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A Multi-Dimensional Age-Dependent Branching Process—Subcritical Case

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1. INTRODUCTION

We consider a population consisting of n different types of particles, each particle living and reproducing independently of others. A particle of type i lives for a random length of time ℓ_i distributed according to the law $P(\ell_i \leqslant t) = G_i(t)$, and at the time of its death is replaced by a random number of offspring $(j_1, j_2, ..., j_n)$ of various types. Let

$$h_i(s) = \sum_J p_i(J)\, s_1^{j_1} \cdots s_n^{j_n}, \qquad s = (s_1, s_2, ..., s_n),$$
$$J = (j_1, j_2, ..., j_n),$$

denote the generating function of the probabilities $p_i(J)$. Also, let

$$Z(t) = (Z_1(t), Z_2(t), ..., Z_n(t))$$

be the random vector giving the number of particles at time t. The nature of $Z(t)$ depends mainly on the moment matrix $M = (m_{ij})$ where

$$m_{ij} = \frac{\partial h_i(s)}{\partial s_j} \,\Big|\, s = (1, 1, ..., 1).$$

We shall always assume that M is irreducible. Let ρ be the Perron-Frobenius root of M. We say that the above described process is a subcritical process if $\rho < 1$. The case when M is a positive matrix with $\rho > 1$ has been extensively studied by Mode [1, 2]. Practically no results are available for $\rho \leqslant 1$ in the multi-dimensional case. Vinogradov [7] has obtained an asymptotic form of the probability of extinction in the one-dimensional case. It may also be mentioned that results are available for the case of Galton-Watson processes with discrete and continuous time when $\rho < 1$ [3, 12].

The purpose of this paper is to study the subcritical multi-dimensional age-dependent process and get results analogous to those in [3], using Haar's Tauberian theorem [4].

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2. On a System of Integral Equations

Frequently in the analysis of the age-dependent branching process we come across a system of integral equations of the type

$$A_i(t) = f_i(t) + \int_0^t \sum_{j=1}^n a_{ij} A_j(t-u)\, dG_i(u), \qquad i = 1, 2,\ldots, n \tag{2.1}$$

where $A = (a_{ij})$ is an irreducible matrix of non-negative elements, $G_i(t)$ is a distribution function, and $f_i(t)$ is a bounded function. Let us define the convolution operation

$$G_1 * G_2(t) = \int_0^t G_1(t-u)\, dG_2(u).$$

If $a_{ij} < \infty$ and $G_i(0+) = 0$ for all i and j, then among all functions bounded on finite intervals in $[0, \infty)$ (2.1) has a unique bounded solution which may be represented as

$$A_i(t) = f_i(t) + \sum_{j=1}^n f_j * F_{ij}(t) \tag{2.2}$$

where

$$F_{ij}(t) = \sum_{r=1}^{\infty} \sum_{i_1,\ldots,i_{r-1}} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{r-1}j} G_i * G_{i_1} * \cdots * G_{i_{r-1}}(t)$$

with each i_k running from 1 to n (see [1]).

Using the techniques in [4] it is possible to derive an asymptotic form of $A_i(t)$. To this end, set

$$G_i{}^*(\lambda) = \int_0^\infty e^{-\lambda t}\, dG_i(t)$$

for all real λ for which the integral converges, and

$$H(\lambda) = (a_{ij} G_i{}^*(\lambda)).$$

Suppose a real root of the determinantal equation $|I - H(\lambda)| = 0$ exists. Then there is a real number α with the following properties. (i) The Perron-Frobenius root of the matrix $H(\alpha)$ is 1 and corresponding to this root there are positive left and right eigenvectors $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)'$ such that $\eta H(\alpha) = \eta$, $H(\alpha)\mu = \mu$, $\eta\mu = 1$, and $\max_i \mu_i = 1$.

(ii) α is the root of the determinantal equation $|I - H(\lambda)| = 0$ with largest real part and has multiplicity one.

A proof of (i) and (ii) for the case of positive matrices with $\rho > 1$ may be found in [1, 5]. The proof is similar for the case of irreducible matrices.

