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A granularity-based framework of deduction, induction, and abduction

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ABSTRACT

In this paper, we propose a granularity-based framework of deduction, induction, and abduction using variable precision rough set models proposed by Ziarko and measure-based semantics for modal logic proposed by Murai et al. The proposed framework is based on α-level fuzzy measure models on the basis of background knowledge, as described in the paper. In the proposed framework, deduction, induction, and abduction are characterized as reasoning processes based on typical situations about the facts and rules used in these processes. Using variable precision rough set models, we consider β-lower approximation of truth sets of nonmodal sentences as typical situations of the given facts and rules, instead of the truth sets of the sentences as correct representations of the facts and rules. Moreover, we represent deduction, induction, and abduction as relationships between typical situations.

1. Introduction

Reasoning processes in our daily life consist of various styles of reasoning under uncertainty, such as logical reasoning with some nonmonotonicity, probabilistic reasoning, and reasoning with ambiguity and vagueness; for example, implying conclusions logically from information we currently possess, finding rules from observations, and speculate reasons behind observed (or reported) facts. In general, logical aspects of these types of reasoning processes are divided into the following three categories:

- Deduction: A reasoning process for concluding specific facts from general rules.
- Induction: A reasoning process for providing general rules from specific facts.
- Abduction: A reasoning process for providing hypotheses that explain the given facts.

Moreover, when we consider these types of reasoning processes, we consider not *all* possible scenarios or situations that match the propositions used in them, but some *typical* scenarios or situations. For example, suppose we consider the following deduction: from the propositions "the sun rises in the east" and "if the sun rises in the east, then the sun sets in the west," we conclude that "the sun sets in the west." In this deduction process, we do not consider all days when the sun rose in the east, and we may consider only a small number of examples of days when the sun rose in the east as typical situations. Moreover, because the sun set in the west on any typical day when the sun rose in the east, we conclude that the sun sets in the west. In other words, typical situations in which the sun rises in the east are also typical situations in which the sun sets in

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the west. This example indicates that considering the relationship between typical situations captures aspects of deduction, induction, and abduction in our daily life.

Consequently, in this paper, we consider the semantic characterization of deduction, induction, and abduction by the possible world semantics of modal logic. In possible world semantics, each nonmodal sentence that represents a fact is characterized by its truth set, i.e., the set of *all* possible worlds in which the nonmodal sentence is true in the given model. We consider the truth set of a nonmodal sentence as the correct representation of the given fact. However, as we have discussed, we need to treat typical situations related to facts, and treating only the truth sets of nonmodal sentences that represent facts is not suitable, because these truth sets correspond to all situations that match the facts. Thus, we need to represent typical situations based on some theory.

To represent typical situations about the facts, we consider introducing rough set theory to the possible world semantics of modal logic. Rough set theory [11,12] provides a theoretical basis of set-theoretic approximation and reasoning based on data. The variable precision rough set (VPRS) models proposed by Ziarko [14] are an extension of rough set theory, which enables us to treat probabilistic or inconsistent information in the framework of rough sets. In terms of the relationship between rough set theory and modal logic, it is well known that lower (upper) approximation in rough sets and necessity (possibility) as interpreted by Kripke models of modal logic are closely related. From the viewpoint of reasoning based on rough set theory, Murai et al. have proposed a framework of granular reasoning [9,10], which represents reasoning processes by controlling the granularity of equivalence classes. Moreover, we have discussed the relationship between granularity and background knowledge in reasoning processes [6]. From the viewpoint of rough sets, we consider characterizing typical situations of the given facts by the lower approximations of the truth sets of nonmodal sentences that represent the given facts. Moreover, we also need to consider misunderstandings about facts.

Similar to the case of the relationship between Pawlak's rough sets and Kripke models, we can consider the relationship between VPRS models and the measure-based semantics of modal logic proposed by Murai et al. [7,8]. Measure-based semantics provides an interpretation of modal sentences using fuzzy measures assigned to possible worlds. On the other hand, the definition of β-lower approximations in VPRS models is based on conditional probabilities, i.e., a special case of fuzzy measures. This relationship indicates that we can treat the concept of typical situations characterized by β-lower approximations as a modality of modal logic interpreted by measure-based semantics.

Moreover, as we have discussed, consideration of the relationship between typical situations represents logical aspects of deduction, induction, and abduction. Thus, considering the relationship between typical situations and all possible situations, we propose characterizing deduction, induction, and abduction as the following processes based on typical situations:

- Deduction: A reasoning process for providing a conclusion from a general rule and a condition that holds in the given typical situation.
- Induction: A reasoning process for concluding a general rule from typical situations.
- Abduction: A reasoning process for providing a hypothesis that explains a fact in the given typical situation.

Combining the above discussions, we propose a unified framework of deduction, induction, and abduction using granularity based on VPRS models and measure-based semantics for modal logic. Note that this paper is a revised and extended version of a conference paper [5].

The paper is structured as follows. In Section 2, we briefly review rough sets and VPRS. In Section 3, Kripke models and measure-based models are introduced as the basis of the formulation presented in the paper. In Section 4, we introduce α-level fuzzy measure models based on background knowledge to illustrate characteristics of typical situations by a modal operator of propositional modal logic. In Section 5, we characterize three types of reasoning processes known as deduction, induction, and abduction in the framework of α-level fuzzy measure models on the basis of background knowledge as reasoning processes based on typical situations. Finally, we present our conclusions in Section 6.

2. Rough set theory

2.1. Rough sets

In this subsection, we briefly review the foundations of Pawlak's rough set theory and VPRS models. This subsection is based on [13].

Let U be a nonempty, finite set of objects called the universe of discourse and R be an equivalence relation on U called an indiscernibility relation. For any element $x \in U$, the equivalence class of x with respect to R is defined as

$$[x]_R \stackrel{\text{def}}{=} \{y \in U \mid xRy\}. \tag{1}$$

The equivalence class $[x]_R$ is the set of objects that cannot be discerned from x with respect to R. The quotient set $U/R \stackrel{\text{def}}{=} \{[x]_R \mid x \in U\}$ provides a partition of U. According to Pawlak [12], any set $X \subseteq U$ represents a *concept*, and a set of concepts is called *knowledge* about U. Thus, we consider that R provides knowledge about U as the quotient set U/R.

