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The Role of the Tensor Product in the Splitting of Abelian Groups

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INTRODUCTION

The splitting problem in abelian groups, in its simplest form, asks when is an abelian group a direct sum of a torsion group and a torsion-free one. This is of course a special case of the problem of describing all extensions of torsion-free groups by torsion groups. It is a hard problem of long standing, and, despite numerous attempts, only fragmentary information is available: for instance, see the papers in the bibliography. In this paper we develop what seems to be a new approach to solving the splitting problem.

The outline of the paper is as follows: The notation and terminology are collected in Section 1.

In Section 2 we quickly reduce the splitting problem to two problems, the splitting of those whose quotient by their torsion subgroup is reduced, and those whose quotient by their torsion subgroup is divisible. For simplicity, we concentrate on groups whose torsion subgroups are primary. (See, however, the remark at the end of the section). The main result of this section, Theorem 1.2, shows that decomposing a group of the first kind as the tensor product of two other groups is at least as good a reduction as decomposing it into the direct sum of two groups. For we show that if $A = B \otimes C$, then A splits if, and only if, both B and C do. Note that rank A = (rank B) $\times$ (rank C), whereas in $A = K \oplus L$, rank A = rank K + rank L, so that usually the ranks of B and C are smaller than those of K and L.

In Section 3, we investigate groups whose quotient by their torsion subgroup is divisible. It turns out that for this class of groups it is fruitful to define the notion of splitting length $l(x)$ of a group X. This is defined in Section 3 as the least positive integer n, such that $X \otimes \cdots \otimes X$, n times, splits. This has the following properties: Denote by C_n the class of those groups of splitting length n, a group being in C_∞ if it is not in any C_n, $n < \infty$. Then,

(1) Every group is in some class by definition.

(2) Each C_n is nonempty.

(3) Groups in class C_n are simpler, with respect to failure to split, than those in C_{n+1} (and, indeed C_1 is the class of groups which split). The meaning of "simpler" may be clarified by saying that $l(x)$ is proportional to the size of the subgroups S of the torison subgroup of X required for X/S to lie in class C_1. This is borne out by the main theorem of this section which, loosely expressed, relates the splitting of $X \otimes X$ to that of X/S.

The main result of Section 4 is to show that each class C_n is nonempty.

Finally, we wish to remark that this is a first paper on the subject, and so there is no need for us to pose questions. As he progresses, the reader will find himself being stared in the face by a host of questions which are left unanswered.

Section 1

In this section we comment briefly about notation and terminology.

First and foremost "group" means "abelian group" throughout this paper. "Map" means "homomorphism."

The symbol $T(G)$ is used to denote the torsion subgroup of group G. The symbol $[x]$ means the cyclic group generated by x.

Very importantly, "G splits" means $T(G)$ is a direct summand of G.

Unless otherwise clear from the context, we use "quotient" to mean "a group modulo its torsion subgroup".

We occasionally use "rank" to mean "torsion-free rank," that is, the rank of the group modulo its torsion subgroup.

Recall that where A, B are any abelian groups and $R \subset A$, $S \subset B$ are arbitrary subsets, the notation $\langle R, S\rangle$ designates the subgroup of $A \otimes B$ generated by elements $r \otimes s \in A \otimes B$, $r \in R$, $s \in S$.

For the definitions of "cotorsion," "adjusted," and homological terms we refer to Harrison's paper, [8] and Mac Lane [10].

Section 2

The purpose of this section is to present some reduction theorems and observations which may provide motivation for the results which follow. Some results are mentioned in this section which are of interest in themselves and others are given for completemess.

We begin with

LEMMA 2.1. *Let $f : A \to A/T(A)$ be the natural homomorphism, and let $A/T(A) = D \oplus R$. Then A splits if, and only if, both $f^{-1}(D)$ and $f^{-1}(R)$ split.*

Proof. Suppose A splits. Then $A = F \oplus T(A)$ where F is torsion free. Let $F_D = \{a \in F : f(a) \in D\}$. Since $T(A) \subseteq f^{-1}(D)$, we have $f^{-1}(D) = F_D \oplus T(A)$ which shows that $f^{-1}(D)$ splits. Similarly $f^{-1}(R)$ splits.

Conversely, suppose that $f^{-1}(D)$ and $f^{-1}(R)$ both split. Since $T[f^{-1}(D)] = T(A) = T[f^{-1}(R)]$ we have $f^{-1}(D) = F_D \oplus T(A)$ and $f^{-1}(R) = F_R \oplus T(A)$ where F_D and F_R are torsion-free subgroups of A. Clearly, $A = F_D + F_R + T(A)$. Now suppose $a \in (F_D + F_R) \cap T(A)$. Then

$$0 = a + T(A) = f_D + f_R + T(A) = [f_D + T(A)] + [f_R + T(A)],$$

and since $A/T(A), f^{-1}(D)$, and $f^{-1}(R)$ are the given direct sums $f_D = f_R = 0$ and the proof is complete.

An observation which reduces the splitting problem into two cases is

COROLLARY 2.1. *The splitting problem for arbitrary Abelian groups may be reduced to the problem for those groups G with $G/T(G)$ reduced and those groups G with $G/T(G)$ is divisible.*

Proof. Let A be any group. Write $A/T(A) = D \oplus R$ where D is divisible and R is reduced and apply Lemma 2.1. Then A splits if, and only if, both $f^{-1}(D)$ and $f^{-1}(R)$ do.

In much of what follows we will often make the simplifying assumptions "p divisible" instead of "divisible" and "p reduced" instead of "reduced" and $T(A)$ primary instead of torsion, but we shall always make these distinctions clear as we go along so as to make most of the statements of the results self-contained.

Remark. Referring to $G/T(G)$ as the quotient of G, notice that

$$\{\text{groups with } p \text{ reduced quotient}\} \subseteq \{\text{groups with reduced quotient}\}$$

and

$$\{\text{groups with divisible quotient}\} \subseteq \{\text{groups with } p\text{-divisible quotient}\}$$

Notice that those groups whose quotients are both divisible and reduced must be torsion groups.

Our next observation shows that tensoring by p-reduced groups does not affect the splitting of a group A with $T(A)$ p primary:

Remark. The authors are indebted to the referee for the shortened proofs of Theorems 2.1 and 2.3 which follow.