If $\rho > 1$ α exists and $\alpha > 0$, and if $\rho = 1$, $\alpha = 0$. If $\rho < 1$ α may or may not exist depending on $G_i(t)$, $i = 1, 2, \ldots, n$. It is not difficult to see that if $G_i^*(\lambda)$ exists and continuous in $(-a, \infty)$ for some $a > 0$ and $G_i^*(\lambda) \to \infty$ as $\lambda \to -a+$ $(i = 1, 2, \ldots, n)$ then α exists for $\rho < 1$. The distribution functions with the following densities are examples of this situation.

(i) $g_i(x) = \dfrac{a^{b_i}}{\Gamma b_i} x^{b_i - 1} e^{-ax}, \qquad 0 \leqslant x < \infty$

(ii) $g_i(x) = ce^{-x^2}, \qquad 0 \leqslant x < \infty.$

There are cases when $\rho < 1$ and α does not exist. Obviously this is true of distribution functions such that $\int_0^\infty e^{\epsilon t}\, dG_i(t) = \infty$ for every $\epsilon > 0$. Chistyakov [6] has studied this case for the one dimensional process. An easy extension of his results to the multi-dimensional case is possible when $G_i(t) = G(t)$ $(i = 1, 2, \ldots, n)$. However, we will not attempt to do this here.

THEOREM 2.1. *Let the following conditions hold*

(i) *A is irreducible.*

(ii) *α exists.*

(iii) *$\int_0^\infty t^r e^{-\alpha t}\, dG_i(t) < \infty$ for an integer $r \geqslant 2$.*

(iv) *Density $g_i(t)$ of $G_i(t)$ exists and $t^k e^{-\alpha t} g_i(t)$ $(k = 0, 1, \ldots, r-2)$ are of bounded variation in $(0, \infty)$ $(i = 1, 2, \ldots, n)$.*

(v) *The functions $f_i(t)$ are of the form $f_i(t) = \sum_{j=1}^n a_j f_{ij}(t)$ where a_j's are constants, $e^{-\alpha t} f_{ij}(t)$'s are bounded and non-negative functions for $t \geqslant 0$ and satisfy the conditions*

$$\lim_{t\to\infty} t^{r-2} e^{-\alpha t} f_{ij}(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} t^{r-2} \int_t^\infty e^{-\alpha x} f_{ij}(x)\, dx = 0.$$

Let $B(\lambda)$ be the adjoint of the matrix $I - H(\lambda)$,

$$\Delta(\lambda) = |I - H(\lambda)|, \qquad \Delta'(\lambda) = \frac{d\Delta(\lambda)}{d\lambda}, \qquad f_j^*(\lambda) = \int_0^\infty e^{-\lambda t} f_j(t)\, dt,$$

and

$$f^*(\lambda) = (f_1^*(\lambda), f_2^*(\lambda), \ldots, f_n^*(\lambda)).$$

Then

$$\lim_{t\to\infty} t^{r-2} \{e^{-\alpha t} A_i(t) - a_i\} = 0$$

where a_i is the i-th element in the vector

$$a = \frac{1}{\Delta'(\alpha)} B(\alpha) f^*(\alpha).$$

Proof. The proof is essentially the same as that in [4] except for some obvious modifications needed for the multi-dimensional case. Hence we shall sketch the proof omitting the details.

Let us first assume that $\rho = 1$ so that $\alpha = 0$. Let $G^*(\lambda) = 1 - \Delta(\lambda)$. It can be shown that

$$G^*(\lambda) = \sum_s (-1)^{N(s)} G_s^*(\lambda) \Delta_s ,$$

where s is a non-empty subset of $(1, 2,..., n)$, $N(s)$ is the number of elements in s,

$$G_s^*(\lambda) = G^*_{(i_1,...,i_r)}(\lambda) = G^*_{i_1}(\lambda) \cdots G^*_{i_r}(\lambda),$$

A_s is the square submatrix of A corresponding to the rows and columns in the set s and $\Delta_s = | A_s |$, the determinant of A_s.