The ordered pair (U, R) is called an approximation space, and it provides the basis of approximation in rough set theory. For any set of objects $X \subseteq U$, the *lower approximation* $\underline{R}(X)$ and the *upper approximation* $\overline{R}(X)$ of X by R are defined as

$$\underline{R}(X) \stackrel{\text{def}}{=} \{x \in U \mid [x]_R \subseteq X\}, \tag{2}$$

$$\overline{R}(X) \stackrel{\text{def}}{=} \{x \in U \mid [x]_R \cap X \neq \emptyset\}. \tag{3}$$

$\underline{R}(X)$ of X is the set of objects that are certainly included in X. On the other hand, $\overline{R}(X)$ of X is the set of objects that may be included in X.

If $\underline{R}(X) = X = \overline{R}(X)$, we consider that X is R-definable; else, if $\underline{R}(X) \subset X \subset \overline{R}(X)$, we consider that X is R-rough. The concept X being R-definable means that we can denote X correctly using background knowledge based on R. On the other hand, when X is R-rough, we cannot denote the concept correctly based on background knowledge.

2.2. Variable precision rough set models

The VPRS models proposed by Ziarko [14] are an extension of Pawlak's rough set theory, which provides a theoretical basis to treat probabilistic or inconsistent information in the framework of rough sets.

VPRS is based on the majority inclusion relation. Let $X, Y \subseteq U$ be any subsets of U. The majority inclusion relation is defined by the following measure $c(X, Y)$ of the relative degree of misclassification of X with respect to Y,

$$c(X, Y) \stackrel{\text{def}}{=} \begin{cases} 1 - \dfrac{|X \cap Y|}{|X|}, & \text{if } X \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $|X|$ represents the cardinality of the set X. It is easy to confirm that $X \subseteq Y$ holds, if and only if $c(X, Y) = 0$.

Formally, the majority inclusion relation $\stackrel{\beta}{\subseteq}$ with a fixed *precision* $\beta \in [0, 0.5)$ is defined using the relative degree of misclassification as follows:

$$X \stackrel{\beta}{\subseteq} Y \stackrel{\text{def}}{\Longleftrightarrow} c(X, Y) \leqslant \beta, \tag{5}$$

where the precision β provides the limit of permissible misclassification.

Let $X \subseteq U$ be any set of objects, R be an indiscernibility relation on U, and the degree $\beta \in [0, 0.5)$ be a precision. The *β-lower approximation* $\underline{R}_\beta(X)$ and the *β-upper approximation* $\overline{R}_\beta(X)$ of X are defined as follows:

$$\underline{R}_\beta(X) \stackrel{\text{def}}{=} \left\{ x \in U \,\middle|\, [x]_R \stackrel{\beta}{\subseteq} X \right\} = \left\{ x \in U \,\middle|\, c([x]_R, X) \leqslant \beta \right\}, \tag{6}$$

$$\overline{R}_\beta(X) \stackrel{\text{def}}{=} \left\{ x \in U \,\middle|\, c([x]_R, X) < 1 - \beta \right\}. \tag{7}$$

As mentioned previously, the precision β represents the threshold degree of misclassification of elements in the equivalence class $[x]_R$ to the set X. Thus, in VPRS, misclassification of elements is allowed if the ratio of misclassification is less than β. Note that the β-lower and -upper approximations with $\beta = 0$ correspond to Pawlak's lower and upper approximations.

Table 1 represents some properties of the β-lower and -upper approximations. The symbols "○" and "×" indicate whether a property is satisfied ("○") or may not be satisfied ("×") in the case of $\beta = 0$ and $0 < \beta < 0.5$, respectively. For example, by the definition of the β-lower approximation in (6), it is easy to confirm that the property **T**. $\underline{R}_\beta(X) \subseteq X$ is not guaranteed to be satisfied in the case of $0 < \beta < 0.5$. Note that symbols assigned to properties such that **T** correspond to axiom schemas in modal logic (for details, see [1]). In the next subsection, we briefly review modal logic and the relationship between rough set theory and modal logic.

Table 1
Some properties of β-lower and upper approximations.

Properties		$\beta = 0$	$0 < \beta < 0.5$
Df $\Diamond$.	$\overline{R}_\beta(X) = \underline{R}_\beta(X^c)^c$	○	○
M.	$\underline{R}_\beta(X \cap Y) \subseteq \underline{R}_\beta(X) \cap \underline{R}_\beta(Y)$	○	○
C.	$\underline{R}_\beta(X) \cap \underline{R}_\beta(Y) \subseteq \underline{R}_\beta(X \cap Y)$	○	×
N.	$\underline{R}_\beta(U) = U$	○	○
K.	$\underline{R}_\beta(X^c \cup Y) \subseteq (\underline{R}_\beta(X)^c \cup \underline{R}_\beta(Y))$	○	×
D.	$\underline{R}_\beta(X) \subseteq \overline{R}_\beta(X)$	○	○
P.	$\underline{R}_\beta(\emptyset) = \emptyset$	○	○
T.	$\underline{R}_\beta(X) \subseteq X$	○	×
B.	$X \subseteq \underline{R}_\beta(\overline{R}_\beta(X))$	○	×
4.	$\underline{R}_\beta(X) \subseteq \underline{R}_\beta(\underline{R}_\beta(X))$	○	○
5.	$\overline{R}_\beta(X) \subseteq \underline{R}_\beta(\overline{R}_\beta(X))$	○	○
	$X \subseteq Y \Rightarrow \underline{R}_\beta(X) \subseteq \underline{R}_\beta(Y)$	○	○
	$R \subseteq R' \Rightarrow \underline{R}_\beta(X) \supseteq \underline{R'}_\beta(X)$	○	×

3. Possible world semantics for modal logic

3.1. Kripke models

Propositional modal logic (hereafter called simply modal logic) extends classical propositional logic by two unary operators $\Box$ and $\Diamond$ (called modal operators), and for any proposition p, it provides the following statements: $\Box p$ (p is necessary) and $\Diamond p$ (p is possible).