THEOREM 2.1. *Suppose M is a group where $M/T(M)$ is not p divisible and A is any group with $T(A)$ p primary. Then $M \otimes A$ splits implies that A splits.*

Proof. Let $B/T(M)$ be p-basic subgroup (see [7]) of $M/T(M)$. Then $B = T(M) \oplus F$ where F is free. Also $F \neq 0$ for $M/T(M)$ is not p-divisible. Now $M/B \cong (M/T(M))/(B/T(M))$ is p-torsion free and p divisible. Hence, the exact sequence

$$0 \to T(A) \to A \to \frac{A}{T(A)} \to 0$$

gives exact sequence

$$0 \to \operatorname{Tor}\left(\frac{M}{B}, T(A)\right) \to \operatorname{Tor}\left(\frac{M}{B}, A\right) \to 0,$$

but $\operatorname{Tor}[M/B, T(A)] = 0$, since $T(A)$ is p primary and M/B p-torsion free. Hence, $\operatorname{Tor}(M/B, A) = 0$. Now the exact sequence

$$0 \to B \to M \to \frac{M}{B} \to 0$$

gives exact sequence

$$0 \to A \otimes B \xrightarrow{e} A \otimes M \to A \otimes \frac{M}{B} \to 0.$$

Using the facts that M/B is p divisible, $T(A)$ is p primary, and Exercise 16, p. 256 of [6], we have

$$e[T(A \otimes B)] = T(A \otimes M).$$

Since $A \otimes M$ splits and $e[T(A \otimes B)] = T(A \otimes M)$, $A \otimes B \cong e(A \otimes B)$ also splits.

Since Z is a direct summand of B, we have that $A \cong A \otimes Z$ is a direct summand of $A \otimes B$ and, therefore, A splits as stated.

As a special case where $T(M) = 0$, we have

COROLLARY 2.2. *Let A be any abelian group with $T(A)$ p-primary. Let M be torsion free, and not p divisible. Then A splits if and only if $A \otimes M$ splits.*

As an important application of Corollary 2.2 we have the following:

COROLLARY 2.3. *The splitting question for any group A with $T(A)$ p primary reducrs to the splitting question for two groups G and H, one with $G/T(G)$ p reduced, the other such that $H/T(H)$ is divisible.*

Proof. Let $R_{-p} = \{a/b \in Q : (b, p) = 1\} \subseteq Q$. Clearly, $pR_{-p} < R_{-p}$ and so $R_{-p} = M$ and A satisfy the hypotheses of Corollary 2.2. Hence, A splits if, and only if $R_{-p} \otimes A$ splits. (Note that for any A, $T(R_{-p} \otimes A)$ is p primary). Next apply Corollary 2.1 to $R_{-p} \otimes A$ and set $G = f^{-1}(R)$ and $H = f^{-1}(D)$. Now it suffices to show that $G/T(G)$ is p reduced. Since G in this case is divisible by every prime $q \neq p$ and $G/T(G)$ is reduced, $G/T(G)$ must be p reduced as stated.

A major consequence of Theorem 2.1 is the following

THEOREM 2.2. *For each $i = 1,..., n$ let A_i be any group having the property that each $T(A_i)$ is p primary and that no $A_i/T(A_i)$ is p divisible. Then $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ splits if, and only if, each A_i splits.*

Proof. If each A_i splits then $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ splits, simply because the tensor product of torsion groups, and of torsion and torsion free groups, is torsion while the tensor product of torsion free groups is torsion free.

To prove the converse, first suppose $n = 2$. Set $A_1 = M$ and $A_2 = A$ and assume that our conditions are satisfied on M and A, and that $M \otimes A$ splits. Then, by Theorem 2.1, A (that is, A_2) splits. Interchanging the roles of A_1 and A_2, we conclude in the same way that A_1 splits. This completes the proof of the theorem for the case $n = 2$.

The case of general $n > 2$ follows, by induction, using the "p-divisible quotient" case of the following lemma:

LEMMA 2.2. *For any groups A and B, $G = A \otimes B$ is such that $G/T(G)$ is p divisible if, and only if, one of A, B is such that $A/T(A)$ or $B/T(B)$ is p divisible.*

Remark. Another way of stating this lemma which makes it more palatable is:

For any groups A and B, $G = A \otimes B$ has a p divisible quotient if, and only if, at least one of A, B does.

Proof. Let $C = A/T(A)$, $D = B/T(B)$. Recall (for example, from Fuchs, p. 252) that $(A \otimes B)/T(A \otimes B) \simeq C \otimes D$. Hence, if C or D is p divisible, so is $(A \otimes B)/T(A \otimes B)$.

To prove the converse: If neither C nor D is p divisible, then $pC \neq C$ and $pD \neq D$.

Now we have a natural homomorphism of $C \otimes D$ onto $C/pC \otimes D/pD$. The latter is not the 0 group, and is a direct sum of cyclic groups of order p. Hence it cannot be p-divisible. But it is a homomorphic image of $(A \otimes B)/T(A \otimes B)$, so the latter group is not p divisible.

Several corollaries of Theorem 2.2 are worth noting.

COROLLARY 2.4. *For any group $A = B_1 \otimes B_2$ such that $T(B_1)$, $T(B_2)$ p primary and $A/T(A)$ not p divisible, A splits if, and only if, B_1 and B_2 both do.*

Proof. The assumption that $A/T(A)$ is not p divisible implies the same for B_1 and B_2. The conclusion follows by Theorem 2.2.

This corollary shows that for groups with p-reduced quotients, "decomposition" as a tensor product shares a useful property with decomposition as a direct sum: if $A = B_1 \oplus B_2$ or $A = B_1 \otimes B_2$, A splits if, and only if, B_1 and B_2 both do.

The next observation is important for our notion of splitting length in what follows.

COROLLARY 2.5. *If $T(A)$ is p primary, and $A/T(A)$ is not p divisible, and A doesn't split, then neither does $A \otimes A \otimes \cdots \otimes A$ for any number of factors.*

Remark. It may be of interest to observe that in the proof of Theorem 2.2, we can actually show that, under the assumption of that Theorem, not only is each A_i isomorphic to a p-pure subgroup of $A_1 \otimes A_2 \otimes \cdots \otimes A_n$, but indeed if $\{i_1, ..., i_m\}$ is any subset of $\{1, ..., n\}$, then $A_{i_1} \otimes \cdots \otimes A_{i_m}$ is also isomorphic to such a subgroup.

The next lemma relates the splitting of a group A to that of A/W where the subgroup $W \subseteq T(A)$.

LEMMA 2.3. *Let A be a group and W a subgroup of $T(A)$. Let $f : A \to A/W$ be the natural homomorphism. If A splits, then A/W splits, and, hence, can be written as $T(A/W) \oplus F$; and the subgroup $f^{-1}(F)$ defined thereby also splits. Conversely, if A/W splits: $A/W \cong T(A/W) \oplus F$, and if $f^{-1}(F)$ also splits, then A splits.*

Proof. Write $f^{-1}(F)$ as $f^{-1}(F) = T(f^{-1}(F)) \oplus H$ where H is torsion free. Then it is easy to see that $A = T(A) \oplus H$. If A splits, then clearly A/W and $f^{-1}(F)$ split.

