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# Quantum complexity of parametric integration

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## Abstract

We study parametric integration of functions from the class  $C^r([0, 1]^{d_1+d_2})$  to  $C([0, 1]^{d_1})$  in the quantum model of computation. We analyze the convergence rate of parametric integration in this model and show that it is always faster than the optimal deterministic rate and in some cases faster than the rate of optimal randomized classical algorithms.

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*Keywords:* Quantum computing; Parametric integration; Information-based complexity

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## 1. Introduction

Summation and integration are the most famous numerical problems that achieved a speed-up in the quantum model of computation, compared to the optimal convergence rates of deterministic and randomized algorithms in the classical case.

In this paper, we study the problem of parametric integration, where the integral depends on a parameter. Therefore, the solution is now a function, so the problem carries features of both integration and approximation.

We will consider the problem from the point of view of complexity theory and provide an analysis for the class of  $r$ -times continuously differentiable functions. For this class, we determine the order of the minimal error (up to a logarithmic gap) by deriving matching upper and lower complexity bounds.

In Section 2, we present the required notions from quantum information-based complexity (IBC) theory, recall related previous results and formulate the main result. Section 3 is devoted to the proof of the upper bound. In Section 4, we prove the lower bound, and in the final Section 5 we give some comments on the results.

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## 2. Preliminaries

In this section, we formulate the problem which is investigated. Then we give the basic definitions of quantum IBC theory, state some useful technical results and finally formulate the main result.

### 2.1. Problem formulation

Let  $D_1 = [0, 1]^{d_1}$  and  $D_2 = [0, 1]^{d_2}$  with  $d_1, d_2 \geq 0$ . On the domain  $D = D_1 \times D_2$  we define the function class  $C^r(D)$  for an integer  $r \geq 1$  as the set of all functions  $f(s, t)$ , for which all partial derivatives up to order  $r$  exist and are continuous. Let  $\alpha$  be a multiindex; then the norm  $\|\cdot\|_r$  on  $C^r(D)$  is defined by

$$\|f\|_r := \max_{|\alpha| \leq r} \|f^{(\alpha)}\|_{C(D)}.$$

Let  $C(D_1)$  be the space of continuous functions on  $D_1$  with the supremum norm. We consider the solution operator

$$S: C^r(D) \rightarrow C(D_1),$$

$$f \mapsto (Sf)(s) = \int_{D_2} f(s, t) dt. \quad (1)$$

This means, we study parametric integration: Integrate the family of functions  $f(s, t)$  parametrized by  $s \in D_1$  over  $t \in D_2$ . The limiting cases where either  $d_1 = 0$  (pure integration) or  $d_2 = 0$  (pure approximation) are formally included because they represent classical problems of numerical mathematics. The aim of this paper is to study the intermediate cases where  $d_1 \neq 0$  and  $d_2 \neq 0$ .

### 2.2. Quantum setting

We use the terminology developed by Heinrich in [4], which is a translation of IBC methods to the quantum model of computation. In order to be as self-contained as possible, we summarize the quantum IBC notions needed in this paper.

First, we briefly recall the standard notation of quantum computing. Let  $H_1$  be the two-dimensional complex Hilbert space  $\mathbb{C}^2$  and

$$H_m = H_1 \otimes \cdots \otimes H_1$$

be the Hilbertian tensor product of  $m$  copies of  $H_1$ . We use the following notation,

$$\mathbb{Z}[0, N) := \{0, \dots, N-1\}$$

for  $N \in \mathbb{N}$ . Let  $\mathcal{C}_m = \{|i\rangle : i \in \mathbb{Z}[0, 2^m)\}$  be the set of unit basis vectors of  $H_m$ , also called classical states or basis states, and let  $\mathcal{U}(H_m)$  denote the set of unitary operators on  $H_m$ .

Let  $\mathcal{F}(D, K)$  be the set of mappings  $f : D \rightarrow K$ . Now we introduce the notion of a quantum query. For  $F \subset \mathcal{F}$  a quantum query is given by a tuple

$$Q = (m, m', m'', Z, \tau, \beta),$$

where  $m, m', m'' \in \mathbb{N}$ ,  $m' + m'' \leq m$ ,  $Z \subseteq \mathbb{Z}[0, 2^{m'}]$  is a nonempty subset, and

$$\tau : Z \rightarrow D,$$

$$\beta : K \rightarrow \mathbb{Z}[0, 2^{m''}]$$

are arbitrary mappings. The mapping  $\tau$  is the coding from basis states of one register of the quantum computer to the domain of  $f \in F$ , whereas  $\beta$  is the coding of the function values from the range of  $f$  to basis states of a second register of the quantum computer. Such a tuple  $Q$  defines a query mapping

$$Q : F \rightarrow \mathcal{U}(H_m),$$

$$f \rightarrow Q_f$$

by

$$Q_f |i\rangle |x\rangle |y\rangle = \begin{cases} |i\rangle |x \oplus \beta(f(\tau(i)))\rangle |y\rangle & \text{if } i \in Z, \\ |i\rangle |x\rangle |y\rangle & \text{otherwise,} \end{cases} \quad (2)$$

where  $|i\rangle \in \mathcal{C}_{m'}$ ,  $|x\rangle \in \mathcal{C}_{m''}$ ,  $|y\rangle \in \mathcal{C}_{m-m'-m''}$  (if  $m = m' + m''$ , we drop the last component) and  $\oplus$  means addition modulo the respective power of 2, here modulo  $2^{m''}$ . The total number of qubits needed for  $Q$  is  $m(Q) = m$ .

Suppose we are given a mapping  $S : F \rightarrow G$ , where  $G$  is a normed space (in this context  $S$  is a general mapping). We want to approximate  $S(f)$  for  $f \in F$  with the help of a quantum computer. To do so, we formally define the notion of a quantum algorithm. A quantum algorithm on  $F$  with no measurement is a tuple

$$A = (Q, (U_j)_{j=0}^n),$$

where  $Q$  is a quantum query on  $F$ ,  $n \in \mathbb{N}_0$  and  $U_j \in \mathcal{U}(H_m)$  ( $j = 0, \dots, n$ ), with  $m = m(Q)$ . Given such an  $A$  and  $f \in F$  we define  $A_f \in \mathcal{U}(H_m)$  by

$$A_f = U_n Q_f U_{n-1} \dots U_1 Q_f U_0.$$

By  $n_q(A) := n$  we denote the number of queries and by  $m(A) = m = m(Q)$  the number of qubits used by  $A$ . We also introduce the following notation. Let  $A_f(x, y)$  for  $x, y \in \mathbb{Z}[0, 2^m]$  be given by

$$A_f |y\rangle = \sum_{x \in \mathbb{Z}[0, 2^m]} A_f(x, y) |x\rangle.$$

Hence,  $(A_f(x, y))_{x,y}$  is the matrix of the transformation  $A_f$  in the canonical basis  $\mathcal{C}_m$ .

