give a very precise description of the limit process based on the boundary value problem corresponding to the differential generator associated to (2). The method of [2,3] is based on an explicit computation of the solution of the boundary value problem and on the behaviour of the explicit solution when $\varepsilon \to 0^+$. Also a more specific study has been done in [8], in the case where the ODE reduces to

$$x'(t) = \operatorname{sgn}\big(x(t)\big)\big|x(t)\big|^\gamma, \quad t \geqslant 0,$$

where $\gamma \in (0, 1)$ and $\operatorname{sgn} : \mathbb{R} \mapsto \{-1, 0, +1\}$ denotes the usual sign function. In [8], a representation of the density of the solutions to (2) is given, and furthermore the authors obtain an expansion—with respect to ε—of the density in term of eigenvalues and eigenfunctions of a suitable Schrödinger operator. This allows in particular to obtain a rate of convergence of density of (2) to the density of the limit process which is pathwisely a solution to the ODE (1).

Our approach is of a completely different nature. First we want to deal with the multidimensional continuous case where it is not possible to obtain an explicit computation of the solution of the boundary value problem associated with the second order operator corresponding to (2). Second and mainly because we do not suppose the continuity of the function f and consequently we have no existence result for classical solutions of the ODE (1). So we use a generalized notion of solution due to Filippov [7] that we recall now:

Definition 1. Let us consider a function $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ to which we associate the following set-valued map – called Filippov's regularization of F_f

$$F_f(x) := \bigcap_{\lambda(N)=0} \bigcap_{\delta>0} co f\big((x + \delta B) \setminus N\big);$$

the first intersection is taken over all sets of $\mathbb{R}^d$, being negligible with respect to the Lebesgue measure λ, B is the closed unit ball and co denotes the closed convex hull.

An absolutely continuous solution $t \in [0, +\infty) \mapsto x(t) \in \mathbb{R}^d$ is a Filippov solution of (1) if and only if it is a solution of the following differential inclusion

$$x'(t) \in F_f\big(x(t)\big), \quad t \geqslant 0, \qquad x(0) = x. \tag{3}$$

As we will see in the first section the set-valued map F_f is upper semi continuous with compact convex values. This implies that the differential inclusion (3) has a nonempty set of (local) solutions (cf. [1,4]).

The plan of the paper is as follows. In the first section, we recall some properties of Filippov's regularization and we give some new representations of the Filippov map. Section 2 is devoted to the convergence of the density of the solutions of the SDEs. In Section 3, we discuss some examples and applications.

1. On Filippov's set-valued map

In this section, we state some facts concerning Filippov's map F_f associated with the function f. We summarize them in the following proposition. Although some of them are already known we prove all of them for sake of completeness.

Proposition 2. *Let $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a measurable and (locally) bounded function. Then*

(i) *There exists a set N_f negligible under the Lebesgue measure such that for any $x \in \mathbb{R}^d$:*

$$F_f(x) = \bigcap_{\delta>0} co f\big((x+\delta B) \setminus N_f\big). \tag{4}$$

(ii) *For almost all $x \in \mathbb{R}^d$, we have $f(x) \in F_f(x)$.*

(iii) *The set valued map F_f is the smallest upper semi continuous[1] set-valued map F with closed convex values such that $f(x) \in F(x)$, for almost all $x \in \mathbb{R}^d$.*

(iv) *The map $x \mapsto F_f(x)$ is single-valued if and only if there exists a continuous function g which coincides almost everywhere with f. In this case we have $F_f(x) = \{g(x)\}$ for almost all $x \in \mathbb{R}^d$.*

(v) *If a function $\tilde{f}$ coincides almost everywhere with f then $F_f(x) = F_{\tilde{f}}(x)$ for all $x \in \mathbb{R}^d$.*

(vi) *There exists a function $\bar{f}$ which is equal almost everywhere to f and such that*

$$F_f(x) = \bigcap_{\delta>0} co \bar{f}\big((x+\delta B)\big).$$

(vii) *We have*

$$F_f(x) := \bigcap_{\tilde{f}=f \text{ a.e.}} \bigcap_{\delta>0} co \tilde{f}\big((x+\delta B)\big), \tag{5}$$

where the first intersection is taken over all functions $\tilde{f}$ being equal to f almost everywhere.