The next observation shows that the splitting of a group A is reduced to the splitting of a subgroup M of A whose torsion subgroup is a direct sum of cyclic groups.

COROLLARY 2.6. *Let B be a basic subgroup of $T(A)$. Then (A/B) decomposes as $A/B = T(A)/B \oplus M/B$; and A splits iff M splits.*

Proof. Use Lemma 2.3, setting $W = B$ and $F = M/B$; the proof is straightforward.

A reduction theorem, to groups of torsion -free rank one, for those groups whose quotients are p divisible is

THEOREM 2.3. *Let A be any abelian group such that $T(A)$ is p primary, suppose that $A/T(A)$ is p divisible, and let $f : A \to A/T(A)$ be the natural homomorphism. Then A splits if, and only if, $f^{-1}(C)$ splits whenever C is any p-pure rank one subgroup of $A/T(A)$.*

Proof. The "only if" part is obvious.

To prove the converse, observe that $\text{Ext}(B, T) \cong \text{Ext}(R_{-p} \otimes B, T)$ where B is torsion free, T p primary and $R_{-p} = \{a/b \in Q : (p, b) = 1\}$. The isomorphism is the map which takes equivalence class $[0 \to T \to X \to B \to 0]$ onto

$$[0 \to T \cong R_{-p} \otimes T \to R_{-p} \otimes X \to R_{-p} \otimes B \to 0].$$

Therefore,

$$\epsilon : 0 \to T(A) \to A \xrightarrow{f} \frac{A}{T(A)} \to 0$$

splits if, and only if,

$$\epsilon^* : 0 \to R_{-p} \otimes T(A) \to R_{-p} \otimes A \xrightarrow{f^*} R_{-p} \otimes \frac{A}{T(A)} \to 0$$

splits. Since $A/T(A)$ is torsion free and p divisible $R_{-p} \otimes A/T(A) = \sum_{\lambda \in \Lambda} Q_\lambda$ where Q_λ denotes a copy of rationals for each $\lambda \in \Lambda$. Hence, ϵ^* is equivalent to a vector $\langle \epsilon_\lambda^* \rangle$ representing an element in $\prod_{\lambda \in \Lambda} \text{Ext}[Q_\lambda, T(A)]$. The hypothesis that $f^{-1}(C)$ splits whenever C is a rank one p-pure subgroup of $A/T(A)$ implies that $f^{*-1}(Q_\lambda)$ splits for each λ, i.e., $\epsilon_\lambda^* = 0$ for each $\lambda \in \Lambda$ and thus $\epsilon^* = 0$. Hence, $\epsilon = 0$ and A splits.

The next two observations compare the torsion subgroups of G and $G \otimes G$.

LEMMA 2.4. *Let G be any mixed group for which $G/T(G)$ is divisible. Then $T(G \otimes G)$ is a direct sum of cyclic groups.*

Proof. The pure exact sequence

$$0 \to T(G) \to G \to \frac{G}{T(G)} \to 0$$

yields

$$0 \to G \otimes T(G) \to G \otimes G \to G \otimes \frac{G}{T(G)} \to 0$$

and

$$0 \to T(G) \otimes T(G) \to G \otimes T(G) \to 0.$$

From the last two sequences we see that $T(G \otimes G) \cong T(G) \otimes T(G)$ and it is well known that this last group is a direct sum of cyclic groups.

An easy consequence of Lemma 2.4 is

COROLLARY 2.7. *Any cotorsion adjusted mixed group G is such that $T(G \otimes G)$ is a direct sum of cyclic groups.*

Proof. $G/T(G)$ is divisible.

Along these lines we have the following n.a.s.c.: for $T(G)$ and $T(G \otimes G)$ to be a direct sum of cyclic groups together.

THEOREM 2.4. *Let G be a mixed group with $T(G)$ p primary, G reduced and $G/T(G)$ p reduced. Then $T(G \otimes G)$ is a direct sum of cyclic groups, if, and only if, $T(G)$ is.*

Proof. Let $T(G)$ be a direct sum of cyclic groups. Then put $\tilde{G} = \text{Ext}(Q/Z, G)$ and we have $\tilde{G}/G$ is torsion-free divisible. This means that $T(G) = T(\tilde{G})$. Now the pure exact sequence

$$0 \to G \to \tilde{G} \to \frac{\tilde{G}}{G} \to 0$$

yields

$$0 \to G \otimes G \to G \otimes \tilde{G} \to G \otimes \frac{\tilde{G}}{G} \to 0$$

and

$$0 \to G \otimes \tilde{G} \to \tilde{G} \otimes \tilde{G} \to \tilde{G} \otimes \frac{\tilde{G}}{G} \to 0,$$

where the groups on the right all torsion-free divisible. Hence, $T(G \otimes G) \cong T(G \otimes \tilde{G}) \cong T(\tilde{G} \otimes \tilde{G})$. But by Harrison we have $\tilde{G} = T(\tilde{G}) \oplus C = A \oplus C$ where A is adjusted and C torsion-free. Now $\tilde{G} \otimes \tilde{G} \cong (A \otimes A) \oplus (A \otimes C) \oplus (C \otimes A) \oplus (C \otimes C)$. The torsion subgroup on the right is a direct sum of the torsion subgroups of its summands each of which is a direct sum of cyclic groups since $T(G)$ is.

For the converse, notice that $G/T(G) \neq 0$ being p reduced allows us to select a p-pure cyclic subgroup $[y] \cong Z$ in G. To see this pick an element in $G/T(G)$ of p-height zero and jump back and forth via the natural map. Next consider the p-pure exact sequence

$$0 \to [y] \to G \to \frac{G}{[y]} \to 0$$

from which we get the exact sequence $0 \to [y] \otimes G \to G \otimes G$ whence we have that $T(G)$ must be a direct sum of cyclic groups. Note the last sequence is left exact since $[y]$ is p-pure in G and the tensoring G has only p torsion.