A quantum algorithm on  $F$  with output in  $G$  with  $k$  measurements is a tuple

$$A = ((A_l)_{l=0}^{k-1}, (b_l)_{l=0}^{k-1}, \varphi),$$

where  $k \in \mathbb{N}$ , and  $A_l$  ( $l = 0, \dots, k-1$ ) are quantum algorithms on  $F$  without measurement. We set  $m_l = m(A_l)$ . Then  $b_0 \in \mathbb{Z}[0, 2^{m_0}]$  and for  $1 \leq l \leq k-1$ ,  $b_l$  is a

function

$$b_l : \prod_{i=0}^{l-1} \mathbb{Z}[0, 2^{m_i}] \rightarrow \mathbb{Z}[0, 2^{m_l}],$$

and  $\varphi$  is a function with values in  $G$ ,

$$\varphi : \prod_{l=0}^{k-1} \mathbb{Z}[0, 2^{m_l}] \rightarrow G.$$

The function  $\varphi$  combines the outputs of the algorithms  $A_l$  to give a final result. The functions  $b_l$  determine the starting state of the next algorithm  $A_l$  depending on the results of the previous algorithms.

We also say that  $A$  is a quantum algorithm with measurement(s), or just a quantum algorithm.

Let  $\mathcal{P}_0(G)$  be the set of all probability measures on  $G$  whose support is a finite set. The output of  $A$  on input  $f \in F$  will be an element  $A(f) \in \mathcal{P}_0(G)$  (we use the same symbol  $A$  for the mapping  $A : F \rightarrow \mathcal{P}_0(G)$ ). We define  $A(f)$  via a sequence of random variables  $(\xi_{l,f})_{l=0}^{k-1}$  (we assume that all random variables are defined over a fixed—suitably large—probability space  $(\Omega, \Sigma, \mathbb{P})$ ). Let now  $f \in F$  be fixed and let  $\xi_{l,f}$  be such that

$$\mathbb{P}(\xi_{0,f} = x) = |A_{0,f}(x, b_0)|^2$$

and, for  $1 \leq l \leq k-1$ ,

$$\mathbb{P}(\xi_{l,f} = x | \xi_{0,f} = x_0, \dots, \xi_{l-1,f} = x_{l-1}) = |A_{l,f}(x, b_l(x_0, \dots, x_{l-1}))|^2.$$

This defines the distribution of  $(\xi_{l,f})_{l=0}^{k-1}$  uniquely. Let us define for  $x_0 \in \mathbb{Z}[0, 2^{m_0}], \dots, x_{k-1} \in \mathbb{Z}[0, 2^{m_{k-1}}]$

$$p_{A,f}(x_0, \dots, x_{k-1}) = |A_{0,f}(x_0, b_0)|^2 |A_{1,f}(x_1, b_1(x_0))|^2 \cdots |A_{k-1,f}(x_{k-1}, b_{k-1}(x_0, \dots, x_{k-2}))|^2.$$

It follows that

$$\mathbb{P}(\xi_{0,f} = x_0, \dots, \xi_{k-1,f} = x_{k-1}) = p_{A,f}(x_0, \dots, x_{k-1}).$$

Finally, we define the output  $A$  on input  $f$  as

$$A(f) = \text{dist}(\varphi(\xi_{0,f}, \dots, \xi_{k-1,f})),$$

the distribution of  $\varphi(\xi_{0,f}, \dots, \xi_{k-1,f})$ .

The number  $n_q(A) := \sum_{l=0}^{k-1} n_q(A_l)$  is called the number of queries used by  $A$ . This is the crucial quantity for our query complexity analysis.

Now we define the error of a quantum algorithm  $A$ : Let  $0 < \theta < 1$ , and let  $\zeta$  be a random variable with distribution  $A(f)$ . Then the (probabilistic) quantum error of  $A$  for  $S$  on input  $f$  with failure parameter  $\theta$  is defined by

$$e(S, A, f, \theta) := \inf\{\varepsilon \mid \mathbb{P}(\|S(f) - \zeta\|_G > \varepsilon) \leq \theta\}.$$

Then we put

$$e(S, A, F, \theta) := \sup_{f \in F} e(S, A, f, \theta)$$

and

$$e_n^q(S, F, \theta) := \inf_A \{e(S, A, f, \theta) \mid n_q(A) \leq n\}.$$

We will consider these quantities for the fixed error probability 1/4 and set

$$e(S, A, f) = e(S, A, f, 1/4), \quad e(S, A, F) = e(S, A, F, 1/4),$$

and we define the  $n$ th minimal query error of the problem class  $F$  and the mapping  $S$  by

$$e_n^q(S, F) := e_n^q(S, F, 1/4). \tag{3}$$

This means that we will analyze the error rate at given cost. There is a close connection between  $e_n^q$  and the  $\varepsilon$ -complexity of a problem, which is defined by

$$\text{comp}_\varepsilon^q(S, F) := \min\{m \mid e_m^q(S, F) \leq \varepsilon\}.$$

The two quantities satisfy the following relation: For all  $n \in \mathbb{N}_0$ ,  $\varepsilon > 0$  we have

$$e_n^q(S, F) \leq \varepsilon \Leftrightarrow \text{comp}_{\varepsilon_1}^q(S, F) := \min\{m \mid e_m^q(S, F) \leq \varepsilon_1\} \leq n \quad \forall \varepsilon_1 > \varepsilon.$$

### 2.3. Tools from quantum complexity

For our analysis of parametric integration in the quantum model we will need some statements from quantum IBC, which are now summarized:

**Lemma 2.1.** *Let  $F \subset \mathcal{F}(D, \mathbb{R})$ ,  $l \in \mathbb{N}_0$  and let  $S_k : F \rightarrow G$  ( $k = 0, \dots, l$ ) be mappings, where  $G$  is a normed space. Define  $S : F \rightarrow G$  by  $S(f) = \sum_{k=0}^l S_k(f)$ . Let  $\theta_0, \dots, \theta_l \geq 0$ ,  $n_0, \dots, n_l \in \mathbb{N}_0$  and put  $n = \sum_{k=0}^l n_k$ . Then*

$$e_n^q\left(S, F, \sum_{k=0}^l \theta_k\right) \leq \sum_{k=0}^l e_{n_k}^q(S_k, F, \theta_k). \tag{4}$$

This is a generalization of Lemma 2 from Heinrich [6], which can be proved by the same technique by just replacing the absolute value with the norm on  $G$ .

