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Rank One Lattice Type Vertex Operator Algebras and Their Automorphism Groups
II. *E*-Series

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Let L be the A_1 root lattice and let G be a finite subgroup of $\mathrm{Au}(V)$, where $V = V_L$ is the associated lattice VOA (in this case, $Aut(V) \cong PSL(2, \mathbb{C})$). The fixed point sub-VOA, V^G, was studied previously by the authors, who found a set of generators and determined the automorphism group when G is cyclic (from the "A-series") or dihedral (from the "D-series"). In the present article, we obtain analogous results for the remaining possibilities for G, that it belong to the "E-series": $G \cong Alt_4$, Alt_5, or Sym_4. For such L and G, the above V_L^G may be rational VOAs. © 1999 Academic Press

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1. INTRODUCTION

This paper is a sequel to [DG] in which we determined a set of generators and the full automorphism groups of $V_{L_{2n}}^+$ where L_{2n} is a rank 1 lattice spanned by an element of squared length $2n$ and $V_{L_{2n}}^+$ is the fixed points of the lattice VOA $V_{L_{2n}}$ under an automorphism of $V_{L_{2n}}$ lifting the -1 isometry of L_{2n}. In this paper we determine a set of generators and the full automorphism group of V_L^G when $L = L_2$ is the root lattice of type A_1 and G is an automorphism group of type tetrahedral, octahedral, or icosahedral.

The graded dimensions of $V_{L_{2n}}$, $V_{L_{2n}}^+$, and the three $V_{L_{2n}}^G$ realize all the partition functions of rank 1 rational conformal field theories: such functions (but not the VOAs themselves) are classified in the physics literature [G, K]. It is unknown whether two inequivalent rank 1 rational VOAs may have the same graded dimension and it is also unknown whether all the VOAs above are rational. Certainly, the $V_{L_{2n}}$ are rational (see [D, DLM2]) and some progress has been made toward showing that the $V_{L_{2n}}^+$ are rational, namely the finiteness of the number of isomorphism types of irreducible modules has been proven [DN1].

It is well known that the finite subgroups of $PSL(2,\mathbb{C})$ are labeled by the simply laced Lie algebras. If G is of type A or D, $V_{L_2}^G$ is $V_{L_{2n}}$ or $V_{L_{2n}}^+$ for some n. Since the full automorphism groups for all lattice vertex operator algebras and $V_{L_{2n}}^+$ have been determined in [DN2, DG], the results in this paper complete the determination of generators and full automorphism groups for this set of vertex operator algebras of rank 1. Using the results from [DN2, DG] and the present paper, one can easily see that the set of isomorphism types

$$\mathscr{S} = \left\{ V_{L_{2n}}, V_{L_{2n}}^+, V_{L_2}^G \mid n \geq 1, G = Alt_4, Sym_4, Alt_5 \right\}$$

is closed in the sense that, for any $V \in \mathscr{S}$ and a finite subgroup G of $Aut(V)$, $V^G \in \mathscr{S}$.

The paper is organized as follows. In Section 2 we review the invariant theory for the subgroups of $PSL(2,\mathbb{C})$ of E-series following [S]. In Section 3 we determine the generators and the automorphism groups of $V_{L_2}^G$ for $G \cong Alt_4, Sym_4, Alt_5$ which are the subgroups of $PSL(2,\mathbb{C})$ of type E.

We assume that the reader has some familiarity with the definition of vertex operator algebra and vertex operator algebras associated to even positive definite lattices as presented in [B, FLM, DG].

The following notation will be used throughout the paper.

W_m The m-dimensional irreducible module for $SL(2,\mathbb{C})$.

p_m The projection from a finite dimensional $SL(2,\mathbb{C})$-module onto its W_m-homogeneous component.

T A particular copy of the finite group $2Alt_4$ in $SL(2,\mathbb{C})$.
S A particular copy of the finite group $2Sym_4$ in $SL(2,\mathbb{C})$.
I A particular copy of the finite group $2Alt_5$ in $SL(2,\mathbb{C})$.
V^H The fixed points of the action of a group H on a module V.