Remark. Notice that using the same technique as in the proof of the foregoing theorem it can be shown that if X and Y are groups satisfying the hypotheses on G in Theorem 2.4, we have that $T(X \otimes Y)$ is a direct sum of cyclic groups if, and only if, both $T(X)$ and $T(Y)$ are. (Actually we may replace "p reduced" by "not p divisible" on $X/T(X)$ and $Y/T(Y)$.) Using induction and the proof of Corollary 2.4 we obtain:

THEOREM 2.5. *For each $i = 1, 2,..., n$ let A_i be any group having the property that $T(A_i)$ is p primary and that $A_i/T(A_i)$ is not p divisible, and that A_i is reduced. Then $T(A_1 \otimes A_2 \otimes \cdots \otimes A_n)$ is a direct sum of cyclic groups if, and only if, each $T(A_i)$ is.*

Setting each $A_i = G$ in Theorem 2.4 we have easily

COROLLARY 2.8. *Let G satisfy the hypotheses on the A_i in Theorem 2.5, and let $G^n = G \otimes \cdots \otimes G$ (n factors). Then for $n = 1, 2, 3,...$, $T(G^n)$ is a direct sum of cyclic groups if, and only if, $T(G)$ is.*

Remark. The following example shows that a word of caution is necessary: That splitting in the p-primary case in some sense does not imply splitting in general. To see this consider the following example: Let $\{p_i\}$ be the set of primes and set $G = \prod_{i=1}^{\infty} C(p_i)$. Notice that $T = \bigoplus_{i=1}^{\infty} C(p_i)$ is the rotsion subgroup of G such that G/T is divisible. The interesting thing is that each $G_p = T_p = C(p)$ is a direct summand of G. But the torsion subgroup T is *not* a direct summand. Along these lines see also [12].

SECTION 3

In this section we study conditions for $B_1 \otimes B_2$ to split where B_i are groups such that $T(B_i)$ is p primary and $B_i/T(B_i)$ is p divisible.

In Section 2 we established a rather satisfactory role for the tensor product in the splitting of an abelian group whose quotient is reduced. Now we consider the other main class of groups, namely those for which the quotient is p divisible. Here as one might forsee from the fact that the tensor product of torsion groups is a direct sum of cyclic groups, tensoring plays the role of a simplifying operation. This is definitely so for groups of finite splitting length, while for those of infinite splitting length it at least points to a more intrinsic complication. Perhaps certain algebraic operations involving "limits" provide the appropriate tool for their study. We begin with a few lemmas.

The following fact is standard:

LEMMA 3.1. *Suppose A is an abelian group such that $T(A)$ is p-primary, and has a p-pure torsion free subgroup B such that A/B is a torsion group. Then A splits.*

Proof. Define the subgroup F of A:

$$F = \{a \in A \mid \text{for some integer } m,\ (m, p) = 1,\ ma \in B\}.$$

We claim F is torsion free, pure (not only p pure) in A, and A/F is a torsion group. The first and third facts are obvious. We prove the purity as follows:

Suppose $a \in A$, $ca \in F$, $c \neq 0$. We may write c as $c = np^i$, where $(n, p) = 1$. By the definition of F, for some integer m, $(m, p) = 1$, $m(ca) \in B$. By the p purity of B in F, there exists $b \in B$ such that $p^i(b - mna) = 0$. Now $(b - mna)$ has order prime to mn, hence, there exists $t \in T(A)$ such that $mnt = (b - mna)$. Since $p^i(mnt) = 0$, so does $p^i t = 0$ already, since $T(A)$ is p primary. Let $a' = a + t$. Then $mna' = mna + (b - mna) = b \in B$, so $a' \in F$; and $c(a' - a) = ct = np^i t = 0$.

We shall now show that $A = T(A) \oplus F$. Given $a \in A$, then, for some nonzero integer c, $ca \in F$ (since A/F is torsion). F being pure and torsion free, there is a unique $a' \in F$ such that $c(a' - a) = 0$. The required splitting is: $a = (a - a') + a'$.

A rather interesting fact is found in

LEMMA 3.2. *Let C and D be any abelian groups, and assume that $C \otimes D$ splits into $C \otimes D = T(C \otimes D) \oplus F$ where $F \neq 0$. Then F contains a non-zero element of the form $c \otimes d$, $c \in C$, $d \in D$.*

Proof. Let $f : C \to C/T(C)$ and $g : D \to D/T(D)$ be the natural homomorphisms, and let x and y be elements of C and D, respectively, having infinite order. Then it is well known that $(f \otimes g)(x \otimes y)$ has infinite order in $C/T(C) \otimes D/T(D)$, where $(f \otimes g)(x \otimes y) = f(x) \otimes g(y)$. Hence, $x \otimes y$ may be written in the form $t + z$, $t \in T(C \otimes D)$ and $0 \neq z \in F$. If order $t = n$, then $nx \otimes y$ obviously is in F and is not zero. Set $c = nx$ and $d = y$.

Remark. It is interesting to note that if in Lemma 3.2 we have $C = D$, and rank $[C/T(C)] = 1$, then F contains a nonzero element of the form $c \otimes c$, $c \in C$. For let nx and y be as in the proof of Lemma 3.2. Since the torsion free rank of C is one, there exist nonzero integers m and k such that $m(nx) = ky$. Then $mk(nx \otimes y)$ is in F and we may take $c = mnx = ky$.

In the following lemma we make two simple observations, needed in some subsequent proofs, on the preservation of p divisibility.

LEMMA 3.3. *Assume $X/T(X)$ is p divisible. Let Y be any abelian group, and A any subgroup of X. Then:*

(1) *$(X \otimes Y)/T(X \otimes Y)$ is p divisible, and*

(2) *$(X/A)/T(X/A)$ is p divisible.*

Proof of (1). Let $x \otimes y$ be any generator of $X \otimes Y$. Since $X/T(X)$ is p divisible, there exists x' in X such that

$$px' - x \in T(X).$$

Then

$$p(x' \otimes y) - (x \otimes y) = (px' - x) \otimes y \in \langle T(X), Y \rangle \subseteq T(X \otimes Y).$$

Proof of (2). Let $x \in X$ be a representative of any element $x + A$ of X/A. There exists x' in X such that

$$px' - x \in T(X).$$

Then

$$p(x' + A) - (x + A) = (px' - x) + A \in T(X) + A \subseteq T\frac{X}{A}.$$

The next observation is one of those "technical" lemmas that will come in handy.

LEMMA 3.4. *Let X_p be a p-primary torsion group, and let Y be such that $Y/T(Y)$ is p divisible. Then the natural map $i : X_p \otimes T(Y) \to X_p \otimes Y$ is an isomorphism.*

Proof. From the exact sequence

$$0 \to T(Y) \to Y \to \frac{Y}{T(Y)} \to 0$$

and the fact that $X_p \otimes [Y/T(Y)] = 0$, we have (using the half-exactness of the $\otimes$ functor) the exactness of

$$X_p \otimes T(Y) \to X_p \otimes Y \to 0;$$

so i is onto.

To show that i is one-to-one, we need only observe that $T(Y)$ is pure in Y, and that, as is well known, for any pure subgroup A of a group B and for any group G, the induced map $G \otimes A \to G \otimes B$ is one-to-one.