**Lemma 2.2.** *Let  $\emptyset \neq F \subseteq \mathcal{F}(D, K)$  and  $\emptyset \neq \tilde{F} \subseteq \mathcal{F}(\tilde{D}, \tilde{K})$ . Let  $\Gamma : F \rightarrow \tilde{F}$  be of the following form: there exist  $\kappa, m^* \in \mathbb{N}$  and mappings*

$$\eta_j : \tilde{D} \rightarrow D \quad (j = 0, \dots, \kappa - 1),$$

$$\beta : K \rightarrow \mathbb{Z}[0, 2^{m^*}),$$

$$\rho : \tilde{D} \times \mathbb{Z}[0, 2^{m^*})^\kappa \rightarrow \tilde{K},$$

such that for  $f \in F$  and  $s \in \tilde{D}$

$$(\Gamma(f))(s) = \rho(s, \beta \circ f \circ \eta_0(s), \dots, \beta \circ f \circ \eta_{\kappa-1}(s)).$$

Given a quantum algorithm  $\tilde{A}$  from  $\tilde{F}$  to  $G$ , there is a quantum algorithm  $A$  from  $F$  to  $G$  with

$$n_q(A) = 2\kappa n_q(\tilde{A})$$

and for all  $f \in F$

$$A(f) = \tilde{A}(\Gamma(f)).$$

Consequently, if  $\tilde{S} : \tilde{F} \rightarrow G$  is any mapping and  $S = \tilde{S} \circ \Gamma$ , then for each  $n \in \mathbb{N}_0$

$$e_{2\kappa n}^q(S, F) \leq e_n^q(\tilde{S}, \tilde{F}). \tag{5}$$

The proof of Lemma 2.2 can be found in Heinrich [6].

We finally state some calculation rules for the query error:

**Lemma 2.3.** *Let  $S, T : F \rightarrow G$  be mappings,  $n \in \mathbb{N}_0$  and  $e_n^q(S, F)$  be finite. Then it holds that*

$$(i) \quad e_q^n(T, F) \leq e_q^n(S, F) + \sup_{f \in F} \|T(f) - S(f)\|. \tag{6}$$

(ii) *For  $\lambda \in \mathbb{R}$  it holds that*

$$e_q^n(\lambda S, F) = |\lambda| e_q^n(S, F). \tag{7}$$

(iii) *If  $K = \mathbb{R}$  and  $S$  is a linear operator from  $\mathcal{F}(D, K)$  to  $G$ , then for all  $\lambda \in \mathbb{R}$  we have*

$$e_q^n(S, \lambda F) = |\lambda| e_q^n(S, F). \tag{8}$$

The proof of this lemma can be found in Heinrich [4].

We now cite a method how to increase the success probability of a quantum algorithm. Let  $M \in \mathbb{N}$  and  $\psi_0 : \mathbb{R}^M \rightarrow \mathbb{R}$  be the median of  $M$  numbers. For a quantum algorithm  $A$ , we define  $\psi_0(A^M) := \psi_0(A, \dots, A)$  to be the median of the results of  $M$  repetitions of  $A$ .

**Lemma 2.4.** *Let  $T : F \rightarrow \mathbb{R}$  be a mapping and  $A$  a quantum algorithm. Then*

$$e(T, \psi_0(A^M), f, e^{-M/8}) \leq e(T, A, f, \frac{1}{4}). \tag{9}$$

A proof of this Lemma can be found in Heinrich [4].

### 2.4. Main results

First we recall the known results for the special cases where either  $d_1 = 0$  or  $d_2 = 0$ .

We use the asymptotic notation  $a_n \preceq b_n$  for sequences of nonnegative real numbers  $a_n$  and  $b_n$ , which means that there exists some constant  $c > 0$  and some  $n_0 \in \mathbb{N}$  such that  $a_n \leq cb_n$  for all  $n \geq n_0$ . If  $a_n \preceq b_n$  and  $b_n \preceq a_n$  then we write  $a_n \asymp b_n$ . We often use the same symbol  $c$  for possibly different constants. In particular, the needed constants may depend on  $d_1, d_2$  and  $r$ .

Let  $\mathcal{B}(G)$  denote the unit ball of a normed space  $G$ , i.e.

$$\mathcal{B}(G) := \{g \in G \mid \|g\|_G \leq 1\}.$$

Then we have the following two theorems that are important for our analysis. The first considers quantum integration and was proved by Novak [9].

**Theorem 2.5.** *Let  $S^{\text{int}} : C^r(D_2) \rightarrow \mathbb{R}$  be the integration operator. Then*

$$e_n^q(S^{\text{int}}, \mathcal{B}(C^r(D_2))) \asymp n^{-r/d_2-1}. \tag{10}$$

Let  $A_{\text{int}}(\cdot, n)$  be a sequence of quantum algorithms which is of optimal order, that is

$$e(S^{\text{int}}, A_{\text{int}}(\cdot, n), \mathcal{B}(C^r(D_2))) \leq cn^{-r/d_2-1}.$$

The second theorem is concerned with approximation.

**Theorem 2.6.** *Let  $S^{\text{appr}}$  denote the function approximation problem, that is the embedding operator from  $C^r(D_1)$  to  $C(D_1)$ . Then*

$$\begin{aligned} e_n^{\text{det}}(S^{\text{appr}}, \mathcal{B}(C^r(D_1))) &\asymp e_n^{\text{mc}}(S^{\text{appr}}, \mathcal{B}(C^r(D_1))) \\ &\asymp e_n^q(S^{\text{appr}}, \mathcal{B}(C^r(D_1))) \asymp n^{-r/d_1}. \end{aligned} \tag{11}$$

Here  $e_n^{\text{det}}$  and  $e_n^{\text{mc}}$  are the minimal deterministic and Monte Carlo error. A detailed definition can be found in [7]. The proof of the rate for the quantum case is due to Heinrich [5].