Proof. We define N_f as the complement of set of points of approximate continuity of f. Recall that the points $x \in \mathbb{R}^d$ of approximate continuity of f are such that

$$\forall \varepsilon > 0, \quad \lim_{r\to 0^+} \frac{\lambda\{y \in (x+rB),\ |f(y) - f(x)| > \varepsilon\}}{\lambda(x+rB)} = 0.$$

[1] Recall that a set valued map G is upper semi continuous at a point x if and only if for any $\varepsilon > 0$ there exists $\alpha > 0$ such that for every $y \in x + \alpha B$ we have $G(y) \subset G(x) + \varepsilon B$.

Following [5] we observe that

$$\frac{\lambda\{y \in (x+rB),\ |f(y)-f(x)| > \varepsilon\}}{\lambda(x+rB)} \leqslant \frac{1}{\varepsilon\lambda(x+rB)} \int\limits_{x+rB} \big|f(y)-f(x)\big|\,dy.$$

The right-hand side tends to 0 for all the Lebesgue points of f. Moreover almost every point is a Lebesgue point ([5–7]), so N_f is a set of Lebesgue measure equal to 0.

(i) Consider $x \in \mathbb{R}^d$. Fix N a set of measure 0, we claim that

$$\bigcap_{\delta>0} cof\big((x+\delta B)\setminus N_f\big) = \bigcap_{\delta>0} cof\big((x+\delta B)\setminus (N_f \cup N)\big). \tag{6}$$

One inclusion is trivial, we prove the other one. Take $z \in \bigcap_{\delta>0} cof((x+\delta B)\setminus N_f)$. Then there exist a decreasing sequence $\delta_n \downarrow 0^+$, points $x_n^i \in (x+\delta_n B)\setminus N_f$, $\lambda_n^i \geqslant 0$, for $i = 1, 2\ldots, N_n$, with $\sum_{i=1}^{N_n} \lambda_n^i = 1$, such that

$$\sum_{i=1}^{N_n} \lambda_n^i f\big(x_n^i\big) = z. \tag{7}$$

Because x_i^n, $i = 1, 2, \ldots, N_n$, are points of approximate continuity of f, there exists $0 < r_n < \delta_n$ such that

$$\forall i = 1, 2, \ldots, N_n, \quad \lambda\left\{y \in \big(x_n^i + r_n B\big),\ \big|f(y) - f\big(x_n^i\big)\big| > \frac{1}{n}\right\} \leqslant \frac{1}{2}\lambda(r_n B).$$

Thus, because N is of measure 0, for any $i = 1, 2, \ldots, N_n$, there exists

$$y_n^i \in \big(x_n^i + r_n B\big) \Big\backslash \left(N_f \cup N \cup \left\{y \in \big(x_n^i + r_n B\big),\ \big|f(y) - f\big(x_n^i\big)\big| > \frac{1}{n}\right\}\right).$$

This yields

$$\big|f\big(y_n^i\big) - f\big(x_n^i\big)\big| \leqslant \frac{1}{n},$$

and, consequently,

$$\left|\sum_{i=1}^{N_n} \lambda_n^i f\big(y_n^i\big) - z\right| \leqslant \frac{1}{n},$$

which, in view of (7), implies that

$$\lim_n \sum_{i=1}^{N_n} \lambda_n^i f\big(y_n^i\big) = z.$$

So $z \in \bigcap_{\delta>0} cof((x+\delta B)\setminus (N_f \cup N))$ and we have obtained our claim (6).

Since the set N of measure 0 is arbitrary, the proof of (i) is complete.

(ii) Fix a point x where f is approximatively continuous (i.e. $x \notin N_f$) and consider a sequence $r_n \downarrow 0$. Then there exists a sequence $\varepsilon_n \downarrow 0^+$ such that

$$\lambda\left\{y \in (x + r_n B),\ \big|f(y) - f(x)\big| > \frac{1}{n}\right\} \leqslant \varepsilon_n \lambda(r_n B).$$