Let $g(t)$ be the function whose Laplace transform is $G^*(\lambda)$. It is easy to see that $g(t)$ exists, since $G^*(\lambda)$ is the Laplace transform of a linear combination of convolutions of densities. Consider the system of equations

$$C_{ij}(t) = \delta_{ij} \int_0^t g(t-u)\, g(u)\, du + \int_0^t \sum_j a_{ij} C_{ij}(t-u)\, dG_i(u). \tag{2.3}$$

Let

$$C_j(t) = (C_{1j}(t), C_{2j}(t),..., C_{nj}(t))'$$

and

$$\delta_j = (\delta_{1j}, \delta_{2j}, ..., \delta_{nj})'.$$

Taking Laplace transforms in (2.3),

$$C_j^*(\lambda) = \frac{B(\lambda)\, \delta_j [G^*(\lambda)]^2}{\Delta(\lambda)} \qquad \text{for } \lambda > 0. \tag{2.4}$$

Applying Haar's Tauberian theorem to (2.4) we find that there exist constants c_{ij} such that

$$\lim_{t\to\infty} t^{r-2}\{C_j(t) - c_j\} = 0, \qquad j = 1, 2,..., n \tag{2.5}$$

where

$$c_j = (c_{1j}, c_{2j}, ..., c_{nj})'.$$

Taking Laplace transforms in (2.1),

$$A^*(\lambda) = \frac{B(\lambda) f^*(\lambda)}{\Delta(\lambda)} \tag{2.6}$$

where

$$A^*(\lambda) = (A_1{}^*(\lambda), A_2{}^*(\lambda), \ldots, A_n{}^*(\lambda))'.$$

From (2.4) and (2.6),

$$A^*(\lambda) = B(\lambda) f^*(\lambda) + B(\lambda) f^*(\lambda)\, G^*(\lambda) + C^*(\lambda) f^*(\lambda) \tag{2.7}$$

where

$$C^*(\lambda) = (C_{ij}^*(\lambda)).$$

From (2.5) and (2.7) we find, as in [4], that there exists a vector $a = (a_1, a_2, \ldots, a_n)$ such that

$$\lim_{t\to\infty} t^{r-2}\{A(t) - a\} = 0.$$

Using a result on page 187 Widder [10] the vector a can be obtained as

$$a = \lim_{\lambda\to 0^+} \lambda A^*(\lambda) = \frac{B(0) f^*(0)}{\Delta'(0)}.$$

If $\rho \neq 1$, set

$$D_i(t) = e^{-\alpha t} A_i(t),$$
$$k_i(t) = e^{-\alpha t} f_i(t),$$
$$d_{ij} = a_{ij} G_i{}^*(\alpha),$$

and

$$H_i(t) = \frac{1}{G_i{}^*(\alpha)} \int_0^t e^{-\alpha u}\, dG_i(u).$$

Multiplying (2.1) by $e^{-\alpha t}$, we see that

$$D_i(t) = k_i(t) + \int_0^t \sum_j d_{ij} D_j(t-u)\, dH_i(u).$$

But the Perron-Frobenius root of the matrix (d_{ij}) is one, and hence

$$\lim_{t\to\infty} t^{r-2}\{D_i(t) - d_i\} = 0$$

where

$$d_i = \lim_{\lambda\to 0^+} \lambda D_i{}^*(\lambda) = \lim_{\lambda\to 0^+} \lambda A_i{}^*(\lambda + \alpha).$$

From this the theorem follows.

3. Limiting Distribution of $Z(t)$

Let $\epsilon_i = (\delta_{i1}, \ldots, \delta_{in})$ and $Z(0) = \epsilon_i$. It is well known that the generating functions $F_i(s, t)$ $i = 1, 2, \ldots, n$ of $Z(t)$ given $Z(0) = \epsilon_i$ satisfy the system of integral equations

$$F_i(s, t) = s_i[1 - G_i(t)] + \int_0^t h_i[F(s, t - u)]\, dG_i(u), \qquad i = 1, 2, \ldots, n \tag{3.1}$$

where

$$F(s, t) = (F_1(s, t), F_2(s, t), \ldots, F_n(s, t)).$$

For a discussion of (3.1) the reader is referred to [8].

