Oscillation of a forced super-linear second order differential equation with impulses

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Abstract

Some oscillation criteria are established for a forced super-linear second order differential equation with impulses. Those results extend some well-known results for the equation without impulses, which are different from most existing ones in that they are based only on information on a sequence of subintervals of \([t_0, \infty)\), rather than on the whole half-linear interval.

Keywords: Oscillation; Impulses; Forced term; Second order differential equation; Super-linear

1. Introduction

Consider a forced super-linear second order differential equation with impulses

\[
\begin{cases}
(r(t)x'(t))' + p(t)|x(t)|^{\alpha-1}x(t) = q(t), & t \geq t_0, t \neq \tau_k, k = 1, 2, \ldots, \\
x(\tau_k^+) = a_kx(\tau_k), & x'(\tau_k^+) = b_kx'(\tau_k), \\
x(t_0^+) = x_0, & x'(t_0^+) = x_0',
\end{cases}
\]

where \(\{\tau_k\}\) denotes the impulse moments sequence with \(0 \leq t_0 = \tau_0 < \tau_1 < \cdots < \tau_k < \cdots, \lim_{k \to \infty} \tau_k = +\infty\), and

\[
x'(\tau_k) = \lim_{h \to 0-} \frac{x(\tau_k + h) - x(\tau_k)}{h}, \quad x'(\tau_k^+) = \lim_{h \to 0+} \frac{x(\tau_k + h) - x(\tau_k^+)}{h}.
\]

Here, we always assume that the following conditions hold:

(A1) \(\alpha > 1\) is a constant, \(r(t) : [t_0, \infty) \to (0, \infty)\) is a continuous function, \(p(t), q(t)\) are real valued continuous functions defined on \([t_0, \infty);\)

(A2) \(b_k \geq a_k > 0\) are constants, \(k = 1, 2, \ldots\)

Let \(J \subset \mathbb{R}\) be an interval, we define

\[
\text{PC}(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t) \text{ is continuous everywhere except some } \tau_k\text{s at which } x(\tau_k^+) \text{ and } x(\tau_k^-) \text{ exist and } x(\tau_k^-) = x(\tau_k)\}.
\]

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\textbf{Definition 1.1.} A function \( x \in PC([t_0, \infty), \mathbb{R}) \) is called a solution of (1.1) if

\begin{enumerate}[(a)]  
\item \( x(t_0^+) = x_0, \ x'(t_0^+) = x'_0; \)
\item \( x(t) \) satisfies \( (r(t)x'(t))^\prime + p(t)x(t)^{\alpha-1}x(t) = q(t), \) when \( t \in [t_0, \infty), \ t \neq \tau_k; \)
\item \( x(t_k^+) = a_kx(\tau_k), \ x'(t_k^+) = b_kx'(\tau_k), \) and for any \( \tau_k, \) we always assume that both \( x(t) \) and \( x'(t) \) are left continuous.
\end{enumerate}

\textbf{Definition 1.2.} A solution of (1.1) is said to be non-oscillatory if this solution is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

We note that impulsive differential equations are an adequate mathematical apparatus for the simulation of processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, etc. For further applications and questions concerning the existence and uniqueness of solutions of impulsive differential equations, see [1].

Based on the oscillatory behavior of the forcing term, Wong [2] obtained oscillation criteria for the linear nonhomogeneous equation

\[ (r(t)x'(t))^\prime + p(t)x(t) = q(t). \tag{1.2} \]

On the other hand, Nsar [3] considered the forced super-linear differential equation

\[ x''(t) + p(t)|x(t)|^{\alpha-1}x(t) = q(t), \tag{1.3} \]

and established an interval oscillation theorem for (1.3).

The results of Wong and Nsar are very interesting since they make use of the “oscillatory intervals” of \( q(t), \) and Leighton’s variational principles [4] for the oscillation of the associated nonhomogeneous equation. Naturally, the question arises: Is it possible to extend the results of Wong and Nsar to (1.1)? The object of this paper is to give an affirmative answer to this question. Indeed, motivated by the ideas of Wong [2], Nsar [3] and Kong [5], we establish the sufficient conditions for the oscillation of all solutions of (1.1), which utilize the oscillatory behavior of forcing term “\( q(t) \)” on intervals. Those results extend some well-known results for the equation without impulses, which are different from most existing ones in that they are based only on the information on a sequence of subintervals of \([t_0, \infty),\) rather than on the whole half-line interval.

For more results on the forced second order differential equations without impulses, we refer to [4,6–12] and the references therein.