Hence

$$f\big((x+r_nB)\setminus N_f\big)$$
$$\supset f\left((x+r_nB)\Big\backslash\left(N_f\cup\left\{y\in(x+r_nB),\ \big|f(y)-f(x)\big|>\frac{1}{n}\right\}\right)\right)\neq\emptyset$$

and consequently

$$f(x)\in\frac{1}{n}B+f\left((x+r_nB)\Big\backslash\left(N_f\cup\left\{y\in(x+r_nB),\ \big|f(y)-f(x)\big|>\frac{1}{n}\right\}\right)\right)$$
$$\subset co f\big((x+r_nB)\setminus N_f\big)+\frac{1}{n}B.$$

By taking the intersection on n, and using (i), we obtain

$$\forall x\in\mathbb{R}^d\setminus N_f,\quad f(x)\in F_f(x).$$

(iii) From the expression of F_f obtained in (ii), it appears clearly that F_f is upper semicontinuous with compact convex nonempty values and that $f(x)\in F(x)$ for almost all x.

Consider another set-valued map G upper semicontinuous with compact convex values such that for some N_G of measure 0 we have:

$$f(x)\in G(x),\quad \forall x\in\mathbb{R}^d\setminus N_G.$$

Fix $y\in\mathbb{R}^d$. From the upper semicontinuity of G, there exists a sequence $\delta_n\downarrow 0^+$ with

$$G(y+\delta_nB)\subset G(y)+\frac{1}{n}B,\quad \forall n\geqslant 1.$$

Clearly,

$$f\big((y+\delta_nB)\setminus(N_f\cup N_G)\big)\subset G(y+\delta_nB)\subset G(y)+\frac{1}{n}B,$$

an consequently, because $G(y)$ is a compact convex set of $\mathbb{R}^d$, this yields

$$\bigcap_{\geqslant 1} co f\big((y+\delta_nB)\setminus(N_f\cup N_G)\big)\subset G(y).$$

From (6), we obtain $F_f(y)\subset G(y)$. The proof of (iii) is achieved.

(iv) Assume that $F_f(x)=\{g(x)\}$ for all $x\in R^d$. Because $x\mapsto\{g(x)\}$ is upper semicontinuous as a set-valued map, this yields that the function g is continuous. Furthermore, from (iii), $g(x)=f(x)$ for almost every x.

Conversely, suppose that there exists some g continuous which coincide with f on the complement of some negligible set N. By (6), we have for any $x\in\mathbb{R}^d$,

$$F_f(x)=\bigcap_{\delta>0} co f\big((x+\delta B)\setminus(N_f\cup N)\big).$$

So

$$\bigcap_{\delta>0} co f\big((x+\delta B)\setminus(N_f\cup N)\big)=\bigcap_{\delta>0} co g\big((x+\delta B)\setminus(N_f\cup N)\big).$$

The right-hand side of the above equality reduces to $\{g(x\}$ thanks to the continuity of g.

(v) Suppose that $\tilde f(x)=f(x)$ for any $x\in\mathbb{R}^d\setminus\tilde N$, where $\tilde N$ is a negligible set. Then for any set N of measure 0 we have for any x

$$\bigcap_{\delta>0} co f\big((x+\delta B)\setminus(\tilde N\cup N)\big)=\bigcap_{\delta>0} co \tilde f\big((x+\delta B)\setminus(\tilde N\cup N)\big).$$

By taking the intersection over all sets N of null measure, we obtain $F_f(x) = F_{\tilde{f}}(x)$, which proves our claim.

(vi) Let us define $\bar{f}$ by setting $\bar{f}(x) = f(x)$ if $x \notin N_f$ and if $x \in N_f$ we choose $\bar{f}(x)$ as being any element of $F_f(x)$. Clearly $\bar{f}$ coincides with f on $\mathbb{R}^d \setminus N_f$. Now fix $y \in \mathbb{R}^d$.

One has from (6)

$$F_f(y) = \bigcap_{\delta>0} co f\big((y+\delta B)\setminus N_f\big) = \bigcap_{\delta>0} co \bar{f}\big((y+\delta B)\setminus N_f\big) \subset \bigcap_{\delta>0} co \bar{f}\big((y+\delta B)\big).$$

Conversely, for any $\delta > 0$,

$$\bar{f}(y+\delta B) = \big\{\bar{f}(z),\ z \in y+\delta B\big\}$$

and we deduce from the very definition of $\bar{f}$ that

$$\bar{f}(z) \subset \bigcap_{\eta>0} co f\big((z+\eta B)\setminus N_f\big) \subset \bigcap_{\eta>0} co f\big((y+(\eta+\delta)B)\setminus N_f\big).$$

Consequently

$$\bigcap_{\delta>0} co \bar{f}(y+\delta B) \subset \bigcap_{\delta>0,\eta>0} co f\big(\big(y+(\delta+\eta)B\big)\setminus N_f\big).$$

The right-hand side of the above relation is equal to $F_f(y)$. Our proof is ended.