An easy consequence of the foregoing lemma is

COROLLARY 3.1. *Under the assumptions of Lemma* 3.4, *$X_p \otimes Y$ is a direct sum of p-primary cyclic groups.*

Proof. It is well-known that the tensor product of two torsion groups is a direct sum of cyclic groups. The corollary follows immediately.

DEFINITION. Let A be an abelian group, x an element of A, and p be a a prime. A sequence $x_1, x_2, x_3, \ldots$ of elements of A will be called a p sequence for x if, and only if, $px_1 = x$ and $px_{n+1} = x_n$, $n = 1, 2, 3, \ldots$.

Observe that if A is torsion free and $pA = A$, then each element of A has a p sequence which is unique.

An easy result which uses the idea of a p sequence and is fundamental in this paper is

PROPOSITION 3.1. *Let A be a group where $A/T(A)$ is of rank 1, $T(A)$ is p primary, and let $0 \neq a \in A$ be an element of infinite order having a p sequence $\{a_n\}$ in A. Then A splits.*

Proof. Let B be the subgroup of A generated by $\{a, a_1, a_2, \ldots\}$. Notice that B is torsion-free. ($ma_n = 0$ yields $mp^n a_n = 0$ or $ma = 0$, thus m must be 0.) Since B is p divisible, B is p pure in A, and A/B is torsion since A is of torsion-free rank 1. Now by Lemma 3.1 we have that A splits as stated.

An interesting p sequence is displayed in

LEMMA 3.5. *Let X and Y be groups and be such that $X/T(X)$ has rank 1. Let Y be such that $Y/T(Y)$ is p divisible. Assume that $X \otimes Y$ splits: $X \otimes Y = T(X \otimes Y) \oplus F$. Consider any $z_1 = x \otimes y$ in F. Then there exists a p sequence $z_2, z_3, z_4, \ldots$ for z_1 in F; and each z_i can be written in the form $z_i = x \otimes b_i$, $b_i \in Y$.*

Proof. Assuming $z_i \in F$ is determined, we construct z_{i+1}. Since $Y/T(Y)$ is p divisible, so is $(X \otimes Y)/T(X \otimes Y)$ (by assertion (1) of Lemma 3.3, with the roles of X and Y interchanged). But this factor group is isomorphic to F; so there exists $z_{i+1} \in F$ such that

$$pz_{i+1} = z_i .$$

(Since F is torsion free, z_{i+1} is in fact unique.)

It remains to be proved that the z_i can be represented in the form claimed.

Consider $f: X \otimes Y \to (X/[x]) \otimes Y$. Clearly $f(x \otimes y) = 0$. Since $p^i z_i = x \otimes y$, we also have $p^i f(z_i) = 0$; so each $f(z_i)$ lies in the p-primary component of $(X/[x]) \otimes Y$, which component is naturally isomorphic to $X_p \otimes Y$ where X_p is the p-primary component of the torsion group $X/[x]$.

By Corollary 3.1, $X_p \otimes Y$ is a direct sum of p-primary cyclic groups. On the other hand, each z_i, and, hence, each $f(z_i)$, has infinite p height. This is impossible unless every $f(z_i) = 0$.

Finally, it follows from the half-exactness of the $\otimes$ functor that the ker of f is precisely the set of elements of $X \otimes Y$ which are of the form $x \otimes b$ with $b \in Y$. Hence each z_i is of this form.

Another one of those technical lemmas which turns out to be very useful is

LEMMA 3.6. *Let X and Y be groups such that $X/T(X)$ is p-divisible, $T(Y)$ is p-primary and $X \otimes Y$ splits. Let α be a homomorphism of X onto E where E is torsion free p divisible. Then* $(\ker \alpha) \otimes Y$ *splits.*

Proof. Since E is torsion free, $\ker \alpha$ is pure so that

$$0 \to (\ker \alpha) \otimes Y \to X \otimes Y \to E \otimes Y \to 0 \text{ is exact.}$$

Since E is torsion free p divisible and $T(Y)$ is p primary, $E \otimes Y$ is torsion free. This means that $T[(\ker \alpha) \otimes Y] \simeq T(X \otimes Y) = T$ so that

$$0 \to \frac{(\ker \alpha) \otimes Y}{T} \to \frac{X \otimes Y}{T}$$

is exact whence

$$\operatorname{Ext}\left[\frac{X \otimes Y}{T}, T\right] \xrightarrow{\beta} \operatorname{Ext}\left[\frac{(\ker \alpha) \otimes Y}{T}, T\right] \to 0$$

is exact. Since the extension giving rise to $(\ker \alpha) \otimes Y$ is the β image of the extension giving rise to $X \otimes Y$ (and the latter is zero since $X \otimes Y$ splits) we have that $(\ker \alpha) \otimes Y$ splits.

And now with the help of Lemma 3.6 we can give a slightly better version of Lemma 3.5, as

LEMMA 3.7. *Let X and Y be groups such that $T(X)$ and $T(Y)$ are p primary and $Y/T(Y)$ is p divisible. Further suppose that $X \otimes Y$ splits as $X \otimes Y = T(X \otimes Y) \oplus F$ where F is torsion free and $0 \neq x \otimes y \in F$. Then for some integer n, $x \otimes ny$ has a p sequence $\{z_i\}$ where each z_i has the form $z_i = x \otimes y_i$, $y_i \in Y$.*

Proof. Embed $x + T(X)$ in a pure subgroup S of $X/T(X)$ of rank 1. Consider the sequence of maps

$$X \to \frac{X}{T(X)} \to \frac{X/T(X)}{S} = E \to 0$$

and let α be the composition: $\alpha : X \to E$. Clearly, $\ker \alpha$ has torsion free rank 1, contains $T(X)$, and $(\ker \alpha)/T(X)$ is p divisible. By Lemma 3.6, $(\ker \alpha) \otimes Y$ splits into $T[(\ker \alpha) \otimes Y] \oplus F'$. Since $\ker \alpha$ has rank 1 and for some integer n, $x \otimes ny \in F'$, Lemma 3.5 applies and we have the desired p sequence.

We now have enough tools to manufacture a proof of one of the main results in this paper. But first a

Definition. *Consider abelian groups* X, Y, *and their tensor product* $X \otimes Y$. *For any* $x \in X$, *we define*

$$\bar{x} = \{y \in Y \mid x \otimes y = 0\}.$$

Now we can prove:

Theorem 3.1. *Let* X, Y *be abelian groups such that* $T(X)$, $T(Y)$ *are* p *primary and* $X/T(X)$, $Y/T(Y)$ *are* p *divisible. Then* $X \otimes Y$ *splits if, and only if,* $Y/\bar{x}$ *splits for every* $x \in X$.