Let $\mathcal{L}_{ML}(\mathcal{P})$ be a set of sentences constructed from a given at most countably infinite set of atomic sentences $\mathcal{P} = \{\mathsf{p}_1, \ldots, \mathsf{p}_n(,\ldots)\}$; constant sentences $\top$ (truth) and $\bot$ (falsity); logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditionality), $\leftrightarrow$ (biconditionality), and $\neg$ (negation); and modal operators $\Box$ (necessity) and $\Diamond$ (possibility) by the following construction rules:

$$\mathsf{p} \in \mathcal{P} \Rightarrow \mathsf{p} \in \mathcal{L}_{ML}(\mathcal{P}), \top, \bot \in \mathcal{L}_{ML}(\mathcal{P}), p \in \mathcal{L}_{ML}(\mathcal{P}) \Rightarrow \neg p, \Box p, \Diamond p \in \mathcal{L}_{ML}(\mathcal{P}),$$
$$p, q \in \mathcal{L}_{ML}(\mathcal{P}) \Rightarrow p \wedge q, p \vee q, p \rightarrow q, p \leftrightarrow q \in \mathcal{L}_{ML}(\mathcal{P}).$$

We say that a sentence is modal if it contains at least one modal operator; else, we say it is nonmodal.

In this paper, we consider possible world semantics to interpret sentences used in modal logic. A Kripke model, one of the most popular frameworks of possible world semantics, is the following triple:

$$\mathcal{M} = (U, R, v), \tag{8}$$

where U is a set of possible worlds, R is a binary relation on U called an accessibility relation, and $v : \mathcal{P} \times U \rightarrow \{0, 1\}$ is a valuation function that assigns a truth value to each atomic sentence $\mathsf{p} \in \mathcal{P}$ at each world $w \in U$. An atomic sentence p is defined as true at a possible world x by the given Kripke model $\mathcal{M}$ if and only if $v(\mathsf{p}, x) = 1$. We say that a Kripke model is finite if its set of possible worlds is a finite set.

$\mathcal{M}, x \models p$ indicates that the sentence p is true at the possible world $x \in U$ by the Kripke model $\mathcal{M}$. Interpretation of nonmodal sentences is similar to the case of classical propositional logic, as follows:

$$\mathcal{M}, x \models \mathsf{p}\ (\mathsf{p} \in \mathcal{L}_{ML}(\mathcal{P})) \iff v(\mathsf{p}, x) = 1,$$
$$\mathcal{M}, x \models \neg p \iff \mathcal{M}, x \not\models p,$$
$$\mathcal{M}, x \models p \wedge q \iff \mathcal{M}, x \models p \text{ and } \mathcal{M}, x \models q,$$
$$\mathcal{M}, x \models p \vee q \iff \mathcal{M}, x \models p \text{ or } \mathcal{M}, x \models q,$$
$$\mathcal{M}, x \models p \rightarrow q \iff \mathcal{M}, x \not\models p \text{ or } \mathcal{M}, x \models q,$$
$$\mathcal{M}, x \models p \leftrightarrow q \iff \mathcal{M}, x \models p \rightarrow q \text{ and } \mathcal{M}, x \models q \rightarrow p.$$

In possible world semantics using Kripke models, on the other hand, we use accessibility relations to interpret modal sentences. $\Box p$ is true at x if and only if p is true at every possible world y accessible from x. Conversely, $\Diamond p$ is true at x if and only if there is at least one possible world y accessible from x, and p is true at y. Formally, interpretation of modal sentences is defined as follows:

$$\mathcal{M}, x \models \Box p \stackrel{\text{def}}{\iff} \forall y \in U (xRy \Rightarrow \mathcal{M}, y \models p), \tag{9}$$
$$\mathcal{M}, x \models \Diamond p \stackrel{\text{def}}{\iff} \exists y \in U (xRy \text{ and } \mathcal{M}, y \models p). \tag{10}$$

For any sentence $p \in \mathcal{L}_{ML}(\mathcal{P})$, the truth set is the set of possible worlds in which p is true by the Kripke model $\mathcal{M}$, and the truth set is defined as follows:

$$\|p\|^{\mathcal{M}} \stackrel{\text{def}}{=} \{x \in U \mid \mathcal{M}, x \models p\}. \tag{11}$$

We say that a sentence p is true in a Kripke model $\mathcal{M}$, if and only if p is true at every possible world in $\mathcal{M}$. We denote $\mathcal{M} \models p$ if p is true in $\mathcal{M}$.

When the accessibility relation R in the given Kripke model $\mathcal{M}$ is an equivalence relation, for any possible world $x \in U$, the set of possible worlds that are accessible from x by R is identical to the equivalence class $[x]_R$. Thus, in this case, we can rewrite the definition of modal sentence interpretation as

$$\mathcal{M}, x \models \Box p \iff [x]_R \subseteq \|p\|^{\mathcal{M}}, \tag{12}$$
$$\mathcal{M}, x \models \Diamond p \iff [x]_R \cap \|p\|^{\mathcal{M}} \neq \emptyset. \tag{13}$$

Therefore, when R is an equivalence relation, the following correspondence relationship holds between Pawlak's lower approximation and necessity, and that between Pawlak's upper approximation and possibility:

$$\underline{R}(\|p\|^{\mathcal{M}}) = \|\Box p\|^{\mathcal{M}}, \tag{14}$$
$$\overline{R}(\|p\|^{\mathcal{M}}) = \|\Diamond p\|^{\mathcal{M}}. \tag{15}$$

The modal system *S5* is well known to be sound and complete with respect to the class of all Kripke models with equivalence relations as accessibility relations (for details, see [1]). The modal system *S5* consists of all inference rules and axiom schemas of propositional logic and the following inference rules and axiom schemas:

Df$\Diamond$. $\Diamond p \leftrightarrow \neg\Box\neg p$, **RN**. $\dfrac{p}{\Box p}$,

K. $\Box(p \to q) \to (\Box p \to \Box q)$, **T**. $\Box p \to p$,

5. $\Diamond p \to \Box\Diamond p$.

All other axiom schemas listed in Table 1; that is, axiom schemas **M**, **C**, **N**, **P**, **D**, **B**, and **4** are theorems of *S5*. Therefore, from the viewpoint of modal logic, Pawlak's rough set theory is characterized by the modal system *S5*.

3.2. Measure-based semantics

Instead of using accessibility relations to interpret modal sentences, measure-based semantics of modal logic uses fuzzy measures [7,8]. A function $\mu : 2^U \to [0,1]$ is called a *fuzzy measure* on U if the function μ satisfies the following three conditions:

(1) $\mu(U) = 1$,
(2) $\mu(\emptyset) = 0$,
(3) $\forall X, Y \subseteq U,\ X \subseteq Y \Rightarrow \mu(X) \leqslant \mu(Y)$,

where 2^U represents the power set of U.