We state another important result, which is in fact the key to the integration result mentioned above. Let

$$L_\infty^N := \{g : \{0, \dots, N-1\} \rightarrow \mathbb{R}\},$$

with the norm  $\|g\|_{L_\infty^N} = \max_i |g(i)|$ . Then we get the optimal rate for quantum summation, where the upper bound is from Brassard et al. [1] and the lower bound from Nayak and Wu [8], with the extension that is used for our purpose coming from Heinrich [4]:

**Theorem 2.7.** *Let  $S_N : L_\infty^N \rightarrow \mathbb{R}$  be given by  $S_N g = \frac{1}{N} \sum_{i=0}^{N-1} g(i)$ . Then for  $n < N$  there are constants  $c_1, c_2$  not depending on  $N$  such that*

$$c_1 n^{-1} \leq e_n^q(S_N, \mathcal{B}(L_\infty^N)) \leq c_2 n^{-1}. \tag{12}$$

Note that the convergence rate does not depend on the number of summands, so we can choose  $N$  to be large enough to satisfy any needed precision for the

approximation of an integral by such a weighted sum. We denote by  $A_{\text{sum}}(\cdot, n, N)$  a sequence of quantum algorithms with this convergence rate, meaning that

$$e(S_N, A_{\text{sum}}(\cdot, n, N), \mathcal{B}(L_\infty^N)) \leq c_2 n^{-1}.$$

Finally, we consider the rates of parametric integration. From now on let  $F$  denote the unit ball of  $C^r(D_1 \times D_2)$ . In order to enable comparison, we also recall the results in the deterministic and Monte Carlo setting. A detailed analysis for the Monte Carlo setting can be found in Heinrich and Sindambiwe [7].

**Theorem 2.8.** *The minimal errors of the operator  $S$  from (1) behave in the following way:*

$$e_n^{\text{det}}(S, F) \asymp n^{-r/(d_1+d_2)}, \quad (13)$$

$$e_n^{\text{mc}}(S, F) \asymp \begin{cases} n^{-(r+d_2/2)/(d_1+d_2)} (\log n)^{1/2} & \text{if } r > d_1/2, \\ n^{-r/d_1} (\log n)^{r/d_1} & \text{if } r < d_1/2 \end{cases} \quad (14)$$

and

$$n^{-1/2} (\log n)^{1/2} \asymp e_n^{\text{mc}}(S, F) \asymp n^{-1/2} (\log n)^{3/2} \quad \text{if } r = d_1/2.$$

The following theorem settles the quantum query complexity of parametric integration and answers the question when and how much better quantum algorithms are (as compared to deterministic and Monte Carlo methods). The comparison is discussed in detail in Section 5.

**Theorem 2.9.** *For  $d_1 \neq 0, d_2 \neq 0$  and  $r > 0$  the operator  $S$  from (1) satisfies*

$$n^{-\frac{r+d_2}{d_1+d_2}} \asymp e_n^q(S, F) \asymp n^{-\frac{r+d_2}{d_1+d_2}} (\log n)^{\frac{r+d_2}{d_1+d_2}} \quad \text{if } r > d_1, \quad (15)$$

$$n^{-r/d_1} \asymp e_n^q(S, F) \asymp n^{-r/d_1} (\log n)^{r/d_1} \quad \text{if } r < d_1 \quad (16)$$

and

$$n^{-1} \asymp e_n^q(S, F) \asymp n^{-1} (\log n)^3 \quad \text{if } r = d_1. \quad (17)$$

Note that in asymptotic statements we leave the logarithm unspecified, whereas in cases in which the basis is essential we write, e.g.,  $\log_2 n$  or  $\ln n$  to indicate base 2 or the natural logarithm. The proof of the above theorem consists of two parts: First we prove an upper bound for the query error  $e_n^q(S, F)$ . Second, we prove a lower bound for  $e_n^q(S, F)$  which has the same order as the upper bound, up to a logarithmic gap.

### 3. Upper bound

Now we assume  $d_1 \neq 0, d_2 \neq 0$  and  $r > 0$ . Let  $k \in \mathbb{N}_0$  be fixed, and let  $\Pi_k^{(1)}$  denote the partition of  $D_1$  into cubes of sidelength  $2^{-k}$  with disjoint interior. Let

$$A_k^{(1)} := \{r^{-1} 2^{-k} (i_1, \dots, i_{d_1}) : 0 \leq i_1, \dots, i_{d_1} \leq r 2^k\}$$



be the equidistant mesh of sidelength  $r^{-1}2^{-k}$  on  $D_1$ . Define

$$n_{1,k} := |A_k^{(1)}| = (r2^k + 1)^{d_1}. \tag{18}$$

Let

$$P_k^{(1)} : C^r(D_1) \rightarrow C(D_1)$$

be the  $d_1$ -dimensional composite Lagrange interpolation of degree  $r$  on  $A_k^{(1)}$ . This means, on each cube  $Q \in \Pi_k^{(1)}$  the function  $P_k^{(1)}f$  is the  $d_1$ -dimensional tensor product Lagrange interpolation over the nodes  $Q \cap A_k^{(1)}$ . Note that the resulting function is an element of  $C(D_1)$ .

For  $f$  fixed  $P_k^{(1)}f$  is uniquely defined by  $\{f(s) : s \in A_k^{(1)}\}$ . Therefore, the operator  $P_k^{(1)}$  will also be interpreted as defined on  $L_\infty(A_k^{(1)})$ , the space of real-valued functions on  $A_k^{(1)}$ , equipped with the maximum norm.

Finally, we also consider the operator  $P_k^{(1)}$  as acting in the space  $C(D_1 \times D_2)$ , meaning that we interpolate with respect to the first component only, leaving the other one fixed. In this case,  $P_k^{(1)}$  is defined by  $(P_k^{(1)}f)(s, t) := (P_k^{(1)}f(\cdot, t))(s)$ .