Proof. Suppose, first, that $Y/\bar{x}$ splits for every $x \in X$. Let $\{x_\lambda\}$ be a maximal linearly independent set in X. Then, in particular, each $Y/\bar{x}_\lambda$ splits. Since $Y/\bar{x}_\lambda$ is isomorphic to $\langle x_\lambda, Y\rangle \subseteq X \otimes Y$, we can write

$$\langle x_\lambda, Y\rangle = T(\langle x_\lambda, Y\rangle) \oplus F_\lambda .$$

$Y/T(Y)$ is p divisible; hence, by Lemma 3.3 so is each $(Y/\bar{x}_\lambda)/T(Y/\bar{x}_\lambda) \cong F_\lambda$.
Let $F = \Sigma_\lambda F_\lambda \subseteq X \otimes Y$.

Since each F_λ is torsion free and p divisible, so is F. We now show that $(X \otimes Y)/F$ is a torsion group. For let $x \in X, y \in Y$ be given. By the maximality property of $\{x_\lambda\}$, we know that there exists a nonzero integer c, and integers c_λ, such that $cx = \Sigma_\lambda c_\lambda x_\lambda$. Hence,

$$c(x \otimes y) = \Sigma_\lambda x_\lambda \otimes (c_\lambda y),$$

the sum including, of course, only finitely many nonzero terms. Each $x_\lambda \otimes (c_\lambda y) \in \langle x_\lambda, Y\rangle$ differs by an element of finite order from an element of F_λ; hence, $c(x \otimes y)$ differs by an element of finite order from an element of F, and so, for a suitable multiple c' of c, $c'(x \otimes y) \in F$, as was claimed.

Thus we have: $T(X \otimes Y)$ is p primary; F is p divisible, hence, p pure in $X \otimes Y$; F is torsion free; and $(X \otimes Y)/F$ is a torsion group. From Lemma 3.1 we conclude that $X \otimes Y$ splits.

To prove the converse, suppose that $X \otimes Y$ splits into $T(X \otimes Y) \oplus F$. We must show that $Y/\bar{x}$ splits for every $x \in X$.

If x is of finite order, then $Y/\bar{x}$ is a torsion group and splits trivially. So let x be of infinite order. Now let $\{y_\lambda\}$ be a maximal torsion free linearly independent set in Y. By multiplying, each y_λ by some nonzero integer n_λ if necessary, we have that each $x \otimes n_\lambda y_\lambda$ is in F. Then Lemma 3.7 tells us that for some multiple m_λ of $n_\lambda y_\lambda$ we have that $x \otimes m_\lambda n_\lambda y_\lambda = x \otimes y_\lambda'$ $(y_\lambda' = m_\lambda n_\lambda y_\lambda)$ has a p sequence in $X \otimes Y$ of the form $x \otimes y'_{\lambda_i}$. Notice that $\{y_\lambda'\}$ is still a maximal independent set in Y. So replacing the y_λ' by y_λ we proceed. Let B be the subgroup of $\langle x, Y\rangle$ generated by the set $\{x \otimes y_{\lambda_i}\}$ where the indices range over all λ and all i. Then B is torsion free p divisible, and, hence, p pure in $\langle x, Y\rangle$. Also $\langle x, Y\rangle/B$ is a torsion group. From Lemma 3.1 we conclude that $\langle x. Y\rangle \cong Y/\bar{x}$ splits as stated.

Remark. Note in the first part of the proof, that $Y/\bar{x}_\lambda$ splits for every x_λ in some maximal torsion free linearly independent set is enough to guarantee that $X \otimes Y$ splits.

An easy consequence of the preceding theorem is

THEOREM 3.2. *Let X and Y be groups such that $T(X)$ and $T(Y)$ are p primary and $X/T(X)$ and $Y/T(Y)$ are p divisible. Let $\{x_\lambda\}$ be a maximal torsion free linearly independent subset of X such that each x_λ is of infinite p height. Then $X \otimes Y$ splits.*

Proof. It suffices to show by the above remark that $Y/\bar{x}_\lambda$ splits for each λ. But $\bar{x}_\lambda = T(Y)$. To see this recall that $\bar{x}_\lambda \subseteq T(Y)$, since x_λ is of infinite order. Now suppose $y \in T(Y)$ of order p^n. Write $x_\lambda = p^n x$ and we have $x_\lambda \otimes y = p^n x \otimes y = x \otimes p^n y = 0$ so that $T(Y) \subseteq \bar{x}_\lambda$. So $Y/\bar{x}_\lambda = Y/T(Y)$ which splits trivially and that's that.

Remark. In the above theorem notice that in applications to rank 1 groups, the existence of a single element x of infinite height and infinite order in X is sufficient.

SECTION 4

In this section we shall study the splitting of tensor products $A \otimes A \otimes \cdots \otimes A$ of certain groups. In view of the reduction Theorem 2.3, we shall be mainly concerned with groups of torsion-free rank one.

To simplify the notation, we shall use A^n to designate $A \otimes \cdots \otimes A$ with n factors.

DEFINITION. A is said to have splitting length σ, denoted $l(A) = \sigma$, if σ is the smallest positive integer such that A^σ splits. (In particular, if A splits its splitting length is 1.) If A^σ splits for no σ, we call the splitting length infinite.

LEMMA 4.1. *Let X, Y be groups with $T(X)$, $T(Y)$ p primary and $X/T(X)$, $Y/T(Y)$ p divisible. Then $l(X \otimes Y) \leqslant \min\{l(X), l(Y)\}$.*

Proof. Suppose $l(X) = n$, so that $X^n = T \oplus F$ where T is torsion and F is torsion-free. Then T must be p primary and F p divisible. Now

$$(X \otimes Y)^n \cong X^n \otimes Y^n = (T \oplus F) \otimes Y^n \cong (T \otimes Y^n) \oplus (F \otimes Y^n).$$

Since the first summand is torsion and the second torsion-free, $l(X \otimes Y) \leqslant n$. Interchanging the roles of X and Y, if necessary, the lemma stands proved.

COROLLARY 4.1. *If $T(A)$ is p primary, then A^n splits if, and only if, $n \geqslant l(A)$.*

Proof. If $T(A)$ is p primary and $A/T(A)$ p divisible, it follows from the above lemma that if A^σ splits then $A^{\sigma+1}$ splits. On the other hand, if $T(A)$ is p primary but $A/T(A)$ is not p divisible, then (by Theorem 2.2) either A splits or no A^σ splits: that is, either $l(A) = 1$, and every A^σ splits, or $l(A) = \infty$, and none does.