If $G_i(0+) = 0$ and $m_{ij} < \infty$ for all i and j, then (3.1) has a unique solution $F(s, t)$ such that $F_i(\underline{1}, t) = 1$ for all $t \geqslant 0$ (see [8]). The integral equations (3.1) may be used as a starting point in the study of age-dependent branching processes.

Let $B_i(t)$ and $D_i(t)$ be the total number of individuals of the i-th type in the population who have been born and have died upto and including time t. If $m_{ij} < \infty$ and $G_i(0+) = 0$ for all (i, j) then $B_i(t) < \infty$ a.s. for all $t \geqslant 0$ [see [11]]. Thus using the relation $Z_i(t) = B_i(t) - D_i(t)$, we find that the sample functions $Z_i(t)$ are continuous from the right so that the process is separable. Hence we can speak of the probability P_i of the event $[Z(t) = 0$ for some $t > 0$ given that $Z(0) = \epsilon_i]$, i.e. the event of extinction of the population. It can be easily shown that $F_i(0, t) \uparrow P_i$ as $t \uparrow \infty$. The probabilities $P_1, P_2, \ldots, P_n$ satisfy the system of equations $s_i = h_i(s_1, s_2, \ldots, s_n)$, $i = 1, 2, \ldots, n$, and are equal to the coordinates of the root of the system of equations lying closest to the origin in the square $0 \leqslant x_i \leqslant 1$ $(i = 1, 2, \ldots, n)$ (see [8]).

We call a system of types $(i_1, i_2, \ldots, i_r)$ a final class if $h_{i_1}(s), h_{i_2}(s), \ldots, h_{i_r}(s)$ are homogeneous linear functions in the variables $s_{i_1}, s_{i_2}, \ldots, s_{i_r}$. In order that $P_i = 1$ $(i = 1, 2, \ldots, n)$ it is necessary and sufficient that (1) $\rho \leqslant 1$ and (2) the system of types $(1, 2, \ldots, n)$ does not contain a single final class (see [8]). Hence it is of interest to study the limiting behavior of the conditional random vector $Z(t)$ given that $Z(t) > 0$. The following theorem gives a precise statement of our results.

Theorem 3.1. *Let the following conditions hold*

(i) *The system of types* $(1, 2, \ldots, n)$ *does not contain a final class.*

(ii) $\mu_{ijk} = \left.\dfrac{\partial^2 h_i(s)}{\partial s_j\, \partial s_k}\right|_{s=\underline{1}} < \infty$ *for all* (i, j, k).

(iii) *$M = (m_{ij})$ is irreducible with $\rho < 1$.*

(iv) *α exists.*

(v) $\int_0^\infty t^3 e^{-\alpha t}\, dG_i(t) < \infty$, $i = 1, 2, \ldots, n$.

(vi) *Density $g_i(t)$ of $G_i(t)$ exists and $e^{-\alpha t} g_i(t)$ and $t e^{-\alpha t} g_i(t)$ are of bounded variation in* $(0, \infty)$ $(i = 1, 2, \ldots, n)$.

(vii) $t e^{-\alpha t} g_i(t) \to 0$ *as* $t \to \infty$, $i = 1, 2, \ldots, n$.

Then the conditional random vector $Z(t)$ given that $Z(0) = \epsilon_i$ and $Z(t) > 0$ converges in distribution to a random vector whose distribution is independent of i.