Let

$$\Pi_k^{(1)} = \{Q_{kj}\}_{j=0}^{2^{d_1 k} - 1},$$

that is  $D_1 = \bigcup_{j=0}^{2^{d_1 k} - 1} Q_{kj}$  and the  $Q_{kj}$  are cubes of sidelength  $2^{-k}$  with disjoint interior. Let  $s_{kj}$  be the point in  $Q_{kj}$  with the smallest Euclidean norm. We define the restriction operator  $R_{kj} : \mathcal{F}(D_1, \mathbb{R}) \rightarrow \mathcal{F}(D_1, \mathbb{R})$  by

$$(R_{kj}g)(s) = \begin{cases} g(2^k(s - s_{kj})) & \text{if } s \in Q_{kj}, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Let  $v = (r + 1)^{d_1}$  and let

$$i = i_1(r + 1)^{d_1 - 1} + i_2(r + 1)^{d_1 - 2} + \dots + i_{d_1 - 1}(r + 1) + i_{d_1}$$

for  $i = 0, \dots, v - 1$  be the representation of  $i$  in base  $r + 1$ . Let now  $\phi_i$  ( $i = 0, \dots, v - 1$ ) be the tensor product Lagrange base polynomials of degree  $r$  on  $D_1$  for the grid  $A_0^{(1)}$ , meaning that  $\phi_i(s) = 1$  at the point  $s = r^{-1}(i_1, \dots, i_{d_1})$  and  $\phi_i(s) = 0$  for all other points in  $A_0^{(1)}$ .

Since we want to take advantage of the fast convergence of quantum summation, we have to find a function whose integral can be approximated by quantum summation. This function needs a small supremum norm to give quantum summation its full impact. For each level,  $k \geq 1$  and a fixed gridpoint  $s \in A_k^{(1)} \setminus A_{k-1}^{(1)}$  we define the detail function  $f_{k,s} \in C^r(D_2)$  as the difference function between  $f$  and its approximation  $P_{k-1}^{(1)}f$ , both functions considered for this fixed  $s$ .

Let us consider the structure of the detail function in dependence of  $f$  and the tensor product Lagrange base polynomials. For fixed  $s \in A_k^{(1)} \setminus A_{k-1}^{(1)}$ , there is a cube

$Q_{k-1, j(s)} \in \Pi_{k-1}^{(1)}$  with  $s \in Q_{k-1, j(s)}$  (if there are several possibilities, choose the one with the smallest index  $j(s)$ ). The detail function has the form

$$f_{k,s}(t) = f(s, t) - (P_{k-1}^{(1)}f)(s, t) = f(s, t) - \sum_{i=0}^{v-1} (R_{k-1, j(s)}\phi_i)(s)f(s_i, t), \tag{20}$$

where

$$s_i = s_{k-1, j(s)} + r^{-1}2^{-(k-1)}(i_1, \dots, i_{d_1}). \tag{21}$$

The following lemma shows that the detail function has a bounded  $\|\cdot\|_r$ -norm.

**Lemma 3.1.** *There is a constant  $c > 0$  such that for any function  $f \in \mathcal{B}(C^r(D))$ , any integer  $k \geq 1$  and any  $s \in \Lambda_k^{(1)} \setminus \Lambda_{k-1}^{(1)}$  we have*

$$\|f_{k,s}\|_r \leq c. \tag{22}$$

**Proof.** The functions  $R_{kj}\phi_i$  ( $i = 0, \dots, v - 1$ ) are the tensor product Lagrange base polynomials on  $Q_{kj}$  for the grid  $Q_{kj} \cap \Lambda_k^{(1)}$  and

$$\|\phi_i\|_{C(D_1)} = \|R_{kj}\phi_i\|_{C(Q_{kj})} \tag{23}$$

for  $k \in \mathbb{N}$  and  $i = 0, \dots, v - 1$ . Since  $\sup_i \|\phi_i\|_{C(D_1)} \leq c'$  we get

$$\|f_{k,s}\|_r \leq \|f(s, t)\|_r + \sum_{i=0}^{v-1} |(R_{k-1, j(s)}\phi_i)(s)| \|f(s_i, t)\|_r \leq 1 + vc'.$$

We now choose  $c = 1 + vc'$  and the statement follows.  $\square$

We will need the following:

**Lemma 3.2.** *Let  $k \geq 0$  be an integer. Then the operator  $P_k^{(1)}$ , considered as acting from  $L_\infty^{n_1, k}$  to  $C(D_1)$  is bounded by a constant which does not depend on  $k$ .*

**Proof.** Let  $z \in L_\infty^{n_1, k}$  with  $\|z\|_{L_\infty^{n_1, k}} \leq 1$ . With the notation from above we infer that for  $j$  such that  $t \in Q_{kj}$  we have

$$(P_k^{(1)}z)(t) = \sum_{i=0}^{v-1} z(i) \cdot (R_{kj}\phi_i)(t),$$

and therefore

$$\|P_k^{(1)}z\|_{C(D_1)} \leq \sum_{i=0}^{v-1} |z(i)| \cdot \|R_{kj}\phi_i\|_{C(Q_{kj})} \leq v c' < c,$$

where  $c$  is the constant from the proof of Lemma 3.1, which is independent of  $k$ .  $\square$

Let us now state the parameters that are needed for the proof of the upper bound. We use a multilevel approach developed by Heinrich (see [3]) which was also used to obtain the optimal Monte Carlo rates for parametric integration in [7]. For  $x \in \mathbb{R}$  the notation  $\lceil x \rceil$  means the smallest integer greater than or equal to, and  $\lfloor x \rfloor$  the greatest one smaller or equal to  $x$ . For  $n \in \mathbb{N}$  we set

$$m := \left\lceil \frac{1}{d_1 + d_2} (\log_2 n) + 1 \right\rceil. \tag{24}$$

The starting level  $\tilde{m}$  is defined by

$$\tilde{m} = \begin{cases} m & \text{if } r \geq d_1, \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

and the final level  $l$  by

$$l := \begin{cases} \lceil (1 + d_2/r)m \rceil & \text{if } r \geq d_1, \\ \lceil (1 + d_2/d_1)m \rceil - p & \text{otherwise, where } p := \lfloor (\log_2 m)/d_1 \rfloor. \end{cases} \tag{26}$$