\section{Main results}

In this section, two intervals \([c_1, d_1]\) and \([c_2, d_2]\) are considered to establish oscillation criteria, so we also assume that

\begin{enumerate}[(A3)]  
\item \( c_j, d_j \not\in \{\tau_k\} \ (j = 1, 2) \) with \( c_1 < d_1 \leq c_2 < d_2, \ p(t) \geq 0 \) for \( t \in [c_1, d_1] \cup [c_2, d_2], \) and \( q(t) \) has different signs in \([c_1, d_1]\) and \([c_2, d_2],\) for instance, let
  \[ q(t) \leq 0 \quad \text{for} \ t \in [c_1, d_1], \quad \text{and} \quad q(t) \geq 0 \quad \text{for} \ t \in [c_2, d_2]. \]
\end{enumerate}

Denote

\[ k(s) = \max\{i : t_0 < \tau_i < s\}, \]

and for \( j = 1, 2, \) let \( \eta_j = \max\{r(t) : t \in [c_j, d_j]\}, \)

\[ \Omega_w(c_j, d_j) = \{w \in C^1[c_j, d_j], w(t) \neq 0, w(c_j) = w(d_j) = 0\}, \]

\[ \Omega_G(c_j, d_j) = \{G \in C^1[c_j, d_j], G(t) \geq 0, \neq 0, G(c_j) = G(d_j) = 0, G(t) = 2g(t)\sqrt{G(t)}\}. \]

\textbf{Theorem 2.1.} Assume that conditions (A1)–(A3) hold, and that there exists \( w(t) \in \Omega_w(c_j, d_j) \) such that

\[ \int_{c_j}^{d_j} \left| p(t) \right|^{\frac{\alpha}{2}} |q(t)|^{1 - \frac{\alpha}{2}} w^2(t) - r(t)w^2(t) \right| \, dt \geq Q(w, c_j, d_j), \tag{2.1} \]
where \( Q(w, c_j, d_j) = 0 \) for \( k(c_j) = k(d_j) \), and
\[
Q(w, c_j, d_j) = \eta_j \left\{ w^2(\tau_{k(c_j)+1}) + \frac{b_{k(c_j)+1} - a_{k(c_j)+1}}{a_{k(c_j)+1}(\tau_{k(c_j)+1} - c_j)} + \sum_{i=k(c_j)+2}^{k(d_j)} w^2(\tau_i) \frac{b_i - a_i}{a_i (\tau_i - \tau_{i-1})} \right\}
\tag{2.2}
\]
for \( k(c_j) < k(d_j) \), \( j = 1, 2 \). Then every solution of (1.1) has at least one zero in \([c_1, d_1] \cup [c_2, d_2] \).

**Proof.** Let \( x(t) \) be a solution of (1.1). Suppose \( x(t) \) does not have any zero in \([c_1, d_1] \cup [c_2, d_2] \). Without loss of generality, we may assume that \( x(t) > 0 \) for \( t \in [c_1, d_1] \). Define
\[
u(t) = -\frac{r(t)x'(t)}{x(t)}, \quad t \in [c_1, d_1].
\tag{2.3}
\]
Then, for \( t \in [c_1, d_1] \) and \( t \neq \tau_k \), we have
\[
u'(t) = p(t)[x(t)]^{\alpha-1} - \frac{q(t)}{x(t)} + \frac{1}{r(t)}u^2(t)
= p(t)[x(t)]^{\alpha-1} + \frac{|q(t)|}{x(t)} + \frac{1}{r(t)}u^2(t).
\]
By Hölder’s inequality, we see that for \( t \in [c_1, d_1] \) and \( t \neq \tau_k \),
\[
p(t)[x(t)]^{\alpha-1} + \frac{|q(t)|}{x(t)} \geq \frac{\alpha - 1}{\alpha} \left( \frac{|q(t)|}{x(t)} \right)^{1-\frac{1}{\alpha}} + \frac{1}{\alpha} \left( [p(t)]^{\frac{1}{\alpha}} [x(t)]^{1-\frac{1}{\alpha}} \right)^{\alpha}
\geq [p(t)]^{\frac{1}{\alpha}} [q(t)]^{1-\frac{1}{\alpha}} + \frac{1}{r(t)}u^2(t), \quad t \in (c_1, d_1), t \neq \tau_k.
\tag{2.4}
\]
And for \( t = \tau_k, k = 1, 2, \ldots \), one has
\[
u(\tau_k^+) = \frac{b_k}{d_k} u(\tau_k).
\tag{2.5}
\]
If \( k(c_1) < k(d_1) \), then there are all impulsive moments in \([c_1, d_1] : \tau_{k(c_1)+1}, \tau_{k(c_1)+2}, \ldots, \tau_{k(d_1)} \). Multiplying by \( w^2(t) \) and integrating on \([c_1, d_1] \) on both sides of (2.4), using integration by parts on the left side, and noting that the condition \( w(c_1) = w(d_1) = 0 \), we obtain
\[
\sum_{i=k(c_1)+1}^{k(d_1)} w^2(\tau_i)(u(\tau_i) - u(\tau_i^+)) \\
\geq \int_{c_1}^{d_1} \left[ [p(t)]^{\frac{1}{\alpha}} [q(t)]^{1-\frac{1}{\alpha}} w^2(t) - r(t)w'^2(t) \right] dt + \int_{c_1}^{\tau_{k(c_1)+1}} \frac{1}{r(t)} [r(t)w'(t) + u(t)w(t)]^2 dt \\
+ \sum_{i=k(c_1)+2}^{k(d_1)} \int_{\tau_{i-1}}^{\tau_i} \frac{1}{r(t)} [r(t)w'(t) + u(t)w(t)]^2 dt + \int_{\tau_{k(d_1)}}^{d_1} \frac{1}{r(t)} [r(t)w'(t) + u(t)w(t)]^2 dt.
\]
Now we claim that
\[
\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i - b_i}{a_i} w^2(\tau_i) u(\tau_i) > \int_{c_1}^{d_1} \left[ [p(t)]^{\frac{1}{\alpha}} [q(t)]^{1-\frac{1}{\alpha}} w^2(t) - r(t)w'^2(t) \right] dt.
\tag{2.6}
\]
Otherwise, it will yield \( r(t)w'(t) + u(t)w(t) = 0 \). From (2.6), we see \( x(t) \) is a multiple of \( u(t) \), i.e., it has zeros at the two points \( c_1 \) and \( d_1 \), which again contradicts our assumption.