2. REPRESENTATIONS OF $SL(2,\mathbb{C})$

In this section we get some special results about tensors of $SL(2,\mathbb{C})$-modules for use in Section 3. We recall that the group $SL(2,\mathbb{C})$ has a unique irreducible module W_m of any finite dimension m [H]. This module contains an m-dimensional integral representation, spanned by a Chevalley basis, of the integral Chevalley group $SL(2,\mathbb{Z})$. We shall write this integral representation as Λ_m We shall need to work with $SL(2,A)$ modules for various choices of a subring A in $\mathbb{C}$. In particular, we write $W_{m,A}$ for the $SL(2,A)$-module $A\otimes_{\mathbb{Z}}\Lambda_m$. We write R for a ring of algebraic integers, and we let bars indicate reduction modulo a prime containing p in R and for the result of tensoring with $\overline{R}$.

We shall be interested in tensor products of pairs of $SL(2,\mathbb{C})$-modules. The decompositions of these tensor products into irreducibles are given by the Clebsch−Gordan formulae: $W_m\otimes W_n\cong W_{m+n-1}\oplus W_{m+n-3}\oplus\cdots\oplus W_{n-m+1}$, which holds whenever $n>m$ [H]. A similar decomposition of $KSL(2,K)$ modules holds for any subfield K in $\mathbb{C}$. The main result of this section is Theorem 2.1, which is needed in Section 3.

THEOREM 2.1. *We have*

(i) $p_{19}(W_9^S\otimes W_{13}^S)\neq 0$.

(ii) $p_{31}(W_{13}^I\otimes W_{21}^I)\neq 0$.

(iii) *The* 1*-dimensional spaces* $p_{13}(W_7^T\otimes W_7^T)$ *and* $p_{13}(W_7^T\otimes W_9^T)$ *are distinct*.

(iv) *The one-dimensional spaces* $p_{13}(W_9^T\otimes W_9^T)$ *and* $p_{13}(W_7^T\otimes W_9^T)$ *are distinct*.

To establish this theorem, we must establish nontriviality for projections of particular subspaces of tensor products of $SL(2,\mathbb{C})$-modules. (The upper bounds of dimension 1 implicit in (iii) and (iv) are immediate consequences of the Clebsch−Gordan formula.) We shall establish these claims by performing explicit computations in analogous modules for a finite group of type $SL(2,\overline{R})$ and lifting the results to characteristic 0. We begin by obtaining conditions under which we can lift statements about the dimension of images of projection maps.

LEMMA 2.2. *Suppose that L and M are finite rank R-torsion free $RSL(2,R)$-modules that are equivalent over the field of fractions of R. Then $\overline{L}$ and $\overline{M}$ are $\overline{R}SL(2,\overline{R})$-modules with identical sets of composition factors.*

Proof. The first argument for 82.1 in [CR] (which establishes an analogous result for representations of a finite group) applies without change. ∎

Let $\Gamma_{m,n}$ denote the set of degrees of irreducible constituents in the Clebsch–Gordan decomposition. Thus $\Gamma_{m,n} = \{m+n-1, m+n-3, \ldots, m+n+1-2\min(m,n)\}$. When $k \in \Gamma_{m,n}$ we write q_k for the composite of the map given by extending the scalars for $W_{m,R} \otimes W_{n,R}$ to the field of fractions, K of R, followed by the projection onto the k-dimensional irreducible constituent of the $KSL(2,R)$-module (the above discussions imply that q_k is well defined).

LEMMA 2.3. *Suppose that $k \in \Gamma_{m,n}$, that $\overline{W_{m,R} \otimes W_{n,R}}$ is completely reducible, and that the degrees of its irreducible constituents are given by $\Gamma_{m,n}$. Then $\overline{\mathrm{Im}(q_k)}$ is irreducible and $\overline{q_k}$: $\overline{W_{m,R} \otimes W_{n,R}} \to \overline{\mathrm{Im}(q_k)}$ is a surjection.*

Proof. Let K be the field of fractions of R. The R-free $RSL(2,R)$-modules $W_{m,R} \otimes W_{n,R}$ and $\oplus_{k\in\Gamma_{m,n}} W_{k,R}$ both extend to $KSL(2,R)$-modules isomorphic to $W_{m,K} \otimes W_{n,K}$. Thus, according to Lemma 2.2, the modular reductions $\overline{W_{m,R} \otimes W_{n,R}}$ and $\oplus_{k\in\Gamma_{m,n}} \overline{W_{k,R}}$ have the same sets of composition factors. However, our hypothesis about the composition factors of the first of these modules shows that $\overline{W_{k,\mathbb{R}}}$ is an irreducible $\overline{R}SL(2,\overline{R})$-module of degree k.