We use  $n_{1,k}$  points for the interpolation on level  $k$ , and we recall that  $n_{1,k} = (r^{2^k} + 1)^{d_1}$ . Let

$$M_k := \lceil 8(k + 3) \ln 2 + 8 \ln n_{1,k} \rceil, \tag{27}$$

then we define the query number for quantum summation as

$$n_{2,k} := \begin{cases} \lceil 2^{d_2 m - \frac{1}{2}(r+d_1)(k-m)} \rceil & \text{if } r \geq d_1, \\ \lceil M_k^{-1} 2^{(d_1+d_2)m - d_1 k - \frac{1}{2}(d_1-r)(l-k)} \rceil & \text{otherwise.} \end{cases} \tag{28}$$

The number of summands for quantum summation in level  $k$  ( $k = \tilde{m}, \dots, l$ ) is defined as

$$N_k := 2^{rkd_2} n_{2,k}^{d_2}. \tag{29}$$

Let us shortly describe the main idea of the proof: In the starting level, we approximate those integrals directly, which correspond to parameters  $s$  on the roughest grid, with the finally needed accuracy. On the finer levels we do the same for the detail functions. Then we interpolate the computed approximations and add them up to get our approximation to the solution function.

Now we prepare the discretization of the functions that will be used on the quantum computer. To do this, we need a mapping from our function class  $F$  to  $L_\infty^{N_k}$ . For  $k \geq \tilde{m}$ , we choose a number  $m^*$  of qubits so large that

$$2^{m^*/2-1} \geq 1$$

and

$$2^{-m^*/2} \leq 2^{-rk} n_{2,k}^{-1}. \tag{30}$$

Then we define

$$\beta: \mathbb{R} \rightarrow \{0, \dots, 2^{m^*} - 1\},$$

$$z \mapsto \begin{cases} 0 & \text{if } z < -2^{m^*/2-1}, \\ \lfloor 2^{m^*/2}(z + 2^{m^*/2-1}) \rfloor & \text{if } -2^{m^*/2-1} \leq z < 2^{m^*/2-1}, \\ 2^{m^*} - 1 & \text{if } z \geq 2^{m^*/2-1}. \end{cases}$$

Furthermore, we define

$$\begin{aligned} \gamma &: \{0, \dots, 2^{m^*} - 1\} \rightarrow \mathbb{R}, \\ y &\mapsto 2^{m^*/2}y - 2^{m^*/2-1}. \end{aligned}$$

On the starting level  $\tilde{m}$  we only have to approximate the integral of  $f$  for fixed  $s \in A_{\tilde{m}}^{(1)}$ , so in this case we just discretize the function  $f(s, \cdot)$ , which means that we have a function  $\eta : \{0, \dots, N_{\tilde{m}} - 1\} \rightarrow D_1 \times D_2$  which is defined by  $\eta(j) = (s, t_j)$ , where the points  $t_j \in D_2$  are node points needed for quantum summation, they will be specified below. Thus, for the starting level we get

$$(\Gamma_{\tilde{m},s}f)(j) = \gamma((\beta \circ f \circ \eta)(j)). \tag{31}$$

Let now  $k > \tilde{m}$  be fixed. As already indicated, we will approximate the integral of the detail function for fixed  $s \in A_k^{(1)} \setminus A_{k-1}^{(1)}$  by quantum summation. For a fixed summation number  $N_k$  from (29) we define

$$\eta_i : \{0, \dots, N_k - 1\} \rightarrow D_1 \times D_2 \quad (i = 0, \dots, v)$$

by

$$\eta_i(j) := (s_i, t_j),$$

where  $s_v = s$  and the points  $s_i$  ( $i = 0, \dots, v - 1$ ) are the points from (21). The points  $t_j \in D_2$  are again node points needed for quantum summation. Finally, we define

$$\rho : \{0, \dots, 2^{m^*} - 1\}^{v+1} \rightarrow \mathbb{R}$$

by

$$\rho(y_0, \dots, y_v) := \gamma(y_v) - \sum_{i=0}^{v-1} (R_{k-1,j(s)}\phi_i)(s)\gamma(y_i). \tag{32}$$

From these mappings we get the operator  $\Gamma_{k,s} : F \rightarrow L_{\infty}^{N_k}$  by

$$\Gamma_{k,s}f := \rho((\beta \circ f \circ \eta_i)_{i=0}^v).$$

This means that

$$(\Gamma_{k,s}f)(j) = \rho((\beta \circ f \circ \eta_i(j))_{i=0}^v).$$

Now we are ready to compute the query error of  $S$ . By Lemma 2.3, we can decompose the query error into

$$e_n^q(S, F) \leq \sup_{f \in F} \|Sf - P_l^{(1)}Sf\| + e_n^q(P_l^{(1)}S, F). \tag{33}$$

So the error splits into a deterministic and a quantum part. Classical polynomial approximation gives for  $g \in C^r(D_1)$  and  $k \in \mathbb{N}_0$  (see e.g. [2, Chapter 3.1]),

$$\|g - P_k^{(1)}g\|_{C(D_1)} \leq c2^{-rk}, \tag{34}$$

so for the deterministic part in (33) we get

$$\|Sf - P_l^{(1)}Sf\|_{C(D_1)} \leq c2^{-rl}. \tag{35}$$

Next we consider the quantum part of (33). Let

$$j = j_1b^{d_2-1} + j_2b^{d_2-2} + \dots + j_{d_2-1}b + j_{d_2}$$

for  $j = 0, \dots, N_k - 1$ , where  $b = 2^{rk}n_{2,k}$ . Let the node points for the quantum summation be defined as

$$t_j := \left(\frac{j_1}{b}, \frac{j_2}{b}, \dots, \frac{j_{d_2}}{b}\right).$$

For  $k > \tilde{m}$  we define the operators  $J_{k,s} : F \rightarrow \mathbb{R}$  by

$$J_{k,s}f := \frac{1}{N_k} \sum_{j=0}^{N_k-1} f_{k,s}(t_j), \tag{36}$$

which is the rectangle rule with  $N_k$  points for  $f_{k,s}$ . Next we define, also for  $k > \tilde{m}$ , operators  $U_{k,s} : F \rightarrow \mathbb{R}$  by