On the other hand, for \( t \in (c_1, \tau_{k(c_1)+1}] \), it holds that
\[
(r(t)x')(t) = q(t) - p(t)|x(t)|^{\alpha-1}x(t) \leq 0.
\]
which follows that \( r(t)x'(t) \) is nonincreasing on \((c_1, \tau_{k(c_1)+1}]\). So, for any \( t \in (c_1, \tau_{k(c_1)+1}] \), one has

\[
x(t) - x(c_1) = x'(\xi)(t - c_1) \geq \frac{r(t)x'(t)}{r(\xi)}(t - c_1), \quad \xi \in (c_1, t),
\]

which implies from \( x(c_1) > 0 \) that

\[
- \frac{r(t)x'(t)}{x(t)} > - \frac{r(\xi)}{t - c_1}.
\]

Let \( t \to \tau_{k(c_1)+1}^- \), it follows that

\[
u(\tau_{k(c_1)+1}) \geq - \frac{r(\xi)}{\tau_{k(c_1)+1} - c_1} \geq - \frac{\eta_1}{\tau_{k(c_1)+1} - c_1}.
\]

Making a similar analysis on \( (\tau_{i-1}, \tau_i), i = k(c_1) + 2, \ldots, k(d_1) \), it is not difficult to get that

\[
u(\tau_i) = - \frac{r(\tau_i)x'(\tau_i)}{x(\tau_i)} \geq - \frac{\eta_1}{\tau_i - \tau_{i-1}}, \quad i = k(c_1) + 2, \ldots, k(d_1).
\]

Here, we must point that (2.7) and (2.8) play a key role in our method for estimating \( u(\tau_i) \), which is different from those without impulses. From (2.7), and (2.8) and (A2), we have

\[
\sum_{i=k(c_1)+1}^{k(d_1)} \frac{b_i - a_i}{a_i} w^2(\tau_i)u(\tau_i) \geq - \eta_1 \left[ w^2(\tau_{k(c_1)+1}) - \frac{k(d_1)}{a_d(\tau_i - \tau_{i-1})} \right],
\]

that is,

\[
\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i - b_i}{a_i} w^2(\tau_i)u(\tau_i) \leq \eta_1 \left[ w^2(\tau_{k(c_1)+1}) - \frac{k(d_1)}{a_d(\tau_i - \tau_{i-1})} \right].
\]

According to (2.6), it leads to

\[
\int_{c_1}^{d_1} \left\{ p(t) \right\}^{\frac{1}{\alpha}} q(t)^{-\frac{1}{\alpha}} w^2(t) - r(t)w^2(t) \, dt < Q(w, c_1, d_1),
\]

which contradicts (2.1).

If \( k(c_1) = k(d_1) \), then \( Q(w, c_1, d_1) = 0 \), and there are no impulsive moments in \([c_1, d_1]\). Similarly to the proof of (2.6), we get

\[
0 > \int_{c_1}^{d_1} \left\{ p(t) \right\}^{\frac{1}{\alpha}} q(t)^{-\frac{1}{\alpha}} w^2(t) - r(t)w^2(t) \, dt.
\]

This again contradicts our assumption.

In the case \( x(t) < 0 \) on \([c_1, d_1]\), we use the function \( y(t) = -x(t) \) as a positive solution of the equation

\[
\begin{cases}
(r(t)x'(t))' + p(t)x(t) = -q(t), & t \geq t_0, t \neq \tau_k, k = 1, 2, \ldots, \\
x(\tau_k^-) = a_k x(\tau_k), & x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, \ldots, \\
x(t_0^+) = x_0, & x'(t_0^+) = x_0'.
\end{cases}
\]

Then we repeat the above procedure on the subinterval \([c_2, d_2]\) in place of \([c_1, d_1]\). So this completes the proof of the theorem.

\[\square\]