Now, $\mathrm{Im}(q_k)$ is an $RSL(2,\mathbb{R})$-module which is K-equivalent to $W_{k,R}$. Hence, by Lemma 2.2, $\overline{\mathrm{Im}(q_k)}$ is also an irreducible $\overline{R}SL(2,\overline{R})$-module of degree k. Since q_k is a surjection from $W_{m,R} \otimes W_{n,R}$ to $\mathrm{Im}(q_k)$, $\overline{q_k}$, the result of tensoring with $\overline{R}$, is also a surjection. ∎

COROLLARY 2.4. *Suppose that $k \in \Gamma_{m,n}$, that $\overline{W_{m,R}} \otimes \overline{W_{n,R}}$ is completely reducible, and that the degrees of its irreducible constituents are given by $\Gamma_{m,n}$. Suppose that $S = \{(m_1,n_1),(m_2,n_2),\ldots,(m_r,n_r)\} \subset W_{m,R} \times W_{n,R}$ has the property that $\overline{m_1} \otimes \overline{n_1}, \overline{m_2} \otimes \overline{n_2}, \ldots, \overline{m_r} \otimes \overline{n_r}$ have linearly independent images under an $\overline{R}SL(2,\overline{R})$-module homomorphism from $\overline{W_{m,R}} \otimes \overline{W_{n,R}}$ onto its irreducible image of degree k. Then $p_k(m_1 \otimes n_1), p_k(m_2 \otimes n_2), \ldots, p_k(m_r \otimes n_r)$ are linearly independent.*

Proof. The modules $\overline{W_{m,R}} \otimes_{\overline{R}} \overline{W_{n,R}}$ and $\overline{W_{m,R} \otimes_R W_{n,R}}$ are naturally isomorphic. Thus we can identify the essentially unique $\overline{R}SL(2,\overline{R})$-module homomorphism from $\overline{W_{m,R}} \otimes_{\overline{R}} \overline{W_{n,R}}$ onto its irreducible image of degree k with the map $\overline{q_k}$ of Lemma 2.3. Independence of images under $\overline{q_k}$ implies independence of the corresponding images under q_k and p_k. ∎

Suppose that F is a finite subgroup of $SL(2,\mathbb{C})$ and that the corresponding character of F can be written over a ring of integers R with $\overline{R} = \mathbf{F}_p$. If the tensor product $\overline{W_n} \otimes \overline{W_m}$ of $SL(2,\overline{R})$-modules meets the conditions of Corollary 2.4, then the p-modular reduction of $p_k(W_n^F \otimes W_m^F)$ is the projection of $\overline{W_n}^F \otimes \overline{W_m}^F$ onto its unique k-dimensional summand. Our strategy for proving Theorem 2.1 is to compute the latter projections. Such work with barred objects is relatively pleasant since it involves linear algebra over the integers modulo a prime.

We write r_k for the projection of an $SL(2,\overline{R})$-module onto an irreducible summand of degree k when such a summand exists and that irreducible has multiplicity 1 in the module (the latter condition implies uniqueness of such a projection)

Here is the computation (which says in part (ii) that "$r_k = \overline{p_k}$," for suitable k). Theorem 2.1 follows from it and Corollary 2.4.