LEMMA 4.2. *Suppose there is a homomorphism f of X onto Y with kernel a torsion group. Then $l(Y) \leqslant l(X)$.*

Proof. Suppose $l(X) = n$, so X^n splits. The homomorphism f induces $f^n : X^n \to Y^n$, whose ker is again torsion. Thus, by Lemma 2.3, Y^n splits.

THEOREM 4.1. *Let X, Y be groups with $T(X)$, $T(Y)$ p primary and $X/T(X)$, $Y/T(Y)$ p divisible and of rank 1. Let $x_0 \in X$ be of infinite order, and $\bar{x}_0 = \{x \in X \mid x \otimes x_0 = 0 \text{ in } X \otimes X\}$. Then $l(X/\bar{x}_0) \leqslant l(X) \leqslant l(X/\bar{x}_0) + 1$.*

Proof. Since $\bar{x}_0 \subseteq T(X)$, the preceding lemma yields the first inequality. For the second inequality, let $l(X/\bar{x}_0) = n - 1$, and consider the map $f : X^{n-1} \to X^n$ determined by

$$f(x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}) = x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes x_0 .$$

The ker of f contains every $x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}$ in which some x_i belongs to $\bar{x}_0$. Hence, the map f can be factored

$$X^{n-1} \xrightarrow{g} \left(\frac{X}{\bar{x}_0}\right)^{n-1} \xrightarrow{h} X^n,$$

where g is induced by the natural quotient map on each factor.

By assumption, $(X/\bar{x}_0)^{n-1}$ splits. The torsion-free summand is of rank one and is p divisible, since $X/T(X)$ is such and $\bar{x}_0$ is torsion. Also $X^n/T(X^n)$ is of rank one and is p divisible. Furthermore, the homomorphism h maps the element a of $(X/\bar{x}_0)^{n-1}$ represented by x_0^{n-1} into $x_0{}^n$ in X^n. From these facts, the following lemma allows us to conclude that X^n splits, yielding the second inequality.

LEMMA 4.3. *Let A, B be groups with $A/T(A)$, $B/T(B)$ of rank one and p divisible. Assume there exists a homomorphism $h : A \to B$ whose image contains an element of infinite order. Then if A splits, so does B.*

Proof. By hypothesis there exists $a \in A$ such that $h(a)$ is of infinite order, and so is itself of infinite order. Since A splits: $A = T(A) \oplus F$, some non-zero multiple na of a belongs to F. Since $F \simeq A/T(A)$, it is torsion-free and p divisible, so na possesses a p sequence in F. The image under h of this p sequence is a p sequence in B. Thus, by Proposition 3.1, B splits.

In the remainder of this section, we shall consider only groups A with $T(A)$ p primary and $A/T(A)$ p divisible.

We begin by exhibiting groups A_2 and A_3 of splitting lengths 2, 3, respectively. They are, in fact, members of a family of groups A_σ, each of splitting length exactly σ, which will be described later in this section; but the computations needed for determining $l(A_2)$ and $l(A_3)$ are considerably simpler than for the general A_σ.

Let A_2 be the abelian group with countably many generators $a_0, a_1, a_2, \ldots$, with the relations $p^i a_i = a_0$ for $i = 1, 2, 3, \ldots$. (This group appears in Fuchs, p. 200, exercise 4.) It can be described more concretely as follows: Let R be the group of rational numbers, and for each i, $i = 1, 2, 3, \ldots$, let $[c_i]$ be a cyclic group of order p^i generated by c_i. Then A_2 is isomorphic to the subgroup of the group $(\oplus_{i=1}^{\infty} [c_i]) \oplus R$ generated by the elements of the form $c_i + p^{-i}$, $i = 1, 2, 3, \ldots$.)

Then A_2 is not p divisible; but $A_2/T(A_2)$ is, being in fact isomorphic to the group R_p of rationals whose demonimators are powers of p. A_2 does not split, since it has no subgroup isomorphic to $A_2/T(A_2)$. But by Theorem 3.2 $A_2 \otimes A_2$ does split, since A_2 contains the element a_0 of infinite order and infinite p-height.

Let A_3, similarly, be the abelian group with countably many generators $a_0, a_1, a_2, \ldots$, with the relations $p^{2i} a_i = p^i a_0$ for $i = 1, 2, 3, \ldots$. We shall use Theorem 3.1 to prove that $A_3 \otimes A_3$ does not split, by calculating $\bar{a}_0$ explicitly and then showing that, not surprisingly, $A_3/\bar{a}_0$ is isomorphic to the group A_2 which is known not to split. Since we will later prove $l(A_\sigma) = \sigma$ in general—although by a slightly different method—the present computation, intended to aid the reader's insight, is not given in complete detail.

Let $c_i = pa_{i+1} - a_i$, for $i = 0, 1, 2, \ldots$. Then c_i is of order p^{2i+1}, that is, exponent $(2i + 1)$; and it is easy to see that $T(A_3)$ is the direct sum of the cyclic subgroups generated by the c_i's.

Let $h(x)$ denote the p height of x. Then it is obvious from the defining relations that $h(p^i a_0) \geqslant 2i$; and it can be shown without difficulty that these are actually equalities.

Our first aim is to determine $\bar{a}_0$, that is, $\{x \in A \mid x \otimes a_0 = 0 \text{ in } A_3 \otimes A_3\}$. Any $x \in \bar{a}_0$ must be of finite order, so expressible as a finite sum $\Sigma n_i c_i$. Now $(\Sigma n_i c_i) \otimes a_0 = 0$ if, and only if, for each i, $c_i \otimes n_i a_0 = 0$; so if the p height of $n_i a_0$ is greater than the exponent of c_i:

$$2e_i \geqslant 2i + 1,$$

where p^{e_i} is the highest power of p dividing n_i. The least n_i for which this occurs is $n_i = p^i$. There fore, $\bar{a}_0$ is generated by

$$\{p^i c_i\} = \{p^{i+1} a_{i+1} - p^i a_i\};$$

or, more conveniently, by the set $\{p^i a_i - a_0\}$.

The group $A_3/\bar{a}_0$ can be determined by adjoining the relations $p^i a_i = a_0$ to those given for A_3. But in fact the new relations imply the old ones; so $A_3/\bar{a}_0$ is defined by the relations $p^i a_i = a_0$, the same as those defining A_2. This proves the isomorphism, and so the nonsplitting of $A_3 \otimes A_3$.