**Corollary 2.1.** Assume that conditions (A1) and (A2) hold. If for any \( T > 0 \) there exist \( c_j, d_j \) satisfying (A3) with \( T \leq c_1 < d_1 < c_2 < d_2, \) and \( w(t) \in \Omega_w(c_j, d_j) \) satisfying (2.1), \( j = 1, 2, \) then (1.1) is oscillatory.
Theorem 2.2. Assume that conditions (A1)–(A3) hold, and there exists a $G \in \Omega_G(c_j, d_j)$ such that
\[
\int_{c_j}^{d_j} \left[ |p(t)|^{\frac{1}{2}} |q(t)|^{1-\frac{1}{2}} G(t) - r(t)h^2(t) \right] dt > R(G, c_j, d_j),
\]  
(2.10)
where $R(G, c_j, d_j) = 0$ for $k(c_j) = k(d_j)$, and
\[
R(G, c_j, d_j) = \eta_j \left[ G(\tau(c_j)) \frac{b_{k(c_j)+1} - a_{k(c_j)+1}}{a_{k(c_j)+1}} + \sum_{i=k(c_j)+2}^{k(d_j)} G(\tau_i) \frac{b_i - a_i}{a_i(\tau_i - \tau_{i-1})} \right]
\]  
(2.11)
for $k(c_j) < k(d_j)$, $j = 1, 2$. Then every solution of (1.1) has at least one zero in $[c_1, d_1] \cup [c_2, d_2]$.

Proof. Similarly to the proof of Theorem 2.1, suppose $x(t) > 0$ for $t \in [c_1, d_1] \cup [c_2, d_2]$. If $k(c_1) < k(d_1)$, let $G(t) \in \Omega_G(c_1, d_1)$, multiplying $G(t)$ throughout (2.4) and integrating over $[c_1, d_1]$, we obtain
\[
\sum_{i=k(c_1)+1}^{k(d_1)} G(\tau_i)[u(\tau_i) - u(\tau_i^+)] 
\geq \int_{c_1}^{d_1} \left[ |p(t)|^{\frac{1}{2}} |q(t)|^{1-\frac{1}{2}} G(t) - r(t)g^2(t) \right] dt + \int_{c_1}^{\tau(c_1)+1} \left[ \frac{G(t)}{r(t)} u(t) + \sqrt{r(t)} g(t) \right]^2 dt 
+ \sum_{i=k(c_1)+2}^{k(d_1)+2} \int_{\tau_i-1}^{\tau_i} \left[ \frac{G(t)}{r(t)} u(t) + \sqrt{r(t)} g(t) \right]^2 dt.
\]
From (2.7), (2.8) and (A2), one has
\[
\sum_{i=k(c_1)+1}^{k(d_1)} G(\tau_i)[u(\tau_i) - u(\tau_i^+)] = \sum_{i=k(c_1)+1}^{k(d_1)} G(\tau_i) \frac{a_i - b_i}{a_i} u(\tau_i) 
\leq \eta_1 \left[ G(\tau(c_1)) \frac{b_{k(c_1)+1} - a_{k(c_1)+1}}{a_{k(c_1)+1}} + \sum_{i=k(c_1)+2}^{k(d_1)} G(\tau_i) \frac{b_i - a_i}{a_i(\tau_i - \tau_{i-1})} \right] 
= R(G, c_1, d_1).
\]
This contradicts (2.10). This contradiction proves that $x(t)$ has at least a zero on $[c_1, d_1]$. If $k(c_1) = k(d_1)$, the proof is similar to that of Theorem 2.1, here it is omitted.

When $x(t) < 0$ on $t \in [c_1, d_1] \cup [c_2, d_2]$, we use $G \in \Omega_G(c_2, d_2)$ and $q(t) \geq 0$ on $[c_2, d_2]$ to derive a similar contradiction. □

Corollary 2.2. Assume that conditions (A1) and (A2) hold. If for any $T > 0$ there exist $c_j, d_j$ satisfying (A3) with $T \leq c_1 < d_1 < c_2 < d_2$, and $G \in \Omega_G(c_j, d_j)$ satisfying (2.10), $j = 1, 2$, then (1.1) is oscillatory.

Remark 2.1. When $a_k = b_k = 1$ for all $k = 1, 2, \ldots$, the impulses in (1.1) disappear. In such a case, (1.2) and (1.3) are particular cases of (1.1), Theorems 2.1 and 2.2 reduce to the main results of Wong [2] and Nsar [3].

Next, we will establish Kemeny-type oscillation criteria for (1.1) following the ideas of Kong [5] and Philos [13]. First, we introduce a class of functions $\Omega_H$ which will be used in the sequel. Let $D = \{ (t, s) : t_0 \leq s \leq t \}$; then function $H \in C(D, \mathbb{R})$ is said to belong to the class $\Omega_H$ if
\[
(H1) \ H(t, t) = 0, \ H(t, s) > 0 \text{ for } t > s; \ (H2) \ H \text{ has partial derivatives } \partial H/\partial t \text{ and } \partial H/\partial s \text{ on } D \text{ such that}
\]
\[
\frac{\partial H}{\partial t} = 2h_1(t, s)\sqrt{H(t, s)} \quad \text{and} \quad \frac{\partial H}{\partial s} = -2h_2(t, s)\sqrt{H(t, s)},
\]
where $h_1$ and $h_2 \in L_{loc}(D, R)$.