(vii) We know from (vi) that

$$F_f(x) = \bigcap_{\delta>0} co \bar{f}\big((x+\delta B)\big) \supset \bigcap_{\tilde{f}=f \text{ a.e.}} \bigcap_{\delta>0} co \tilde{f}\big((x+\delta B)\big).$$

Trivially, one has also

$$F_f(x) = \bigcap_{\tilde{f}=f \text{ a.e.}} \bigcap_{\delta>0} co \tilde{f}\big((x+\delta B)\big) \supset \bigcap_{\tilde{f}=f \text{ a.e.}} \bigcap_{\lambda(N)=0} \bigcap_{\delta>0} co \tilde{f}\big((x+\delta B)\setminus N\big).$$

The right-hand side is $\bigcap_{\tilde{f}=f \text{ a.e.}} F_{\tilde{f}}(x)$ which reduces to $F_f(x)$ by (v).

The proof is complete. □

Recall also [1,4], that the set of absolutely continuous solutions of (3) is nonempty. Furthermore, when the solutions are restricted to a given interval $[0,T]$, the solution set is compact for the uniform convergence topology and it is sequentially compact for the weak-$W^{1,1}([0,T])$ topology.

Remark 3. In the one dimensional case, namely $f : \mathbb{R} \mapsto \mathbb{R}$ one can directly check that

$$\forall x \in \mathbb{R}, \quad F_f(x) = \big[m(f)(x), M(f)(x)\big]$$

where

$$m(f)(x) := \sup_{\delta>0}\Big(\operatorname{ess}\inf_{[x-\delta,x+\delta]} f\Big), \qquad M(f)(x) := \inf_{\delta>0}\Big(\operatorname{ess}\sup_{[x-\delta,x+\delta]} f\Big).$$

2. Limit solutions of the SDE and Filippov's to the ODE

Theorem 4. *Suppose that $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is Lebesgue measurable and satisfies*

$$\| f(x)\| \leqslant M\big(1+|x|\big), \quad \forall x \in \mathbb{R}^d. \tag{8}$$

For any $\varepsilon > 0$, let X_ε be the solution to (2)*. Then, there exists $\varepsilon_n \to 0$ such that X_{ε_n} converges in law, as $\varepsilon_n \to 0$, to some X which belongs almost surely to the set of Filippov's solutions to* (1)*. Furthermore, any cluster point of X_ε is also almost surely in the set of Filippov's solutions.*

Proof. Let us first note that, by classical arguments and Girsanov's transformation, there exists a weak solution $(\Omega_\varepsilon, \mathcal{F}_\varepsilon, P_\varepsilon, X_\varepsilon, W_\varepsilon)$ to

$$X_\varepsilon(t) = x + \int_0^t f\big(X_\varepsilon(s)\big)\,ds + \varepsilon W_\varepsilon(t), \quad t \in [0,T]$$

(see [9,10]). Note that $(\Omega_\varepsilon, \mathcal{F}_\varepsilon, P_\varepsilon, X_\varepsilon, W_\varepsilon)$ is still solution to the same equation with f replaced by $\bar{f}$ of Proposition 2(vi). So without lack of generality, we assume from now on that f satisfies

$$\forall x \in \mathbb{R}^d, \quad F_f(x) = \bigcap_{\delta>0} co\, f(x+\delta B).$$

One can easily show that the family of laws

$$\big\{P_\varepsilon \circ (X_\varepsilon, W_\varepsilon)^{-1},\ \varepsilon > 0\big\}$$

is tight. Hence, by Prokhorov's theorem, there exists a sequence $\varepsilon_n \to 0^+$ with

$$P_{\varepsilon_n} \circ (X_{\varepsilon_n}, W_{\varepsilon_n})^{-1} \to P \circ (X,W)^{-1} \quad \text{in } \mathcal{D}, \quad \text{as } n \to +\infty.$$