Formally, a fuzzy measure model $\mathscr{M}_\mu$ is the following triple,

$$\mathscr{M}_\mu = (U, \{\mu_x\}_{x\in U}, v), \tag{16}$$

where U is a set of possible worlds, and v is a valuation. $\{\mu_x\}_{x\in U}$ is a class of fuzzy measures μ_x assigned to all possible worlds $x \in U$.

In measure-based semantics of modal logic, each degree $\alpha \in (0,1]$ of fuzzy measures corresponds to a modal operator $\Box_\alpha$ [7,8]. Thus, fuzzy measure models can provide semantics of multimodal logic with modal operators $\Box_\alpha$ ($\alpha \in (0,1]$). In this paper, however, we fix a degree α and consider *α-level fuzzy measure models* that provide semantics of modal logic with the two modal operators $\Box$ and $\Diamond$.

Similar to the case of Kripke models, $\mathscr{M}_\mu, x \models p$ indicates that the sentence p is true at the possible world $x \in U$ by the α-level fuzzy measure model $\mathscr{M}_\mu$. Interpretation of nonmodal sentences is identical to that in Kripke models. On the other hand, to define the truth value of modal sentences at each world $x \in U$ in the α-level fuzzy measure model M_μ, we use the fuzzy measure μ_x assigned to the world x instead of accessibility relations. Interpretation of modal sentences $\Box p$ at a world x is defined as follows:

$$\mathscr{M}_\mu, x \models \Box p \overset{\text{def}}{\Longleftrightarrow} \mu_x\left(\|p\|^{\mathscr{M}_\mu}\right) \geqslant \alpha, \tag{17}$$

where μ_x is the fuzzy measure assigned to x. By this definition, interpretation of modal sentences $\Diamond p$ is obtained by dual fuzzy measures as follows:

$$\mathscr{M}_\mu, x \models \Diamond p \Longleftrightarrow \mu_x^*\left(\|p\|^{\mathscr{M}_\mu}\right) > 1 - \alpha, \tag{18}$$

where the dual fuzzy measure μ_x^* of the assigned fuzzy measure μ_x is defined as $\mu_x^*(X) \overset{\text{def}}{=} 1 - \mu_x(X^c)$ for any $X \subseteq U$.

Note that the modal system *EMNP* is sound and complete with respect to the class of all α-level fuzzy measure models [7,8], where the system *EMNP* consists of all inference rules and axiom schemas of propositional logic and the following inference rules and axiom schemas:

Df$\Diamond$. $\Diamond p \leftrightarrow \neg\Box\neg p$, **RE**. $\dfrac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}$,

M. $\Box(p \wedge q) \to (\Box p \wedge \Box q)$, **N**. $\Box\top$, **P**. $\neg\Box\bot$.

4. α-Level fuzzy measure models based on background knowledge

In this section, we introduce α-level fuzzy measure models based on background knowledge to characterize typical situations as a modality of modal logic using granularity based on VPRS and measure-based semantics for modal logic.

4.1. Background knowledge by Kripke models based on approximation spaces

As a basis of reasoning using granularity based on VPRS and measure-based semantics, suppose that we have a Kripke model $\mathscr{M} = (U, R, v)$ consisting of the given approximation space (U, R) and a valuation v. In the Kripke model $\mathscr{M}$, any nonmodal sentence p that represents a fact is characterized by its truth set $\|p\|^{\mathscr{M}}$. When we consider the fact represented by the nonmodal sentence p, we may not consider *all* possible worlds in the truth set $\|p\|^{\mathscr{M}}$. In such cases, we often consider only *typical situations* about the fact p.

To capture such typical situations, we examine the lower approximation of the truth set $\|p\|^{\mathscr{M}}$ by the indiscernibility relation R, and consider each possible world in the lower approximation of the truth set $\|p\|^{\mathscr{M}}$ as a typical situation about p based on background knowledge about U.

Moreover, it may be useful to consider situations that are not typical about the facts as exceptions to typical situations. In this paper, we represent this characteristic using β-lower approximations of the truth sets of sentences that represent facts. Thus, using background knowledge from the Kripke model $\mathscr{M}$, we can consider the following two sets of possible worlds about a fact p:

- $\|p\|^{\mathscr{M}}$: correct representation of fact p
- $\underline{R}_{\beta}(\|p\|^{\mathscr{M}})$: the set of typical situations about p (situations that are not typical may also be included)

4.2. α-Level fuzzy measure models based on background knowledge

Using the given Kripke model as background knowledge, we define an α-level fuzzy measure model to treat typical situations about facts as β-lower approximations in the framework of modal logic.

Definition 1. Let $\mathscr{M} = (U, R, v)$ be a Kripke model that consists of an approximation space (U, R) and a valuation function $v : \mathscr{P} \times U \to \{0, 1\}$, and $\alpha \in (0.5, 1]$ be a fixed degree. An α-level fuzzy measure model $\mathscr{M}_{\alpha}^{R}$ based on background knowledge is the following triple:

$$\mathscr{M}_{\alpha}^{R} \stackrel{\text{def}}{=} \left(U, \{\mu_x^R\}_{x \in U}, v\right), \tag{19}$$

where U and v are the same as in $\mathscr{M}$. The fuzzy measure $\mu_x^R : 2^U \to [0, 1]$ assigned to each $x \in U$ is a probability measure-based on the equivalence class $[x]_R$ with respect to R, defined by

$$\mu_x^R(X) \stackrel{\text{def}}{=} \frac{|[x]_R \cap X|}{|[x]_R|}, \quad \forall X \subseteq U. \tag{20}$$

Similar to the case of Kripke-style models, we denote that a sentence p is true at a world $x \in U$ by an α-level fuzzy measure model $\mathscr{M}_{\alpha}^{R}$ by $\mathscr{M}_{\alpha}^{R}, x \models p$. Truth valuation of modal sentences is defined as

$$\mathscr{M}_{\alpha}^{R}, x \models \Box p \Longleftrightarrow \mu_x^R\left(\|p\|^{\mathscr{M}_{\alpha}^{R}}\right) \geqslant \alpha, \tag{21}$$

$$\mathscr{M}_{\alpha}^{R}, x \models \Diamond p \Longleftrightarrow \mu_x^R\left(\|p\|^{\mathscr{M}_{\alpha}^{R}}\right) > 1 - \alpha. \tag{22}$$

We also denote the truth set of a sentence p in the α-level fuzzy measure model $\mathscr{M}_{\alpha}^{R}$ by $\|p\|^{\mathscr{M}_{\alpha}^{R}}$, which is defined by

$$\|p\|^{\mathscr{M}_{\alpha}^{R}} \stackrel{\text{def}}{=} \{x \in U \mid \mathscr{M}_{\alpha}^{R}, x \models p\}. \tag{23}$$

The constructed α-level fuzzy measure model $\mathscr{M}_{\alpha}^{R}$ from the given Kripke model $\mathscr{M}$ has the following good properties.