The following two lemmas are needed to prove our theorem.
Lemma 2.1. Assume that (A1) holds and \( x(t) \) is a solution of (1.1). If there exist \( \delta_j \in (c_j, d_j) \), \( \delta_j \not\in \{ \tau_k \}, j = 1, 2 \), such that \( x(t) > 0 \) on \( [\delta_1, d_1) \) and \( x(t) < 0 \) on \( [\delta_2, d_2) \), then for any \( H \in \Omega_H \),

\[
\int_{\delta_j}^{d_j} H(d_j, s)[p(s)]^{\frac{1}{2}} |q(s)|^{1 - \frac{1}{2}} ds \leq \sum_{i = k(\delta_j) + 1}^{k(d_j)} H(d_j, \tau_i) \frac{a_i - b_i}{a_i} u(\tau_i) - H(d_j, \delta_j) u(\delta_j)
\]

\[
+ \int_{\delta_j}^{d_j} r(s) h_2^2(d_j, s) ds, \quad j = 1, 2.
\]

(2.12)

Proof. We proceed as in the proof of Theorem 2.1, and multiply (2.4) by \( H(t, s) \), integrate it with respect to \( s \) from \( \delta_j \) to \( t \) for \( t \in [\delta_j, d_j) \) with \( j = 1, 2 \), it follows from (H1) and (H2) that

\[
\int_{\delta_j}^{t} H(t, s)[p(s)]^{\frac{1}{2}} |q(s)|^{1 - \frac{1}{2}} ds \leq \int_{\delta_j}^{t} H(t, s)u'(s) ds - \int_{\delta_j}^{t} H(t, s) \frac{1}{r(s)} u^2(s) ds
\]

\[
= \left( \int_{\delta_j}^{\tau_{k(\delta_j)} + 1} + \int_{\tau_{k(\delta_j)} + 1}^{\tau_{k(\delta_j) + 2}} + \cdots + \int_{\tau_{k(t)}}^{t} \right) H(t, s)du(s) - \int_{\delta_j}^{t} H(t, s) \frac{1}{r(s)} u^2(s) ds
\]

\[
= \sum_{i = k(\delta_j) + 1}^{k(t)} H(t, \tau_i) \frac{a_i - b_i}{a_i} u(\tau_i) - H(t, \delta_j) u(\delta_j) - \left( \int_{\delta_j}^{\tau_{k(\delta_j)} + 1} + \int_{\tau_{k(\delta_j)} + 1}^{\tau_{k(\delta_j) + 2}} + \cdots + \int_{\tau_{k(t)}}^{t} \right)
\]

\[
\times \left\{ r(s) h_2^2(t, s) + \frac{1}{r(s)} \left[ \sqrt{H(t, s)}u(s) - r(s) h_2(t, s) \right]^2 \right\} ds
\]

\[
\leq \sum_{i = k(\delta_j) + 1}^{k(t)} H(t, \tau_i) \frac{a_i - b_i}{a_i} u(\tau_i) - H(t, \delta_j) u(\delta_j) + \int_{\delta_j}^{t} r(s) h_2^2(t, s) ds.
\]

Now we let \( t \to d_j^- \), which yields that (2.12).

Lemma 2.2. Assume that (A1) holds and that \( x(t) \) is a solution of (1.1). If there exist \( \delta_j \in (c_j, d_j) \), \( \delta_j \not\in \{ \tau_k \}, j = 1, 2 \) such that \( x(t) > 0 \) on \( (c_1, \delta_1] \) and \( x(t) < 0 \) on \( (c_2, \delta_2] \), then for any \( H \in \Omega_H \),

\[
\int_{c_j}^{\delta_j} H(s, c_j)[p(s)]^{\frac{1}{2}} |q(s)|^{1 - \frac{1}{2}} ds \leq \sum_{i = k(c_j) + 1}^{k(\delta_j)} H(\tau_i, c_j) \frac{a_i - b_i}{a_i} u(\tau_i) + H(\delta_j, c_j) u(\delta_j)
\]

\[
+ \int_{c_j}^{\delta_j} r(s) h_1^2(c_j, s) ds, \quad j = 1, 2.
\]

(2.13)

Proof. Similarly to the proof of Lemma 2.1, multiplying (2.4) by \( H(s, t) \) for \( j = 1, 2 \), we have

\[
\int_{t}^{\delta_j} H(s, t)[p(s)]^{\frac{1}{2}} |q(s)|^{1 - \frac{1}{2}} ds \leq \int_{t}^{\delta_j} H(s, t)u'(s) ds - \int_{t}^{\delta_j} H(s, t) \frac{1}{r(s)} u^2(s) ds
\]

\[
= \left( \int_{t}^{\tau_{k(t)} + 1} + \int_{\tau_{k(t)} + 1}^{\tau_{k(t) + 2}} + \cdots + \int_{\tau_{k(\delta_j)}}^{\delta_j} \right) H(s, t)du(s) - \int_{t}^{\delta_j} H(s, t) \frac{1}{r(s)} u^2(s) ds
\]