Proof. Using Taylor expansion

$$h_i(s) = 1 + \sum_j m_{ij}(1 - s_j) + \sum_{jk} \bar{\mu}_{ijk}(1 - s_j)(1 - s_k) \tag{3.2}$$

where

$$0 \leqslant \bar{\mu}_{ijk} \leqslant \left.\frac{\partial^2 h_i(s)}{\partial s_j\, \partial s_k}\right|_{s=1}.$$

Let $\Delta_i(s, t) = 1 - F_i(s, t)$. Then from (3.1) and (3.2) we get

$$\begin{aligned} \Delta_i(s, t) = (1 - s_i)(1 - G_i(t)) &+ \int_0^t \sum_j m_{ij} \Delta_j(s, t - u)\, dG_i(u) \\ &- \int_0^t \sum_{jk} \bar{\mu}_{ijk} \Delta_j(s, t - u)\, \Delta_k(s, t - u)\, dG_i(u) \end{aligned} \tag{3.3}$$

For $s = 0$,

$$\begin{aligned} \Delta_i(0, t) = 1 - G_i(t) &+ \int_0^t \sum_j m_{ij} \Delta_j(0, t - u)\, dG_i(u) \\ &- \int_0^t \sum_{jk} \bar{\mu}_{ijk} \Delta_j(0, t - u)\, \Delta_k(0, t - u)\, dG_i(u). \end{aligned}$$

Therefore $\Delta_i(0, t) \leqslant A_i(t)$ where $A_i(t)$ $(i = 1, 2, \ldots, n)$ are the solutions of

$$A_i(t) = [1 - G_i(t)] + \int_0^t \sum_j m_{ij} A_j(t - u)\, dG_i(u).$$

From Theorem 2.1 it follows that $A_i(t) = 0(e^{\alpha t})$. Hence

$$\Delta_i(s, t) \leqslant \Delta_i(0, t) \leqslant A_i(t) \leqslant A e^{\alpha t}$$

for some finite number A and $0 \leqslant s \leqslant \underline{1}$. Now write (3.3) in the form

$$\Delta_i(s, t) = f_i(s, t) + \int_0^t \sum_j m_{ij} \Delta_i(s, t - u)\, dG_i(u)$$

where

$$f_i(s, t) = (1 - s_i)(1 - G_i(t)) - \int_0^t \sum_{j,k} \bar{\mu}_{ijk} \Delta_j(s, t-u) \Delta_k(s, t-u) \, dG_i(u).$$

To apply Theorem 2.1 we must verify that the conditions

$$\lim_{t\to\infty} te^{-\alpha t} f_i(s, t) = 0 \tag{3.4}$$

and

$$\lim_{t\to\infty} t \int_t^\infty e^{-\alpha x} f_i(s, x) \, dx = 0 \tag{3.5}$$

are satisfied. Since

$$\int_0^\infty t^2 e^{-\alpha t} \, dG_i(t) < \infty,$$

we have

$$\lim_{t\to\infty} te^{-\alpha t}[1 - G_i(t)] = 0$$

and

$$\lim_{t\to\infty} t \int_t^\infty e^{-\alpha x}[1 - G_i(x)] \, dx = 0$$

(see page 143 [9]). Also

$$\int_0^t \bar{\mu}_{ijk} \Delta_j(s, t-u) \Delta_k(s, t-u) \, dG_i(u) \leqslant A^2 \mu_{ijk} \int_0^t e^{2\alpha(t-u)} \, dG_i(u).$$

Hence

$$\lim_{t\to\infty} te^{-\alpha t} \int_0^t \bar{\mu}_{ijk} \Delta_j(s, t-u) \Delta_k(s, t-u) \, dG_i(u) \leqslant A^2 \mu_{ijk} \lim_{t\to\infty} \frac{t \int_0^t e^{-2\alpha u} \, dG_i(u)}{e^{-\alpha t}}$$
$$= 0 \qquad \text{by L'Hospital's rule.}$$

Thus $te^{-\alpha t} f_i(s, t) \to 0$ verifying (3.4). Now consider

$$\int_0^\infty te^{-\alpha t} \int_0^t \bar{\mu}_{ijk} \Delta_j(s, t-u) \Delta_k(s, t-u) \, dG_i(u) \, dt$$
$$\leqslant \frac{A^2 \mu_{ijk}}{\alpha^2} \int_0^\infty (1 - \alpha u) e^{-\alpha u} \, dG_i(u) < \infty$$

verifying (3.5).