Theorem 1. *Let $\mathscr{M}$ be a finite Kripke model such that its accessibility relation R is an equivalence relation and $\mathscr{M}_{\alpha}^{R}$ be the α-level fuzzy measure model based on the background knowledge $\mathscr{M}$ defined by (19). For any nonmodal sentence $p \in \mathscr{L}_{ML}(\mathscr{P})$ and any sentence $q \in \mathscr{L}_{ML}(\mathscr{P})$, the following equations are satisfied:*

$$\|p\|^{\mathscr{M}_{\alpha}^{R}} = \|p\|^{\mathscr{M}}, \tag{24}$$

$$\|\Box q\|^{\mathscr{M}_{\alpha}^{R}} = \underline{R}_{1-\alpha}\left(\|q\|^{\mathscr{M}_{\alpha}^{R}}\right), \tag{25}$$

$$\|\Diamond q\|^{\mathscr{M}_{\alpha}^{R}} = \overline{R}_{1-\alpha}\left(\|q\|^{\mathscr{M}_{\alpha}^{R}}\right). \tag{26}$$

Proof. Eq. (24) is clear from the definition of the relationship $\models$.

For (25), it is enough to show that for any sentence $q \in \mathscr{L}_{ML}(\mathscr{P})$, $\mathscr{M}_{\alpha}^{R}, x \models \Box q$ holds if and only if $x \in \underline{R}_{1-\alpha}(\|q\|^{\mathscr{M}})$. Suppose $\mathscr{M}_{\alpha}^{R}, x \models \Box q$ holds. By Definition 1, we have $\mu_x^R\left(\|q\|^{\mathscr{M}_{\alpha}^{R}}\right) \geqslant \alpha$. By the definition of the relative degree of misclassification from (4) and the definition of the fuzzy measure μ_x^R from (20), the property $\mu_x^R\left(\|q\|^{\mathscr{M}}\right) \geqslant \alpha$ holds if and only if $c\left([x]_R, \|q\|^{\mathscr{M}}\right) \leqslant 1 - \alpha$ holds, and therefore, we have $x \in \underline{R}_{1-\alpha}(\|q\|^{\mathscr{M}})$. Eq. (26) is also similarly proved. □

In the next section, we intend to use α-level fuzzy measure models $\mathscr{M}_{\alpha}^{R}$ as the basis of a unified framework for deduction, induction, and abduction based on the concept of typical situations of facts and rules used in these reasoning processes. Thus, as we discussed in Section 4.1, we represent facts and rules in reasoning processes as nonmodal sentences and typical situations of facts and rules as lower approximations of truth sets of nonmodal sentences. From (24) and (25) in Theorem 1, the α-level fuzzy measure model $\mathscr{M}_\alpha^R$ based on background knowledge $\mathscr{M}$ exhibits the characteristics of correct representations of facts by the truth sets of nonmodal sentences and typical situations of the facts by the $(1-\alpha)$-lower approximations of truth sets of nonmodal sentences. Thus, we can denote a modal sentence $\Box p$ as "typically p," and represent the relationship between typical situations by modal sentences. This indicates that the models $\mathscr{M}_\alpha^R$ are a sufficient basis for a unified framework for deduction, induction, and abduction.

Moreover, we have the following soundness properties of systems of modal logic with respect to the class of all α-level fuzzy measure models based on background knowledge.

Theorem 2. *For any α-level fuzzy measure model $\mathscr{M}_\alpha^R$ defined by (19) based on any finite Kripke model $\mathscr{M}$ such that its accessibility relation R is an equivalence relation, the following soundness properties are satisfied in the case of $\alpha = 1$ and $\alpha \in (0.5, 1)$, respectively:*

- *If $\alpha = 1$, then all theorems of system S5 are true in $\mathscr{M}_\alpha^R$.*
- *If $\alpha \in (0.5, 1)$, then all theorems of system EMND45 are true in $\mathscr{M}_\alpha^R$,*

where system EMND45 consists of the inference rules and axiom schemas of the system EMNP and the following axiom schemas:

$$\mathbf{D}.\quad \Box p \to \Diamond p, \quad \mathbf{4}.\quad \Box p \to \Box\Box p, \quad \mathbf{5}.\quad \Diamond p \to \Box\Diamond p.$$

Proof. It is clear from the correspondence relationship between axiom schemas and properties of β-lower approximations shown in Table 1. □

Theorem 2 indicates that the properties of α-level fuzzy measure models based on background knowledge depend on the degree of α. If we fix $\alpha = 1$, we do not allow any exception in typical situations; else, we allow some exceptions depending on α. This is because, if $\alpha = 1$, any α-level fuzzy measure models $\mathscr{M}_\alpha^R$ based on background knowledge satisfy the axiom schema **T**. $\Box p \to p$; else, $\mathscr{M}_\alpha^R$ does not satisfy **T**. Thus, if $\alpha \in (0.5, 1)$, a nonmodal proposition p and a possible world $x \in U$ may exist such that $x \in \|\Box p\|^{\mathscr{M}_\alpha^R}$ but $x \notin \|p\|^{\mathscr{M}_\alpha^R}$; i. e., x is considered a typical situation of p even though p is not true at x in $\mathscr{M}_\alpha^R$.

5. A unified formulation of deduction, induction, and abduction using granularity

In this section, we characterize the reasoning processes of deduction, induction, and abduction in α-level fuzzy measure models on the basis of background knowledge as reasoning processes based on typical situations. In the framework of α-level fuzzy measure models based on background knowledge, these three types of reasoning processes are described as follows:

- Deduction: A reasoning process for providing a conclusion from a general rule and a condition that holds in the given typical situation.
- Induction: A reasoning process for concluding a general rule from typical situations.
- Abduction: A reasoning process for providing a hypothesis that explains a fact in the given typical situation.