\[
= \sum_{i = k(t) + 1}^{k(\delta_j)} H(\tau_i, t) \frac{a_i - b_i}{a_i} u(\tau_i) + H(\delta_j, t) u(\delta_j) - \left( \int_{t}^{\tau_{k(t)} + 1} + \int_{\tau_{k(t)} + 1}^{\tau_{k(t) + 2}} + \cdots + \int_{\tau_{k(\delta_j)}}^{\delta_j} \right)
\]

\[
\times \left\{ r(s) h_2^2(t, s) - \frac{1}{r(s)} \left[ \sqrt{H(s, t)}u(s) - r(s) h_1(t, s) \right]^2 \right\} ds
\]

\[
\leq \sum_{i = k(t) + 1}^{k(\delta_j)} H(\tau_i, t) \frac{a_i - b_i}{a_i} u(\tau_i) + H(\delta_j, t) u(\delta_j) + \int_{t}^{\delta_j} r(s) h_1^2(t, s) ds.
\]

Thus it yields (2.13), only letting \( t \to c_j^+ \).
Theorem 2.3. Assume that conditions (A1)–(A3) hold. Suppose that there are \( \delta_j \in (c_j, d_j) \), \( j = 1, 2 \), and \( H \in \Omega_H \) such that

\[
\frac{1}{H(d_j, \delta_j)} \int_{\delta_j}^{d_j} \left\{ H(d_j, s)[p(s)]^{\frac{1}{2}} |q(s)|^{1-\frac{1}{\alpha}} - r(s)h_1^2(s, c_j) \right\} ds \\
+ \frac{1}{H(\delta_j, c_j)} \int_{c_j}^{\delta_j} \left\{ H(s, c_j)[p(s)]^{\frac{1}{2}} |q(s)|^{1-\frac{1}{\alpha}} - r(s)h_1^2(s, c_j) \right\} ds > P(H, c_j, d_j),
\]

(2.14)

where \( P(H, c_j, d_j) = 0 \) for \( k(c_j) = k(d_j) \), and

\[
P(H, c_j, d_j) = \frac{\eta_j}{H(d_j, \delta_j)} \left( H(d_j, \tau_{k(c_j)+1}) \frac{b_{k(c_j)+1} - a_{k(c_j)+1}}{a_{k(c_j)+1}} + \sum_{i=k(c_j)+2}^{k(d_j)} \frac{H(d_j, \tau_i)}{a_i} \left( \frac{b_i - a_i}{a_i (\tau_i - \tau_{i-1})} \right) \right) \\
+ \frac{\eta_j}{H(\delta_j, c_j)} \left( H(\tau_{k(c_j)+1}, c_j) \frac{b_{k(c_j)+1} - a_{k(c_j)+1}}{a_{k(c_j)+1}} + \sum_{i=k(c_j)+2}^{k(\delta_j)} \frac{H(\tau_i, c_j)}{a_i} \left( \frac{b_i - a_i}{a_i (\tau_i - \tau_{i-1})} \right) \right)
\]

(2.15)

for \( k(c_j) < k(d_j) \), \( j = 1, 2 \). Then every solution of (1.1) has at least one zero in \([c_1, d_1] \cup [c_2, d_2] \).

Proof. Let \( x(t) \) be a solution of (1.1). Suppose \( x(t) \) does not have any zero in \([c_1, d_1] \cup [c_2, d_2] \). Without loss of generality, we may assume that \( x(t) > 0 \) for \( t \in [c_1, d_1] \). From Lemmas 2.1 and 2.2 we see that (2.12) and (2.13) with \( j = 1 \) hold. Dividing (2.12) and (2.13) by \( H(d_1, \delta_1) \) and \( H(\delta_1, c_1) \), respectively, then adding them, we have

\[
\frac{1}{H(d_1, \delta_1)} \int_{\delta_1}^{d_1} \left\{ H(d_1, s)[p(s)]^{\frac{1}{2}} |q(s)|^{1-\frac{1}{\alpha}} - r(s)h_1^2(d_1, s) \right\} ds \\
+ \frac{1}{H(\delta_1, c_1)} \int_{c_1}^{\delta_1} \left\{ H(s, c_1)[p(s)]^{\frac{1}{2}} |q(s)|^{1-\frac{1}{\alpha}} - r(s)h_1^2(s, c_1) \right\} ds \\
\leq \frac{1}{H(d_1, \delta_1)} \sum_{i=k(\delta_1)+1}^{k(d_1)} H(d_1, \tau_i) \frac{a_i - b_i}{a_i} u(\tau_i) + \frac{1}{H(\delta_1, c_1)} \sum_{i=k(c_1)+1}^{k(\delta_1)} H(\tau_i, c_1) \frac{a_i - b_i}{a_i} u(\tau_i).
\]

(2.16)