$$U_{k,s}f = \int_{D_2} f_{k,s}(t) dt. \tag{37}$$

Since the accuracy of the rectangle rule with  $N_k$  points in dimension  $d_2$  is of the order  $N_k^{-1/d_2}$  for functions with bounded first derivatives, we get by (22) and (29)

$$|U_{k,s}(f) - J_{k,s}(f)| \leq c2^{-rk}n_{2,k}^{-1}. \tag{38}$$

By definition of the discretization operator  $\Gamma_{k,s}$  we get for  $|z| \leq 1$

$$\gamma(\beta(z)) \leq z \leq \gamma(\beta(z)) + 2^{-m^*/2},$$

and by (30) this implies that

$$\begin{aligned} & |\Gamma_{k,s}(f)(j) - f_{k,s}(t_j)| \\ & \leq |f(s, t_j) - \gamma(\beta(f(s, t_j)))| + \sum_{i=0}^{v-1} |R_{k-1,j}\phi_i(s)| |f(s_i, t_j) - \gamma(\beta(f(s_i, t_j)))| \\ & \leq c2^{-rk}n_{2,k}^{-1}. \end{aligned} \tag{39}$$

From the discretization accuracy of  $\Gamma_{k,s}$  and (34) we also infer that

$$\begin{aligned} \|\Gamma_{k,s}(f)\|_{L_{\infty}^{N_k}} &\leq \| (f_{k,s}(t_j)) \|_{L_{\infty}^{N_k}} + \| (f_{k,s}(t_j)) - \Gamma_{k,s}f \|_{L_{\infty}^{N_k}} \\ &\leq c2^{-r(k-1)} + cn_{2,k}^{-1}2^{-rk} \leq c_12^{-rk}, \end{aligned} \quad (40)$$

which implies

$$\Gamma_{k,s}(F) \subseteq c_12^{-rk} \mathcal{B}(L_{\infty}^{N_k}). \quad (41)$$

From (39) it also follows that

$$|S_{N_k}\Gamma_{k,s}f - J_{k,s}f| \leq \frac{1}{N_k} \sum_{j=0}^{N_k-1} |(\Gamma_{k,s}f)(j) - f_{k,s}(t_j)| \leq c2^{-rk}n_{2,k}^{-1}. \quad (42)$$

Now we calculate the error of the integration of the  $f_{k,s}$  on  $A_k^{(1)} \setminus A_{k-1}^{(1)}$ . We get with Lemma 2.3, (42), Theorem 2.7, Lemma 2.2 and (41)

$$\begin{aligned} &e_{2^{(v+1)n_{2,k}}}^q(U_{k,s}, F) \\ &\leq \sup_{f \in F} |U_{k,s}(f) - J_{k,s}(f)| + e_{2^{(v+1)n_{2,k}}}^q(J_{k,s}, F) \\ &\leq \sup_{f \in F} |U_{k,s}(f) - J_{k,s}(f)| + \sup_{f \in F} |S_{N_k}\Gamma_{k,s}f - J_{k,s}f| + e_{2^{(v+1)n_{2,k}}}^q(S_{N_k}\Gamma_{k,s}, F) \\ &\leq c2^{-rk}n_{2,k}^{-1} + e_{n_{2,k}}^q(S_{N_k}, c_12^{-rk} \mathcal{B}(L_{\infty}^{N_k})) \\ &\leq c2^{-rk}n_{2,k}^{-1} + c_12^{-rk}e_{n_{2,k}}^q(S_{N_k}, \mathcal{B}(L_{\infty}^{N_k})) \\ &\leq c2^{-rk}n_{2,k}^{-1} + c2^{-rk}n_{2,k}^{-1} \\ &\leq c2^{-rk}n_{2,k}^{-1}. \end{aligned} \quad (43)$$

With the help of this result we can now investigate the error of the operator  $P_l^{(1)}S$ . Since

$$P_l^{(1)}S = P_{\tilde{m}}^{(1)}S + \sum_{k=\tilde{m}+1}^l (P_k^{(1)} - P_{k-1}^{(1)})S, \quad (44)$$

we investigate the error of the operator

$$(P_k^{(1)} - P_{k-1}^{(1)})S : F \rightarrow C(D_1).$$

We define

$$\theta_k := 2^{-(k+3)}, \quad k = \tilde{m}, \dots, l. \quad (45)$$

Then we set

$$\hat{n}_{\tilde{m}} := M_{\tilde{m}}n_{1,\tilde{m}}n_{2,\tilde{m}}, \quad (46)$$

and for  $k = \tilde{m} + 1, \dots, l$  we set

$$\hat{n}_k := M_k(n_{1,k} - n_{1,k-1})2(v + 1)n_{2,k}. \tag{47}$$

Let  $A_{k,s}$  be a quantum algorithm that computes an approximation to  $U_{k,s}$  on  $F$  with the rate from (43) and let  $\zeta_{k,s}$  be a random variable with distribution  $A_{k,s}$ . We define a random variable  $\zeta_k$  with values in  $L_\infty^{n_{1,k}}$  as follows: For  $s \in A_k^{(1)} \setminus A_{k-1}^{(1)}$  we let  $\zeta_k(s)$  be the median of  $M_k$  independent copies of  $\zeta_{k,s}$ , that is, we repeat  $A_{k,s}$   $M_k$  times. For  $s \in A_{k-1}^{(1)}$  we set  $\zeta_k(s) := 0$ . Since  $(U_{k,s}f)(t) = 0$  for  $s \in A_{k-1}^{(1)}$ , by this choice we establish an error of zero in these points. We have

$$(P_k^{(1)} - P_{k-1}^{(1)})(Sf) = P_k^{(1)}((I - P_{k-1}^{(1)})(Sf)) = P_k^{(1)}(U_{k,s})_{s \in A_k^{(1)}}, \tag{48}$$

where on the right-hand side  $P_k^{(1)}$  is considered as acting on  $L_\infty^{n_{1,k}}$ . This means that because we can interchange interpolation with respect to the first component and integration with respect to the second component, we indeed compute an approximation to  $(P_k^{(1)} - P_{k-1}^{(1)})(Sf)$  by  $(P_k^{(1)} - P_{k-1}^{(1)})\zeta_k$ .