We set $Y_\varepsilon(t) := X_\varepsilon(t) - \varepsilon W_\varepsilon(t)$ and observe that Y_ε satisfies

$$Y'_\varepsilon(t) = f\big(X_\varepsilon(t)\big), \quad t \geqslant 0, \qquad Y_\varepsilon(0) = x. \tag{9}$$

Using Skohorod's theorem we can find a new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and stochastic processes $\widetilde{X}_{\varepsilon_n}, \widetilde{W}_{\varepsilon_n}, \widetilde{X}, \widetilde{W}$ defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, such that

(i) $\widetilde{P} \circ (\widetilde{X}_{\varepsilon_n}, \widetilde{W}_{\varepsilon_n})^{-1} = P \circ (X_{\varepsilon_n}, W_{\varepsilon_n})^{-1}$, $n \geqslant 1$, and $\widetilde{P} \circ (\widetilde{X}, \widetilde{W})^{-1} = P \circ (X,W)^{-1}$, and,
(ii) in the topology of the uniform convergence on compacts, $\widetilde{X}_{\varepsilon_n} \to \widetilde{X}$, $\widetilde{W}_{\varepsilon_n} \to \widetilde{W}$, $\widetilde{P}$-a.s.

Hence, for arbitrarily given $T > 0$,

$$\widetilde{X}_{\varepsilon_n} \to \widetilde{X},\ \widetilde{W}_{\varepsilon_n} \to \widetilde{W} \quad \text{in } C\big([0,T], \mathbb{R}^d\big), \qquad \widetilde{P}\text{-a.s.},$$

from where we easily get that

$$\widetilde{Y}_{\varepsilon_n} := \widetilde{X}_{\varepsilon_n} - \varepsilon_n \widetilde{W}_{\varepsilon_n} \to \widetilde{X}, \quad \text{in } C\big([0,T], \mathbb{R}^d\big), \qquad \widetilde{P} \text{ a.s.}$$

Furthermore, from (8) and (9) we have $|\widetilde{\widetilde{Y}}'_{\varepsilon_n}(t)| \leqslant M(1 + |\widetilde{X}_{\varepsilon_n}(t)|$ for every t and n. Thus one can obtain a constant $C > 0$ such that

$$E\Big[\sup_{t\in[0,T]} \big(\big|\widetilde{Y}_{\varepsilon_n}(t)\big|^2 + \big|\widetilde{Y}'_{\varepsilon_n}(t)\big|^2\big)\Big] \leqslant C, \quad \forall n \geqslant 1.$$

From this and from

$$E\Big[\sup_{t\in[0,T]}\big(\big|\widetilde{X}_{\varepsilon_n}(t)-\widetilde{X}(t)\big|^2\big)\Big]\to 0 \quad \text{as } n\to\infty,$$

we infer that

$$E\Big[\sup_{t\in[0,T]}\big(\big|\widetilde{Y}'_{\varepsilon_n}(t)\big|^2\big)\Big]\leqslant C, \quad \forall n\geqslant 1, \quad \text{and}$$

$$E\Big[\sup_{t\in[0,T]}\big(\big|\widetilde{Y}_{\varepsilon_n}(t)-\widetilde{X}(t)\big|^2\big)\Big]\to 0 \quad \text{as } n\to\infty.$$

So up to a subsequence we obtain the weak convergence of $\widetilde{Y}_{\varepsilon_n}$ in the space

$$W^{1,2}:=\big\{Z\in L^2\big([0,T]\times\Omega,\mathbb{R}^d\big),\ Z'\in L^2\big([0,T]\times\Omega,\mathbb{R}^d\big)\big\}.$$

Namely there exists some process U such that $\widetilde{Y}_{\varepsilon_n}\to\widetilde{X}$ in L^2 and $\widetilde{Y}'_{\varepsilon_n}\rightharpoonup U$ in L^2 as $n\to\infty$.

We claim that $U(t)=\widetilde{X}(t)$ for every $t\in[0,T]$, $\widetilde{P}$-a.s.