For the first term in the right side of the above, we replace \( c_1 \) by \( \delta_1 \), and the second term \( d_1 \) by \( \delta_1 \) in (2.7) and (2.8). Then it holds that

\[
\frac{1}{H(d_1, \delta_1)} \int_{\delta_1}^{d_1} \left\{ H(d_1, s)[p(s)]^{\frac{1}{2}} |q(s)|^{1-\frac{1}{\alpha}} - r(s)h_1^2(d_1, s) \right\} ds \\
+ \frac{1}{H(\delta_1, c_1)} \int_{c_1}^{\delta_1} \left\{ H(s, c_1)[p(s)]^{\frac{1}{2}} |q(s)|^{1-\frac{1}{\alpha}} - r(s)h_1^2(s, c_1) \right\} ds \\
\leq \frac{\eta_1}{H(d_1, \delta_1)} \left( H(d_1, \tau_{k(\delta_1)+1}) \frac{b_{k(\delta_1)+1} - a_{k(\delta_1)+1}}{a_{k(\delta_1)+1}} + \sum_{i=k(\delta_1)+2}^{k(d_1)} \frac{H(d_1, \tau_i)}{a_i} \left( \frac{b_i - a_i}{a_i (\tau_i - \tau_{i-1})} \right) \right) \\
+ \frac{\eta_1}{H(\delta_1, c_1)} \left( H(\tau_{k(c_1)+1}, c_1) \frac{b_{k(c_1)+1} - a_{k(c_1)+1}}{a_{k(c_1)+1}} + \sum_{i=k(c_1)+2}^{k(\delta_1)} \frac{H(\tau_i, c_1)}{a_i} \left( \frac{b_i - a_i}{a_i (\tau_i - \tau_{i-1})} \right) \right),
\]

(2.17)

which contradicts (2.14).

When \( x(t) < 0 \), it is easy to reach a similar contradiction and this is omitted here. □

Corollary 2.3. Assume that conditions (A1) and (A2) hold. If for any \( T > 0 \) there exist \( c_j, d_j, \delta_j \) satisfying (A3) with \( T \leq c_1 < d_1 \leq c_2 < d_2 \) and \( \delta_j \in (c_j, d_j) \), and \( H \in \Omega_H(c_j, d_j) \) satisfying (2.14), \( j = 1, 2 \), then (1.1) is oscillatory.
Remark 2.2. The presented results in Corollary 2.3 provide wide possibilities of deriving different oscillation criteria with an appropriate choice of the function $H$, for example with $H(t, s) = (t - s)^2$ which is used in Example 3.2, or

$$H(t, s) = \left[ \int_s^t \frac{d\theta}{\varphi(\theta)} \right]^{\beta - 1},$$

where $\beta > 2$ is a constant and $w(\theta)$ is a positive continuous function on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} \frac{d\theta}{\varphi(\theta)} = \infty.$$

3. Examples

Example 3.1. Consider the following super-linear impulsive differential equation

$$
\begin{align*}
(1 + \sin^2 t)x'(t) + (\beta \cos t)|x(t)|^{\alpha - 1}x(t) &= \sin t, \\
x(t_k^+) &= a_k x(\tau_k), \\
x'(t_k^+) &= b_k x'(\tau_k), \quad k = 1, 2, \ldots.
\end{align*}
$$

Eq. (3.1) is oscillatory if

$$\beta \frac{1}{2} \frac{\Gamma(3 + \frac{1}{\alpha}) \Gamma(4 + \frac{1}{\alpha})}{\Gamma(7)} \geq \frac{3\pi}{4} + \frac{2(b_k - a_k)}{\pi}, \quad k = 1, 2, \ldots,$$

where $a \geq 0, \alpha > 1$, $\Gamma$ is the gamma function, $a_k$ and $b_k$ satisfy (A2).

Indeed, for any $T > 0$, we can choose $n$ large enough such that $T < c_1 = 2n\pi - \frac{\pi}{2}$, $d_1 = c_2 = 2n\pi$, $d_2 = 2n\pi + \frac{\pi}{2}$ and $w(t) = -\sin 2t$. By a simple calculation, it is easy to see that

$$\int_{c_1}^{d_1} \left\{ (p(t))^{\frac{1}{\alpha}} |q(t)|^{1 - \frac{1}{\alpha}} w^2(t) - r(t) w'^2(t) \right\} dt = 4 \beta \frac{1}{2} \int_{2\pi - \frac{\pi}{2}}^{2\pi} \frac{1}{4} \sin t \frac{1}{4} dt - 4 \int_{2\pi - \frac{\pi}{2}}^{2\pi} (1 + \sin^2 2t) dt
= 2 \beta \frac{1}{2} \frac{\Gamma(3 + \frac{1}{\alpha}) \Gamma(4 + \frac{1}{\alpha})}{\Gamma(7)} \frac{3\pi}{2}.
$$

On the other hand, note that $k(c_1) = n - 1$, $k(d_1) = n$, $\eta_1 = 2$, then

$$Q(w, c_1, d_1) = \frac{4(b_n - a_n)}{\pi}.
$$

Thus inequality (3.2) implies that

$$\int_{c_1}^{d_1} \left\{ (p(t))^{\frac{1}{\alpha}} |q(t)|^{1 - \frac{1}{\alpha}} w^2(t) - r(t) w'^2(t) \right\} dt \geq Q(w, c_1, d_1).
$$

Similarly, for $c_2, d_2$, we also can show that

$$\int_{c_2}^{d_2} \left\{ (p(t))^{\frac{1}{\alpha}} |q(t)|^{1 - \frac{1}{\alpha}} w^2(t) - r(t) w'^2(t) \right\} dt \geq Q(w, c_2, d_2).
$$

It follows from Corollary 2.1 that (3.1) is oscillatory.