Proposition 2.5. *Let $p = 101$. Then*:

(i) *The traces for elements of T, S, and I in $SL(2,\mathbb{C})$ reduce in $\overline{R}$ to elements of the prime field $\mathbf{F}_{101}$. So if $Z = T$, S, or I, we may assume that R satisfies $Z \leq SL(2,R)$ and $\overline{R} = \mathbf{F}_{101}$.*

(ii) *The modules $\overline{W_9} \otimes \overline{W_{13}}$, $\overline{W_{13}} \otimes \overline{W_{21}}$, $\overline{W_7} \otimes \overline{W_7}$, $\overline{W_7} \otimes \overline{W_9}$, and $\overline{W_9} \otimes \overline{W_9}$ for $SL(2,101)$ are all completely reducible and the degree sets of their irreducible summands are* $\{5, 7, 9, 11, 13, 15, 17, 19, 21\}$, $\{9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33\}$, $\{1, 3, 5, 7, 9, 11, 13\}$, $\{3,5,7,9,11,13,15\}$, *and* $\{1,3,5,7,9,11,13,15,17\}$. *These degree sets match the degrees in the corresponding versions of the Clebsch–Gordan formula.*

(iii) $r_{19}(\overline{W_9}^S \otimes \overline{W_{13}}^S) \neq 0$.

(iv) $r_{31}(\overline{W_{13}}^I \otimes \overline{W_{21}}^I) \neq 0$.

(v) *The one-dimensional spaces $r_{13}(\overline{W_7}^T \otimes \overline{W_7}^T)$ and $r_{13}(\overline{W_7}^T \otimes \overline{W_9}^T)$ are distinct.*

(vi) *The one-dimensional spaces $r_{13}(\overline{W_9}^T \otimes \overline{W_9}^T)$ and $r_{13}(\overline{W_7}^T \otimes \overline{W_9}^T)$ are distinct.*

Method. For (i), we note that the only character irrationalities that we need to consider involve fifth roots of unity, and that $\mathbf{F}_{101}$ contains fifth roots of unity. Therefore we may work over a suitable ring of integers, R, such that $\overline{R} = \mathbf{F}_{101}$.

For (ii), we compute the decompositions of the tensor products with the Meat-Axe [P]. Moreover, by using the Meat-Axe to determine bases for the irreducible submodules of the dual spaces of these tensor products, we obtain matrices representing all projection maps, r_k, onto irreducible summands of the tensor products.

For (iii), (iv), (v) and (vi), we compute fixed point spaces under finite subgroups of $SL(2, \overline{R})$ with the Meat-Axe. The projections of images of tensor products of these spaces are then determined by using the representations of the projection maps that we computed in (ii). ∎

3. GENERATORS AND AUTOMORPHISMS OF $V_{L_2}^G$

Let $L_2 := \mathbb{Z}\alpha$, $(\alpha, \alpha) = 2$, be the A_1-lattice and let $V := V_{L_2}$ be the VOA of lattice type based on L_2. We let G be a subgroup of $Aut(V) \cong PSL(2, \mathbb{C})$ isomorphic to Alt_4, Sym_4, or Alt_5. The irreducible W_m for $SL(2, \mathbb{C})$ of dimension m may be interpreted as a module for $PSL(2, \mathbb{C})$ when m is odd.

THEOREM 3.1. *$Aut(V^G)$ is the identity if $G \cong Sym_4$ or Alt_5 and is isomorphic to $\mathbb{Z}_2$ if $G \cong Alt_4$. So, in all cases, $Aut(V^G) \cong N_{Aut(V)}(G)/G$.*

It is well known that $SL(2, \mathbb{C})$ acts on $\mathbb{C}[x, y]$ as a group of algebra isomorphisms such that $\mathbb{C}x + \mathbb{C}y$ is a natural $SL(2, \mathbb{C})$-module. One can identify W_{m+1} with the space of degree m homogeneous polynomials. As an $SL(2, \mathbb{C})$-module, $\mathbb{C}[x, y]$ has a decomposition

$$\mathbb{C}[x, y] = \bigoplus_{m \geq 1} W_m.$$

Let $\tilde{G}$ be the preimage of G in $SL(2, \mathbb{C})$ (we may assume $\tilde{G} = T$, S, or I) and A the algebra of invariants for the action of $\tilde{G}$ on $\mathbb{C}[x, y]$. The following proposition can be found in [S]. In all cases, A is a quotient of a polynomial ring with three generators modulo an ideal generated by a single relator, indicated below.