Thus Theorem 2.1 applies and

$$\lim_{t\to\infty} t\{e^{-\alpha t}\Delta_i(s, t) - k_i(s)\} = 0$$

for some function $k_i(s)$. Hence

$$e^{-\alpha t}\Delta_i(s, t) \to k_i(s) \qquad \text{as} \qquad t \to \infty \tag{3.6}$$

uniformly in $0 \leqslant s \leqslant \underline{1}$. Using the expansion

$$h_i(\underline{1} - s) = 1 - \sum_j \bar{m}_{ij} s_j, \qquad 0 \leqslant \bar{m}_{ij} \leqslant m_{ij}$$

we get from (3.1),

$$\Delta_i(s, t) = (1 - s_i)\,[1 - G_i(t)] + \int_0^t \sum_j \bar{m}_{ij}\Delta_j(s, t-u)\, dG_i(u).$$

Multiplying by $e^{-\alpha t}$ and using the fact $\bar{m}_{ij} \to m_{ij}$ as $\Delta_i(s, t) \to 0$ i.e. as $t \to \infty$ we get,

$$k_i(s) = \sum_j m_{ij} G_i^*(\alpha)\, k_j(s).$$

In matrix form

$$k(s) = H(\alpha)\, k(s)$$

where

$$k(s) = (k_1(s), k_2(s), \dots, k_n(s))'.$$

By the definition of α the Perron-Frobenius root of $H(\alpha)$ is one. Hence $k(s)$ must be a multiple of the vector $\mu = (\mu_1, \mu_2, \dots, \mu_n)'$. Thus

$$k_i(s) = f(s)\,\mu_i \tag{3.7}$$

for some function $f(s)$.

Let $F_i^*(s, t)$ be the generating function of the conditional random vector $Z(t)$ given that $Z(0) = \epsilon_i$ and $Z(k) > 0$. Then

$$F_i^*(s, t) = 1 - \frac{\Delta_i(s, t)}{\Delta_i(0, t)} \to 1 - \frac{f(s)}{f(0)} \qquad \text{as} \qquad t \to \infty.$$

Uniformly in $0 \leqslant s \leqslant \underline{1}$ by (3.6) and (3.7). Hence the theorem.

Remark. The proof of the Theorem 3.1 would not be complete unless we show that $f(0) > 0$. This can be done by imposing a condition on μ_{ijk}'s. Consider the equations

$$A_i(t) = [1 - G_i(t)] + \int_0^t \sum_j m_{ij} A_j(t-u)\, dG_i(u)$$

and let

$$c = \max_i \sup_t A_i e^{-\alpha t}.$$

Then $\Delta_i(0, t) \leqslant ce^{\alpha t}$ so that

$$f_i(0, t) \geqslant [1 - G_i(t)] - \int_0^t \sum_{j,k} \mu_{ijk} c^2 e^{2\alpha(t-u)}\, dG_i(u).$$

Taking Laplace transforms we get

$$f_i^*(0, \alpha) \geqslant \frac{1 - G_i^*(\alpha)}{\alpha} + c^2 \sum_{jk} \mu_{ijk} \frac{G_i^*(\alpha)}{\alpha}.$$

Therefore

$$\nu f^*(0, \alpha) \geqslant \frac{1}{\alpha} \sum_i \nu_i [1 - G_i(\alpha)] + \frac{c^2}{\alpha} \sum_{ijk} \mu_{ijk} \nu_i G_i^*(\alpha). \tag{3.8}$$

The expression for a in the Theorem 2.1 can be written as

$$a = b\mu\nu f^*(\alpha) \tag{3.9}$$

where

$$\frac{1}{b} = -\sum_{i,j} m_{ij} \mu_j \nu_i \left. \frac{dG_i^*(\lambda)}{d\lambda} \right|_{\lambda=\alpha}$$

by using a result in [1].

From (3.8) and (3.9) we find that a sufficient condition for $f(0)$ of the Theorem 3.1 to be positive is that

$$c^2 \sum_{ijk} \nu_i \mu_{ijk} G_i^*(\alpha) < \sum_i \nu_i [1 - G_i^*(\alpha)].$$

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