Example 3.2. Consider the following super-linear impulsive differential equation

$$
\begin{align*}
x''(t) + \mu p(t)x^3(t) &= q(t), \\
x(t_k^+) &= a_k x(\tau_k), \\
x'(t_k^+) &= b_k x'(\tau_k),
\end{align*}
$$

(3.3)
where $\tau_{3n+1} = 8n + \frac{1}{2}$, $\tau_{3n+2} = 8n + \frac{3}{2}$, $\tau_{3n+3} = 8n + \frac{9}{2}$, $n = 0, 1, 2, \ldots, \mu > 0$ is a constant, $a_k$ and $b_k$ satisfy (A2). In addition,

$$p(t) = \begin{cases} 
(t - 8n)^3, & t \in [8n, 8n + 2], \\
(8n + 4 - t)^3, & t \in (8n + 2, 8n + 4], \\
(t - 8n - 4)^3, & t \in (8n + 4, 8n + 6), \\
(8n + 8 - t)^3, & t \in (8n + 6, 8n + 8], 
\end{cases}$$

and

$$q(t) = \begin{cases} 
(t - 8n - 2)^3, & t \in [8n, 8n + 4], \\
(8n + 6 - t)^3, & t \in (8n + 4, 8n + 8]. 
\end{cases}$$

For any $T > 0$, we can choose $n$ large enough such that $T < c_1 = 8n < \delta_1 = 8n + 1 < d_1 = 8n + 2$ and $c_2 = 8n + 4 < \delta_2 = 8n + 5 < d_2 = 8n + 6$ satisfying (A3). Let $H(t, s) = (t - s)^2$, then $h_1(t, s) = h_2(t, s) \equiv 1$, and we have

$$\begin{align*}
\frac{1}{H(d_1, c_1)} \int_{\delta_1}^{d_1} & \{H(d_1, s)\mu p(s)\frac{1}{2} |q(s)|^{1 - \frac{1}{2}} - r(s)h_2^2(d_1, s)\}ds \\
+ \frac{1}{H(\delta_1, c_1)} \int_{c_1}^{\delta_1} & \{H(s, c_1)\mu p(s)\frac{1}{2} |q(s)|^{1 - \frac{1}{2}} - r(s)h_1^2(s, c_1)\}ds \\
= \mu \frac{1}{2} \int_{8n+2}^{8n+4} & (8n + 2 - s)^2[(s - 8n)^3]^{\frac{1}{2}}[(8n + 2 - s)^3]^{\frac{1}{2}}ds - 1 \\
+ \mu \frac{1}{2} \int_{8n}^{8n+1} & (s - 8n)^2[(s - 8n)^3]^{\frac{1}{2}}[(8n + 2 - s)^3]^{\frac{1}{2}}ds - 1 \\
= \mu \frac{1}{2} \left( \int_{8n+2}^{8n+4} & (8n + 2 - s)^4(s - 8n)ds + \int_{8n}^{8n+1} (s - 8n)^3(8n + 2 - s)^2ds \right) - 2 \\
= \mu \frac{1}{2} \left( \int_{0}^{1} & (1-t)^4(t + 1)dt + \int_{-1}^{0} (1+t)^3(1-t)^2dt \right) - 2 \\
= \frac{3}{5} \mu \frac{1}{2} - 2.
\end{align*}$$

Note that $\tau_{3n+1} = 8n + \frac{1}{2} \in (c_1, \delta_1)$ and $\tau_{3n+2} = 8n + \frac{3}{2} \in (\delta_1, d_1)$, so it is easy to get

$$P(H, c_1, d_1) = \frac{1}{2} \left( \frac{b_{3n+1} - a_{3n+1}}{a_{3n+1}} + \frac{b_{3n+2} - a_{3n+2}}{a_{3n+2}} \right).$$

Thus (2.14) is satisfied with $j = 1$ if

$$\mu > \left[ \frac{10}{3} \frac{5}{6} \left( \frac{b_{3n+1} - a_{3n+1}}{a_{3n+1}} + \frac{b_{3n+2} - a_{3n+2}}{a_{3n+2}} \right) \right]^3. \quad (3.4)$$

With arguments similar to those for the interval $[c_2, d_2]$, only noting that $\tau_{3n+3} = 8n + \frac{9}{2} \in (c_2, \delta_2)$ and that there is no impulsive moment in $(\delta_2, d_2)$, we know (2.14) is satisfied with $j = 2$ if

$$\mu > \left[ \frac{10}{3} \frac{5}{6} \left( b_{3n+3} - a_{3n+3} \right) \right]^3. \quad (3.5)$$

Therefore (3.3) is oscillatory from Corollary 2.3 if (3.4) and (3.5) hold.

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References