PROPOSITION 3.2. *The algebra has a set of generators as follows (f_n, g_n, and h_n denote homogeneous polynomials of degree n):*

(i) *If $\tilde{G} \cong SL(2, 3)$, $A = \mathbb{C}[f_6, f_8, f_{12}]$, subject to the relation $f_6^4 + f_8^3 + f_{12}^2 = 0$.*

(ii) *If $\tilde{G} \cong 2 \cdot Sym_4$, $A = \mathbb{C}[g_8, g_{12}, g_{18}]$, subject to the relation $g_{18}^2 + g_8^3 g_{12} + g_{12}^3 = 0$.*

(iii) *If $\tilde{G} \cong SL(2, 5)$, $A = \mathbb{C}[h_{12}, h_{20}, h_{30}]$, subject to the relation $h_{12}^5 + h_{20}^3 + h_{30}^2 = 0$.*

Let $L(c, h)$ be the irreducible highest weight module for the Virasoro algebra with central charge c and highest weight h for $c, h \in \mathbb{C}$. The subspace H of highest weight vectors for *Vir* in V is linearly isomorphic to the subspace $\mathbb{C}[x, y]^+$ of even polynomials in $\mathbb{C}[x, y]$, and we may and do assume that this isomorphism $\varphi\colon H \to \mathbb{C}[x, y]^+$ preserves degree. Let $H_{2m+1} := \varphi^{-1}(W_{2m+1})$.

Recall from [DG] that there is an isomorphism

$$(*) \qquad \psi : V \cong \sum_{m \geq 0} W_{2m+1} \otimes L(1, m^2),$$

as modules for $SL(2, \mathbb{C}) \times L(1, 0)$. Let π_{2m+1} be the projection of $\psi(V)$ to $W_{2m+1} \otimes L(1, m^2)$. Note that since $\dim(W_1) = 1$, π_1 can be interpreted as a map onto $L(1, 0)$.

We need the following result from [DM2] (see Lemma 2.3; also see [DM1]):

LEMMA 3.3. *Let K be a compact Lie group which acts continuously on a vertex operator algebra U. Let M, N be two finite dimensional K-submodules of U. Then there exists n such that the linear span of $\sum_{m \geq n} s_m t$, for $s \in M$, $t \in N$, is isomorphic to $M \otimes N$ as K-modules.*

For convenience, we set $v^1 = \varphi^{-1}(f_6)$ (resp. $\varphi^{-1}(g_8)$ or $\varphi^{-1}(h_{12})$), $v^2 = \varphi^{-1}(f_8)$ (resp. $\varphi^{-1}(g_{12})$ or $\varphi^{-1}(h_{20})$), and $v^3 = \varphi^{-1}(f_{12})$ (resp. $\varphi^{-1}(g_{18})$ or $\varphi^{-1}(h_{30})$) if $G \cong Alt_4$ (resp. Sym_4 or Alt_5). Then we have

PROPOSITION 3.4. *The vertex operator algebra V^G is generated by $\{\omega, v^1, v^2, v^3\}$.*

Proof. We prove the result for $\tilde{G} = T$, the other cases being similar. First note that the algebra $\mathbb{C}[x, y]^T$ is generated by f_6, f_8, and f_{12} and the algebra product can factor through the tensor product. The T-invariants of V have the form

$$\psi(V^T) = \bigoplus_{m \geq 0} W^T_{2m+1} \otimes L(1, m^2), \qquad \psi \text{ as in } (*) \text{, above.}$$

We use ψ to transfer the VOA structure of V to the right-hand side of $(*)$, as well as the action of the automorphisms. It is enough to show that if $W^T_{2s+1} \otimes L(1, s^2)$ and $W^T_{2t+1} \otimes L(1, t^2)$ can be generated by the ψ-images of ω and the v^i then so can $W^T_{2(s+t)+1} \otimes L(1, (s+t)^2)$. We assume that $s \geq t$.