5.1. Deduction based on typical situations

Deduction is a reasoning process with the following form:

$p \to q$ p	If P, then Q. P.
q.	Therefore, Q,

where the left side illustrates the formulation of deduction, and the right side illustrates the meaning of the sentence appearing in each deductive step. It is well known that deduction is identical to the inference rule modus ponens used in almost all two-valued logic. Note also that deduction is a logically valid inference, where "logically valid" means that if both the antecedent p and the rule $p \to q$ are true, the consequent q is guaranteed to be true. Hereafter, we assume that all sentences p, q, etc. that represent facts and rules such as $p \to q$ are nonmodal sentences.

Let $\mathscr{M} = (U, R, v)$ be a Kripke model that consists of an approximation space (U, R) and a valuation function v that is given as background knowledge. In the framework of possible world semantics, we can illustrate deduction as follows:

$\mathscr{M} \models p \to q$ $\mathscr{M}, x \models p$	(In any situation,) If P, then Q. (In a situation,) P.
$\mathscr{M}, x \models q$	(In the situation,) Q.

Here, we consider deduction based on typical situations. Let $\mathcal{M}_\alpha^R$ be an α-level fuzzy measure model based on background knowledge with a fixed degree $\alpha \in (0.5, 1]$. True rules are represented by inclusion relationships between truth sets as follows:

$$\mathcal{M}_\alpha^R \models p \rightarrow q \Longleftrightarrow \|p\|^{\mathcal{M}_\alpha^R} \subseteq \|q\|^{\mathcal{M}_\alpha^R}. \tag{27}$$

As we have shown in Table 1, the monotonicity of β-lower approximation is satisfied for all $\beta \in [0, 0.5)$; thus, we have the relationship,

$$\mathcal{M}_\alpha^R \models \Box p \rightarrow \Box q \Longleftrightarrow \|\Box p\|^{\mathcal{M}_\alpha^R} \subseteq \|\Box q\|^{\mathcal{M}_\alpha^R}. \tag{28}$$

If we consider the truth set of $\Box p$ as the set of typical situations of p, then from (28), every element $x \in \|\Box p\|^{\mathcal{M}_\alpha^R}$ is also an element in the truth set of $\Box q$, and therefore, we can conclude that all situations typical of p are also typical of q.

Consequently, using the α-level fuzzy measure model $\mathcal{M}_\alpha^R$, we can characterize deduction based on typical situations by the following valid reasoning:

$\mathcal{M}_\alpha^R \models \Box p \rightarrow \Box q$ $\mathcal{M}_\alpha^R, x \models \Box p$	If (typically) P, then (typically) Q. (Typically) P.
$\mathcal{M}_\alpha^R, x \models \Box q$	(Typically) Q.

Note that the reasoning process of deduction based on typical situations is not affected by a difference in the degree α. This is because property (28) is true for any fixed degree $\alpha \in (0.5, 1]$, and therefore, if a possible world x is a typical situation of a fact p and a modal sentence $\Box p \rightarrow \Box q$ is valid in the α-level fuzzy measure model $\mathcal{M}_\alpha^R$, then x is also a typical situation of the fact q.

As an example of deduction, suppose sentences p and q have the following meanings:

- p: The sun rises in the east.
- q: The sun sets in the west.

Thus, deduction is illustrated as follows:

$\mathcal{M}_\alpha^R \models \Box p \rightarrow \Box q$ $\mathcal{M}_\alpha^R, x \models \Box p$	If the sun rises in the east, then the sun sets in the west. Today, the sun rose in the east.
$\mathcal{M}_\alpha^R, x \models \Box p$	The sun will set in the west today.

5.2. Induction based on typical situations

Induction is a reasoning process with the following form:

p q	P. Q.
$p \rightarrow q.$	Therefore, if P, then Q.

It is well known that induction is not logically valid. However, we often use induction to provide general rules from specific facts.

Induction has the following characteristic: From the fact that all observed objects satisfying a property p also satisfy a property q, we conclude that if objects satisfy p, they also satisfy q. Suppose that the $(1-\alpha)$-lower approximation of the truth set $\|p\|^{\mathcal{M}_\alpha^R}$ of the sentence p illustrates the set of observed objects satisfying p. From the characteristics of induction, we consider that induction based on typical situations needs to have the form

$\mathcal{M}_\alpha^R \models \Box p \rightarrow q$	If observed objects satisfy P, then the objects also satisfy Q.
$\mathcal{M}_\alpha^R \models p \rightarrow q$	If P, then Q.

This form of reasoning is not valid; however, we can consider this reasoning as valid by assuming the property

$$\mathcal{M}_\alpha^R \models \Box p \leftrightarrow p. \tag{29}$$

This assumption means that we consider the set $\|\Box p\|^{\mathscr{M}_\alpha^R}$ of observed objects satisfying p is identical to the set $\|p\|^{\mathscr{M}_\alpha^R}$ of all objects satisfying p; i.e., we generalize from the typical situations of p to all situations of p. This assumption is essential in formulating induction based on typical situations. Combining these processes of reasoning, we characterize induction based on typical situations as follows:

$\mathscr{M}_\alpha^R \models \Box p \to q$ $\mathscr{M}_\alpha^R \models \Box p \leftrightarrow p$	If observed objects satisfy P, then the objects also satisfy Q. Generalization of observation.
$\mathscr{M}_\alpha^R \models p \to q$	If P, then Q.

By repeating observations, we obtain more detailed background knowledge, and assumption (29) may become more probable. As shown in Table 1, in VPRS models, even though the partition becomes finer (that is, the current equivalence relation R changes to another equivalence relation R' such that $R' \subseteq R$), the β-lower approximation may not become large. However, the following situation may result from the more detailed equivalence relation R':

$$\text{For any } q,\ \mathscr{M}_\alpha^R \models \Box p \to q \text{ but } \mathscr{M}_\alpha^{R'} \not\models \Box p \to q. \tag{30}$$

This situation illustrates that by obtaining more detailed background knowledge, we find exceptions in the observed objects such that they do not satisfy q even while satisfying p. Therefore, in the framework of the α-level fuzzy measure model based on background knowledge, induction has nonmonotonicity.