We are now ready to exhibit groups A_σ of arbitrary finite splitting length σ. Each of these will be defined by a system of generators $a_0, a_1, a_2, \ldots$, subject to relations of the form

$$p^{j_i} a_i = p^{k_i} a_0 .$$

DEFINITION. For $\sigma = 2, 3, 4, \ldots$ the group A_σ is the free Abelian group generated by $a_0, a_1, a_2, \ldots$, modulo the subgroup generated by the relators

$$p^{(\sigma-1)i} a_i - p^{(\sigma-2)i} a_0 .$$

Thus, in A_σ, $p^{m+i} a_i = p^m a_0$ if $m \geqq (\sigma - 2)\, i$. In the simplest case, A_2, $p^i a_i = a_0$ for every i.

We also define a group A_1 by the relations

$$p a_{i+1} = a_i .$$

This is isomorphic to the group R_p of rationals whose denominator is a power of p.

In order to achieve our goal of showing that A_σ has splitting length σ, we first study the group $A_\sigma^{\sigma-1}$.

LEMMA 4.4. *Let $\sigma \geqslant 2$, and $\{i, j, \ldots, k\}$ be a set of $\sigma - 1$ nonnegative integers. Then, in $A_\sigma^{\sigma-1}$, we have:*

$$p^{i+j+\cdots+k+(\sigma-2)}(a_i \otimes a_j \otimes \cdots \otimes a_k) = p^{(\sigma-2)}(a_0 \otimes a_0 \otimes \cdots \otimes a_0).$$

Proof. We shall show that if $K \geqslant \max\{i, j, \ldots, k\}$, then

$$p^{i+j+\cdots+k+(\sigma-2)}(a_i \otimes a_j \otimes \cdots \otimes a_k) = p^{(\sigma-1)K+(\sigma-2)}(a_K \otimes a_K \otimes \cdots \otimes a_K).$$

(Since we can, in particular, take $i = j = \cdots = k = 0$, this will prove the lemma.)

For consider $\min(i, j,..., k)$. Suppose this to be, say, i. Then

$$p^{i+j+\cdots+k+\sigma-2}(a_i \otimes \cdots) = p^{0+j+\cdots+k+\sigma-2}(a_0 \otimes \cdots)$$
$$= p^{(i+1)+j+\cdots+k+\sigma-2}(a_{i+1} \otimes \cdots),$$

both equalities holding because

$$(0 + j + \cdots + k + \sigma - 2) \geqslant (\sigma - 2)(i + 1) \geqslant (\sigma - 2)\, i.$$

Thus either the minimum is increased, or the number of places at which the minimum occurs is decreased. Hence, in a finite number of iterations of this process, we reach

$$\cdots = p^{(\sigma-1)K+\sigma-2}(a_K \otimes a_K \otimes \cdots \otimes a_K).$$

From this Lemma we obtain immediately the

COROLLARY 4.2. *A_σ^σ splits.*

Proof. Let $X = A_\sigma^{\sigma-1}$, and $Y = A_\sigma$. It is clear that $T(X)$, $T(Y)$ are p primary, and that $X/T(X)$, $Y/T(Y)$ are each isomorphic to R_p and so of rank 1 and p divisible. The lemma just proved shows that the element

$$p^{(\sigma-2)}(a_0 \otimes a_0 \otimes \cdots \otimes a_0) \in X$$

is of infinite p height. Thus Theorem 3.2 applies, and shows that $X \otimes Y$ splits, as was to be demonstrated.

On the other hand, we can show the

COROLLARY 4.3. *$A_\sigma^{\sigma-1}$ does not split.*

Proof. Let B be the subgroup of $A_\sigma^{\sigma-1}$ generated by the elements of the form

$$(a_i \otimes a_j \otimes \cdots \otimes a_k) - p^{(m-i)+(m-j)+\cdots+(m-k)}(a_m \otimes a_m \otimes \cdots \otimes a_m)$$

where $m = \max\{i, j,..., k\}$.

By the lemma, each of these generators is of finite order, hence B is a subgroup of $T(A_\sigma^{\sigma-1})$. Let $C = A_\sigma^{\sigma-1}/B$. It is easy to see that C is isomorphic to the group defined by a system of generators $\{c_m\}$, subject to the relations

$$p^{(\sigma-1)m+(\sigma-2)}c_m = p^{(\sigma-2)}c_0\,,$$

the isomorphism being given by

$$c_m \leftrightarrow a_m \otimes a_m \otimes \cdots \otimes a_m \bmod B.$$

C does not split: for every element in $C/T(C)$ has a p sequence, while no nonzero element of C does.

Now recall the exact sequence defining C:

$$0 \to B \to A_\sigma^{\sigma-1} \to C \to 0,$$

where $B \subset T(A_\sigma^{\sigma-1})$. If $A_\sigma^{\sigma-1}$ were to split, so would C. Hence $A_\sigma^{\sigma-1}$ does not split.

Finally, we can prove

THEOREM 4.2. *For $\sigma = 1, 2, 3,\ldots$, the group A_σ has splitting length σ.*

Proof. The case $\sigma = 1$ is trivial, since A_1 is torsion-free and so splits. For $\sigma \geqslant 2$, the two corollaries just demonstrated prove the theorem.

REFERENCES

1. R. BAER, The subgroup of the elements of finite order of an abelian group, *Ann. Math.* **37** (1936), 766–781.
2. R. BAER, Die Torsionsuntergruppe einer abelschen Gruppe, *Math. Ann.* **135** (1958), 219–234.
3. ČERNIKOV, Russian, see Ref. [2] of the bibliography of Ref. [6].
4. J. ERDÖS, On the splitting problem of mixed abelian groups, *Publ. Math. Debrecen*, **5** (1958), 364–377.
5. C. B. FOMIN, Über periodische Untergruppen der unendlichen abelschen Gruppen, *Mat. Sbornik*, **2** (1937), 1007–1009.
6. L. FUCHS, "Abelian Groups," pp. 185–204, Pergamon Press, New York, 1960.
7. L. FUCHS, Notes on Abelian Groups II, *Acta. Math. Acad. Sci. Hungar.*, **11** (1960), 117–125.
8. D. K. HARRISON, Infinite abelian groups and homological methods, *Ann. Math.* **69**, No. 2 (1959), pp. 366–391.
9. Kulikov, Russian, see Ref. [4] of the bibliography of Ref. [6].
10. S. MACLANE, "Homology," Academic Press, New York, 1963.
11. A. MISINA, Russian, see Ref. [2] of the bibliography of Ref. [4].
12. J. A. OPPELT, Mixed abelian groups, *Canad. J. Math.* **19**, No. 6 (1967), 1259–1262.
13. E. SASIADA, An application of Kulikov's basic subgroups in the theory of abelian mixed groups, *Bull. Acad. Polon. Sci. Cl. III*, **4** (1956), 411–413.