By Lemma 3.3, $\mathrm{span}\{u_m v \mid u \in W_{2s+1} \otimes L(1, s^2),\ v \in W_{2t+1} \otimes L(1, t^2),\ m \in \mathbb{Z}\}$ is exactly the subspace

$$\bigoplus_{l = 2s-2t+1, 2s-2t+3, \ldots, 2s+2t+1} W_l \otimes L\left(1, \left(\frac{l-1}{2}\right)^2\right)$$

and $\operatorname{span}\{u_m v \mid u \in W^T_{2s+1} \otimes L(1,s^2),\ v \in W^T_{2t+1} \otimes L(1,t^2),\ m \in \mathbb{Z}\}$ is exactly the subspace

$$\bigoplus_{l=2s-2t+1,2s-2t+3,\dots,2s+2t+1} p_l\left(W^T_{2s+1} \otimes W^T_{2t+1}\right) \otimes L\left(1,\left(\frac{l-1}{2}\right)^2\right).$$

Since $p_{2(s+t)+1}(W^T_{2s+1} \otimes W^T_{2t+1}) = W^T_{2(s+t)+1}$, we immediately see that $W_{2(s+t)+1} \otimes L(1,(s+t)^2)$ can be generated by $W^T_{2s+1} \otimes L(1,s^2)$ and $W^T_{2t+1} \otimes L(1,t^2)$. As a result, $W_{2(s+t)+1} \otimes L(1,(s+t)^2)$ can be generated by ω and v^i. ∎

LEMMA 3.5. *For any $k \geq 1$, there is an invariant bilinear form on W_k, and for any $x, y \in W_k$ which are not orthogonal, $p_1(x \otimes y) \neq 0$. This applies to $x = y \neq 0$ in W_k^G whenever* $\dim(W_k^G) = 1$.

Proof. Since W_k is the unique irreducible of dimension k, it is a self-dual module. A nonzero invariant bilinear form is unique up to scalar multiple. Since p_1 maps $W_k \otimes W_k$ onto W_1, this projection is essentially that bilinear form. Since the subspace W_k^G is nonsingularly paired with itself under such a bilinear form, the last statement follows. ∎

Proof of Theorem 3.1. (i) Let $G \cong Sym_4$. Let $\sigma \in Aut(V^G)$. Since σ preserves weights and fixes ω, it stabilizes each H^G_{2m+1} which is the subspace of highest weight vectors of weight m^2 in V^G for the Virasoro algebra. By Proposition 3.2, $W_9^G = \mathbb{C}g_8$, $W_{13}^G = \mathbb{C}g_{12}$, and $W_{19}^G = \mathbb{C}g_{18}$. Thus there are scalars $c_i \in \mathbb{C}$ such that $\sigma g_i = c_i g_i$ for $i = 8, 12, 18$. By Lemmas 3.3 and 3.5, there exists $m_i \in \mathbb{Z}$ such that $0 \neq \pi_1((g_i)_{m_i} g_i) \in L(1,0)$. Since σ is trivial on $L(1,0)$, we immediately have $c_i^2 = 1$ for all i. That is, $c_i = \pm 1$.

From Proposition 3.2, W_{37}^G is two dimensional. Since $g_{18}^2 \in W_{37}^G$, which is regarded as a subspace of $\mathbb{C}[x,y]$, we see from $(*)$ and Lemma 3.3 that there exists $m \in \mathbb{Z}$ such that $0 \neq \pi_{37}((g_{18})_m g_{18}) \in W_{37}^G \otimes L(1,18^2)$. So, σ has an eigenvalue 1 on $W_{37}^G \otimes L(1,18^2)$. Using the actions of the Virasoro operators $L(n)$ for $n \geq 0$ on $\pi_{37}((g_{18})_m g_{18})$, we get an eigenvector in W_{37}^G for σ with respective eigenvalue 1. Similar use of Lemma 3.3 and $(*)$ implies that there exist $n_1, n_2, s_1, s_2, s_3 \in \mathbb{Z}$ such that $\pi_{37}((g_{12})_{n_1}\pi_{25}((g_{12})_{n_2}g_{12})) \in W_{37}^G \otimes L(1,18^2)$ is an eigenvector of σ with eigenvalue c_{12}^3 and $\pi_{37}((g_8)_{s_1}\pi_{29}[(g_8)_{s_2}\pi_{21}[(g_8)_{s_3}g_{12}]])$ is an eigenvector of σ with eigenvalue $c_8^3c_{12}$. As a result W_{37}^G contains three eigenvectors u, v, and w of σ with eigenvalues 1, c_{12}^3 and $c_8^3c_{12}$. From Proposition 3.2, any two of $\{u,v,w\}$ are linearly independent and the three are linearly dependent. Since W_{37}^G is two dimensional, we conclude $c_{12}^3 = c_8^3c_{12} = 1$, whence $c_8 = c_{12} = 1$.