This consideration indicates that, unlike deduction based on typical situations, the degree of $\alpha \in (0.5, 1]$ may affect the result of induction based on typical situations; assumption (29) with $\alpha = 1$ may be more reliable than the assumption with $\alpha \in (0.5, 1)$. This is because, as we have discussed at the end of Section 4.2, if $\alpha = 1$, the modal sentence $\Box p \to p$ is valid in any α-level fuzzy measure model $\mathscr{M}_\alpha^R$ based on background knowledge. On the other hand, if $\alpha \in (0.5, 1)$, this modal sentence may be not true in some observed object $x \in \|\Box p\|^{\mathscr{M}_\alpha^R}$, and such an object x becomes a counterexample of the assumption.

As an example of induction and nonmonotonic reasoning, suppose sentences p and q have the following meanings:

- p: It is a bird.
- q: It can fly.

Thus, induction and nonmonotonic reasoning are illustrated as follows:

$\mathscr{M}_\alpha^R \models \Box p \to q$ $\mathscr{M}_\alpha^R \models \Box p \leftrightarrow p$	All observed birds can fly. (Generalization of observed results)
$\mathscr{M}_\alpha^R \models p \to q$	Therefore, all birds can fly.
The equivalence relation R changes to a more detailed equivalence relation R' by repeating observations	
$\mathscr{M}_\alpha^{R'} \not\models \Box p \to q$	Not all birds can fly.

5.3. Abduction based on typical situations

Abduction is a reasoning process with the following form:

q $p \to q$	Q. If P, then Q.
p.	Therefore, P.

From a fact q and a rule $p \to q$, abduction infers a hypothesis p that produces the fact q. Therefore, abduction is also called hypothesis reasoning. Note that the form of abduction corresponds to affirming the consequent; thus, abduction is not logically valid if the hypothesis p is false and the fact q is true. However, we often use this form of reasoning to generate new ideas.

In general, many rules may exist that produce the fact q, and in such cases, we need to select one rule from many $p_i \to q$ $(p_i \in \{p_1, \ldots, p_n(, \ldots)\})$ that imply q. Thus, using fuzzy measures assigned to typical situations of the fact q, we introduce a selection mechanism to decide which rule to use in abduction.

Similar to the case of deduction, we consider the truth set $\|\Box q\|^{\mathscr{M}_\alpha^R}$ of $\Box q$ as the set of typical situations about q. For each rule $p_i \to q$ that implies the fact q, we consider the following minimal degree of the antecedent p_i in typical situations about q.

Definition 2. Let $p \to q, q \in \mathscr{L}_{ML}(\mathscr{P})$ be nonmodal sentences. The degree $\alpha(p|q)$ of p in typical situations about q is defined as follows:

$$\alpha(p|q) \stackrel{\text{def}}{=} \begin{cases} \min\left\{ \mu_x^R\left(\|p\|^{\mathscr{M}_\alpha^R}\right) \quad x \in \|\Box q\|^{\mathscr{M}_\alpha^R} \right\}, & \text{if } \|\Box q\|^{\mathscr{M}_\alpha^R} \neq \emptyset, \\ 0, & \text{otherwise}. \end{cases} \tag{31}$$

To demonstrate the calculation of the degree $\alpha(p|q)$, we present an example. Let $\mathscr{M} = (U, R, v)$ be a Kripke model that consists of the set of possible worlds $U = \{w1, \ldots, w10\}$, an equivalence relation R, and a valuation function v. The equivalence relation R provides the following three equivalence classes:

$$[w1]_R = \{w1, w2, w3\}, \quad [w4]_R = \{w4, w5, w6, w7\}, \quad [w8]_R = \{w8, w9, w10\}.$$

Moreover, the truth sets of three nonmodal sentences p_1, p_2, and q in $\mathscr{M}$ are:

$$\|p_1\|^{\mathscr{M}} = \{w1, w2, w3, w4, w5, w6\}, \quad \|p_2\|^{\mathscr{M}} = \{w2, w3, w4, w5, w6, w7\},$$
$$\|q\|^{\mathscr{M}} = \{w1, w2, w3, w4, w5, w6, w7, w8\}.$$

Note that both p_1 and p_2 conclude q.

Suppose we fix $\alpha = 0.7$, and consider the α-level fuzzy measure model $\mathscr{M}_\alpha^R$ based on background knowledge $\mathscr{M}$. Here, for the two rules $p_1 \to q$ and $p_2 \to q$, we calculate the degrees $\alpha(p_1|q)$ and $\alpha(p_2|q)$, respectively. The set of typical situations of q in $\mathscr{M}_\alpha^R$ is the set

$$\|\Box q\|^{\mathscr{M}_\alpha^R} = [w1]_R \cup [w4]_R = \{w1, w2, w3, w4, w5, w6, w7\}.$$

For $\alpha(p_1|q)$, we need to calculate the degrees of the truth set $\|p_1\|^{\mathscr{M}}$ by the fuzzy measures μ_x^R defined by (20) as follows:

$$\mu_{wi}^R\left(\|p_1\|^{\mathscr{M}}\right) = \frac{\left|[w1]_R \cap \|p_1\|^{\mathscr{M}}\right|}{\left|[w1]_R\right|} = \frac{|\{w1, w2, w3\}|}{|\{w1, w2, w3\}|} = 1, \quad wi \in [w1]_R,$$
$$\mu_{wj}^R\left(\|p_1\|^{\mathscr{M}}\right) = \frac{\left|[w4]_R \cap \|p_1\|^{\mathscr{M}}\right|}{\left|[w4]_R\right|} = \frac{|\{w4, w5, w6\}|}{|\{w4, w5, w6, w7\}|} = \frac{3}{4}, \quad wj \in [w4]_R.$$

Thus, we have the degree $\alpha(p_1|q)$ as follows:

$$\alpha(p_1|q) = \min\left\{1, \frac{3}{4}\right\} = \frac{3}{4}.$$

Similarly, we also calculate the degree $\alpha(p_2|q) = \frac{2}{3}$.

For any nonmodal sentence $p \to q$, the degree $\alpha(p|q)$ satisfies the following good property.