Let us take any $\phi\in W^{1,2}$. We know that when $n\to\infty$,

$$E\left[\int_0^T \widetilde{Y}'_{\varepsilon_n}(s)\phi(s)\,ds\right]\to E\left[\int_0^T U(s)\phi(s)\,ds\right]. \tag{10}$$

By the integral by part formula, the left-hand side of the above equality is equal to

$$E\big[\widetilde{Y}_{\varepsilon_n}(T)\phi(T)-x\phi(0)\big]-E\left[\int_0^T \widetilde{Y}_{\varepsilon_n}(s)\phi'(s)\,ds\right]$$

$$\to E\big[\widetilde{X}(T)\phi(T)-x\phi(0)\big]-E\left[\int_0^T \widetilde{X}(s)\phi'(s)\,ds\right].$$

The last term is equal to $E[\int_0^T \widetilde{X}'(s)\phi(s)\,ds]$. Hence by (10),

$$E\left[\int_0^T \widetilde{X}'(s)\phi(s)\,ds\right]=E\left[\int_0^T U(s)\phi(s)\,ds\right].$$

This proves our claim ϕ being arbitrary.

Thus, outside a $\widetilde{P}$-null set the process $\widetilde{Y}_{\varepsilon_n}$ converges to $\widetilde{X}$ weakly in $W^{1,2}$.

Let us now fix arbitrarily a $\delta>0$. Then, from (9), there exists a sequence of random processes $\eta_{\varepsilon_n}(=\sup_{k\geqslant n}|\widetilde{X}_{\varepsilon_k}-\widetilde{X}|)\geqslant 0$ converging to 0 uniformly on $[0,T]$, such that

$$\widetilde{Y}'_{\varepsilon_n}(t)=f\big(\widetilde{X}_{\varepsilon_n}(t)\big)\in f\big(\widetilde{X}(t)+\eta_\varepsilon B\big)\subset cof\big(\widetilde{X}(t)+\eta_\delta B\big), \quad \forall\varepsilon_n<\delta.$$

Passing to the limit at the left-hand side of the above formula, we obtain with the help of Mazur's theorem

$$\widetilde{X}'(t)\in cof\big(\widetilde{X}(t)+\eta_\delta B\big), \quad dt\,d\widetilde{P}\text{-a.e.}$$

Hence, taking the following intersection over a sequence $\delta_n\downarrow 0^+$, we get

$$\widetilde{X}'(t)\in\bigcap_{n>0} cof\big(\widetilde{X}(t)+\eta_{\delta_n}B\big)=F_f\big(\widetilde{X}(t)\big), \quad \text{for a.e. } t\geqslant 0, \qquad \widetilde{P}\text{-a.s.}$$

Now we end the proof by observing that

$$1 = \widetilde{P}\big[\widetilde{X}'(t) \in F_f\big(\widetilde{X}(t)\big) \text{ for a.e. } t \geqslant 0\big].$$

Because X and $\widetilde{X}$ have the same law, we can conclude that

$$1 = \widetilde{P}\big[\widetilde{X}'(t) \in F_f\big(\widetilde{X}(t)\big) \text{ for a.e. } t \geqslant 0\big] = P\big[X'(t) \in F_f\big(X(t)\big) \text{ for a.e. } t \geqslant 0\big].$$

Hence, P almost surely X is a pathwise solution to (1). □

Remark 5. Using the same method of proof, we obtain a slightly more general result of the same kind of Theorem 4, if we consider instead solutions to (2), solutions to

$$dX_\varepsilon(t) = f_\varepsilon\big(X_\varepsilon(t)\big)\,dt + \varepsilon\,dW_t, \quad t \geqslant 0, \qquad x(0) = x.$$

For doing this we need the following extra assumption on the functions f_ε: There exists $A \subset \mathbb{R}^d$ negligible for the Lebesgue measure, such that for any compact set $K \subset \mathbb{R}^d$, we have

$$\sup_{x \in K \setminus A} \big|f_\varepsilon(x) - f(x)\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

Remark 6. Conversely, every generalized solution to (1) may not be a limit of solutions to (2). This fact is illustrated by the following easy counter example in dimension $d = 1$. The equation

$$x'(t) = 2\sqrt{\big|x(t)\big|}, \qquad x(0) = 0,$$

has two solutions $x_1 \equiv 0$ and $x_2(t) = t^2$. The constant solution x_1 cannot be a limit of solutions to

$$dX_\varepsilon(t) = 2\sqrt{\big|X_\varepsilon(t)\big|}\,dt + \varepsilon\,dW_t.$$

We refer the reader to [8] for the proof of this fact.

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