It remains to show that $c_{18} = 1$. By Theorem 2.1(i) and Lemma 3.3, there exists $m \in \mathbb{Z}$ such that $\pi_{19}((g_8)_m g_{12}) \in W_{19}^G \otimes L(1, 81)$ is an eigenvector of σ with eigenvalue 1. Since W_{19}^G is one dimensional, we see that $\sigma g_{18} = g_{18}$. Since V^G is generated by ω and g_i (see Proposition 3.4), we conclude that σ is the identity map.

(ii) The proof in the case that $G \simeq Alt_5$ is similar to that in case (i).

(iii) Let $G \cong Alt_4$. In this case V^G is generated by ω and f_6, f_8, f_{12} by Proposition 3.4. Let $\sigma \in Aut(V^G)$. Since $W_7^G = \mathbb{C} f_6$ and $W_9^G = \mathbb{C} f_8$, there exist $c_6, c_8 \in \mathbb{C}$ so that $\sigma f_6 = c_6 f_6$ and $\sigma f_8 = c_8 f_8$. Using the same argument as used in the proof of case (i), we get $c_6 = \pm 1$ and $c_8 = \pm 1$. Note that $W_{13}^G = \mathbb{C} f_6^2 + \mathbb{C} f_{12}$ is two dimensional. From Theorem 2.1(iii) and Lemma 3.3, there exist $m, n \in \mathbb{Z}$ such that $\pi_{13}((f_6)_m f_6), \pi_{13}((f_6)_n f_8) \in W_{13}^G \otimes L(1, 36)$ are linearly independent eigenvectors of σ with eigenvalues 1 and $c_6 c_8$. This implies that V^G is generated by ω, f_6, and f_8 and the automorphism group is isomorphic to one of 1, $\mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Recall that the normalizer $N(G) = N_{Aut(V)}(G)$ is isomorphic to Sym_4. From Lemma 3.2 of [DM1], we know that $V^{N(G)}$ and V^G are different, whence an element of $N(G)/G \cong \mathbb{Z}_2$ acts on V^G as a nontrivial automorphism, denoted by σ. So, $Aut(V^G)$ is either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $Aut(V^G)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then there exists $\tau \in Aut(V^G)$ such that $Aut(V^G)$ is generated by σ and τ. By Theorem 2.4 of [DLM1], V^G can be decomposed

$$V^G = V_{1,1}^G \oplus V_{1,-1}^G \oplus V_{-1,1}^G \oplus V_{-1,-1}^G,$$

where $V_{\mu,\lambda}^G = \{v \in V^G \mid \sigma v = \mu v,\ \tau v = \lambda v\}$ and each is nonzero. Moreover $Y(u, z)v \in V_{\mu_1\mu_2, \lambda_1\lambda_2}^G[[z, z^{-1}]]$ if $u \in V_{\mu_1, \lambda_1}^G$ and $v \in V_{\mu_2, \lambda_2}^G$. It is easy to see that $V^{N(G)} = V_{1,1}^G \oplus V_{1,-1}^G$ and $V^{N(G)}$ has a nontrivial automorphism. This is a contradiction to (i). Thus $\mathrm{Aut}(V^G)$ must be isomorphic to $\mathbb{Z}_2$. This completes the proof. ∎

Remark 3.6. From the proof of Theorem 3.1, we see that, when $G \cong Alt_4$, V^G is generated by ω, f_6, and f_8. This strengthens the result in Proposition 3.4.

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