Proposition 1. *Let $p \to q \in \mathscr{L}_{ML}(\mathscr{P})$ be a nonmodal sentence. For any α-level fuzzy measure model based on background knowledge $\mathscr{M}_\alpha^R$ with the fixed degree $\alpha \in (0.5, 1]$, if the condition $\|\Box q\|^{\mathscr{M}_\alpha^R} \neq \emptyset$ holds, the following property is satisfied:*

$$\mathscr{M}_\alpha^R \models \Box q \to \Box p \iff \alpha(p|q) \geqslant \alpha. \tag{32}$$

Proof. ($\Leftarrow$) Suppose that $\alpha(p|q) \geqslant \alpha$ holds. Because we have $\|\Box q\|^{\mathscr{M}_\alpha^R} \neq \emptyset$ by the assumption of the proposition, there is a possible world $y \in \|\Box q\|^{\mathscr{M}_\alpha^R}$ such that $\alpha(p|q) = \mu_y^R\left(\|p\|^{\mathscr{M}_\alpha^R}\right)$ and $\mu_y^R\left(\|p\|^{\mathscr{M}_\alpha^R}\right) \leqslant \mu_x^R\left(\|p\|^{\mathscr{M}_\alpha^R}\right)$ for all typical situations $x \in \|\Box q\|^{\mathscr{M}_\alpha^R}$. Because $\alpha(p|q) \geqslant \alpha$ holds, $\mu_x^R\left(\|p\|^{\mathscr{M}_\alpha^R}\right) \geqslant \alpha$ for all $x \in \|\Box q\|^{\mathscr{M}_\alpha^R}$. Therefore, we have $\|\Box q\|^{\mathscr{M}_\alpha^R} \subseteq \|\Box p\|^{\mathscr{M}_\alpha^R}$, which leads to $\mathscr{M}_\alpha^R \models \Box q \to \Box p$.

($\Rightarrow$) Suppose that $\mathscr{M}_\alpha^R \models \Box q \to \Box p$ holds. This property implies that $\|\Box q\|^{\mathscr{M}_\alpha^R} \subseteq \|\Box p\|^{\mathscr{M}_\alpha^R}$. Moreover, because we have $\|\Box q\|^{\mathscr{M}_\alpha^R} \neq \emptyset$ by assumption, at least one typical situation of q exists. Thus, for all typical situations $x \in \|\Box q\|^{\mathscr{M}_\alpha^R}$ of q, $\mu_x^R\left(\|p\|^{\mathscr{M}_\alpha^R}\right) \geqslant \alpha$ holds. Therefore, by the definition of the degree $\alpha(p|q)$, we conclude that $\alpha(p|q) \geqslant \alpha$ holds. □

Proposition 1 indicates that we can use the degree $\alpha(p|q)$ as a criterion to select a rule $p \to q$ that implies the fact q. For example, from many rules $p_i \to q$ $(p_i \in \{p_1, \ldots, p_n(, \ldots)\})$ that imply q, we can select a rule $p_j \to q$ with the highest degree $\alpha(p_j|q)$ such that $\alpha(p_j|q) \geqslant \alpha$. In this case, we consider the selected rule $p_j \to q$ as the most universal rule to explain the fact q in the sense that all typical situations of q fit the typical situations of p_j. Thus, in the above example, we select the rule $p_1 \to q$ because we have $\alpha(p_1|q) \geqslant \alpha = 0.7$ but $\alpha(p_2|q) < \alpha$. On the other hand, we can consider the case that no rule satisfies (31) as a situation in which we cannot explain the fact q by the current background knowledge.

Therefore, by selecting the rule $p \to q$ with the highest degree $\alpha(p|q)$ such that $\alpha(p|q) \geqslant \alpha$, we can characterize abduction that infers p from the fact q based on typical situations by the following form of valid reasoning:

$\mathscr{M}_{\alpha}^{R}, x \models \Box q$ $\mathscr{M}_{\alpha}^{R} \models \Box q \rightarrow \Box p$	(Actually) Q. Selection of a rule "if P, then Q."
$\mathscr{M}_{\alpha}^{R}, x \models \Box p$	(Perhaps) P.

By this formulation of abduction based on typical situations, it is clear that the difference of the degree $\alpha \in (0.5, 1]$ affects the result of abduction.

As an example of abduction (or hypothesis reasoning), we consider reasoning based on fortune-telling. Suppose sentences p_1, p_2, and q used in the above example have the following meanings:

- p_1: I wear some red items.
- p_2: My blood type is AB.
- q: I am lucky.

Then, using the α-level fuzzy measure model $\mathscr{M}_{\alpha}^{R}$ based on background knowledge $\mathscr{M}$ in the above example, reasoning based on fortune-telling is characterized by abduction as follows:

$\mathscr{M}_{\alpha}^{R}, x \models \Box q$ $\mathscr{M}_{\alpha}^{R} \models \Box q \rightarrow \Box p$	I am very lucky today! In a magazine, I saw that "wearing red items makes you lucky."
$\mathscr{M}_{\alpha}^{R}, x \models \Box p$	Actually I wear red socks!

6. Conclusion

In this paper, we have introduced an α-level fuzzy measure model based on background knowledge and proposed a unified formulation of deduction, induction, and abduction based on this model. Using the proposed model, we have characterized typical situations of the given facts and rules by $(1-\alpha)$-lower approximation of truth sets of nonmodal sentences that represent the given facts and rules. We have also proven that the system *EMND45* is sound with respect to the class of all α-level fuzzy measure models based on background knowledge. Moreover, we have characterized deduction, induction, and abduction as reasoning processes based on typical situations. In the proposed framework, deduction and abduction are illustrated as valid reasoning processes based on typical situations of facts. On the other hand, induction is illustrated as a reasoning process of generalization based on observations. Furthermore, in the α-level fuzzy measure model based on background knowledge, we have pointed out that induction has nonmonotonicity based on revision of the indiscernibility relation in the given Kripke model as background knowledge and gave an example in which a rule inferred by induction based on typical situations is rejected by refinement of the indiscernibility relation.

Several issues remain to be investigated. One of the most important, we think, is the treatment of iteration of deduction, induction, and abduction in the proposed framework. All reasoning processes we have treated in this paper are one-step reasoning in that the reasoning processes are complete after using either deduction, induction, or abduction only once. Therefore, we need to extend the proposed framework to treat multiple-step reasoning, and we think that this extension will be closely connected to belief revision [2,3] and belief update [4] in the proposed framework. Moreover, choice of the degree $\alpha \in (0.5, 1]$ affects the results of reasoning directly in the sense of whether and to what degree we allow the existence of exceptions to typical situations. However, we assumed that α is given in this paper, and we have not discussed how to choice α. Considering and introducing some criteria for choosing α are also important for the proposed framework.

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