By Lemma 2.4 and (43),

$$\mathbb{P}(|U_{k,s}(f) - \zeta_k(s)| > c2^{-rk}n_{2,k}^{-1}) \leq e^{-M_k/8}. \tag{49}$$

Consequently,

$$\begin{aligned} \mathbb{P}(|U_{k,s}(f) - \zeta_k(s)| \leq c2^{-rk}n_{2,k}^{-1} \quad \forall s \in A_k^{(1)}) \\ \geq 1 - n_{1,k}e^{-M_k/8} \geq 1 - 2^{-(k+3)} = 1 - \theta_k \end{aligned} \tag{50}$$

by (27). From Lemma 3.2, (47), (48) and (50) we obtain for the query error of the operator  $(P_k^{(1)} - P_{k-1}^{(1)})S$ ,

$$e_{\hat{n}_k}^q((P_k^{(1)} - P_{k-1}^{(1)})S, F, \theta_k) \leq c2^{-rk}n_{2,k}^{-1}. \tag{51}$$

We use Lemma 2.1 to calculate the error of  $P_l^{(1)}S$ . From (45), we get  $\sum_{k=\tilde{m}}^l \theta_k \leq 1/4$ ; hence, with Lemma 2.1, Lemma 3.2, (44) and with

$$\tilde{n} := \sum_{k=\tilde{m}}^l \hat{n}_k, \tag{52}$$

we get

$$\begin{aligned} e_{\tilde{n}}^q(P_l^{(1)}S, F, 1/4) \leq e_{\tilde{n}_{\tilde{m}}}^q((P_{\tilde{m}}^{(1)}S, F, \theta_{\tilde{m}}) \\ + \sum_{k=\tilde{m}+1}^l e_{\hat{n}_k}^q((P_k^{(1)} - P_{k-1}^{(1)})S, F, \theta_k). \end{aligned} \tag{53}$$

Now we consider the different cases:

(1)  $r < d_1$ : For the error on the starting level  $\tilde{m} = 0$  we can make direct use of Theorem 2.7 and, in this case using the operator from (31), we get

$$e_{n_0}^q(P_0^{(1)}S, F, \theta_0) \leq c(n_{2,0})^{-1}$$

by Theorem 2.7, Lemma 3.2 and a similar probability argument as above. Now with (53) and (51) we get

$$\begin{aligned} e_n^q(P_l^{(1)}S, F, 1/4) &\leq c(n_{2,0})^{-1} + \sum_{k=1}^l c2^{-rk}n_{2,k}^{-1} \\ &\leq c(\log_2 n) \sum_{k=0}^l 2^{-(d_1+d_2)m+d_1k+\frac{1}{2}(d_1-r)(l-k)-rk} \\ &\leq c(\log_2 n)2^{-(d_1+d_2)m} \sum_{k=0}^l 2^{\frac{1}{2}(d_1-r)(l+k)}. \end{aligned}$$

From this we get with the help of the geometric sum formula

$$e_n^q(P_l^{(1)}S, F, 1/4) \leq c(\log_2 n)2^{-(d_1+d_2)m+(d_1-r)l}$$

and with  $2^{d_1l-d_2m} \asymp 2^{d_1m-\log_2 m}$  we arrive at

$$e_n^q(P_l^{(1)}S, F, 1/4) \leq c2^{-rl} \leq cn^{-r/d_1}(\log_2 n)^{r/d_1}.$$

For the deterministic part of the error we get by the choice of  $l$

$$\|Sf - P_l^{(1)}Sf\| \leq c2^{-rl} \leq cn^{-r/d_1}(\log_2 n)^{r/d_1},$$

which by (33) gives the desired rate for  $r < d_1$ .

(2)  $r \geq d_1$ : To calculate the error on the starting level  $\tilde{m} = m$  we use Theorem 2.5 and (28) and again with the probability argument from above we get

$$e_{n_m}^q(P_m^{(1)}S, F, \theta_m) \leq c(n_{2,m})^{-r/d_2-1} \leq cn^{-(r+d_2)/(d_1+d_2)}. \tag{54}$$



With (53) and (51) we get

$$\begin{aligned}
 e_{\tilde{n}}^q(P_l^{(1)}S, F, 1/4) &\leq cn^{-(r+d_2)/(d_1+d_2)} + \sum_{k=m+1}^l c2^{-rk}n_{2,k}^{-1} \\
 &\leq cn^{-(r+d_2)/(d_1+d_2)} + c \sum_{k=m+1}^l 2^{-rk}2^{-d_2m+\frac{1}{2}(r+d_1)(k-m)} \\
 &\leq cn^{-(r+d_2)/(d_1+d_2)} + c2^{-(r+d_2)m} \sum_{k=m+1}^l 2^{-\frac{1}{2}(r-d_1)(k-m)} \\
 &\leq cn^{-(r+d_2)/(d_1+d_2)} \sum_{k=m+1}^l 2^{-\frac{1}{2}(r-d_1)(k-m)}.
 \end{aligned}$$

For  $r > d_1$ , the sum is bounded by a constant, and for  $r = d_1$  the sum gives an additional factor of  $\log n$ . By the choice of  $l$  we get

$$\|Sf - P_l^{(1)}Sf\| \leq c2^{-rl} \leq cn^{-(r+d_2)/(d_1+d_2)},$$

and with (33) we arrive at

$$e_{\tilde{n}}^q(S, F) \leq cn^{-(r+d_2)/(d_1+d_2)} \tag{55}$$

for  $r > d_1$  and

$$e_{\tilde{n}}^q(S, F) \leq cn^{-1} \log n \tag{56}$$

for  $r = d_1$ .

Finally, we estimate the number of queries  $\tilde{n}$  that are needed to obtain the desired precision. Since the total number of queries is

$$\tilde{n} = M_{\tilde{m}}n_{1,\tilde{m}}n_{2,\tilde{m}} + 2(v+1) \sum_{k=\tilde{m}+1}^l M_k(n_{1,k} - n_{1,k-1})n_{2,k},$$

we get for  $r < d_1$ ,

$$\begin{aligned}
 \tilde{n} &= M_0n_{1,0}n_{2,0} + 2(v+1) \sum_{k=1}^l M_k(n_{1,k} - n_{1,k-1})n_{2,k} \\
 &\leq c \sum_{k=0}^l M_k n_{1,k} n_{2,k} \\
 &\leq c \sum_{k=0}^l 2^{d_1k} 2^{(d_1+d_2)m-d_1k-\frac{1}{2}(d_1-r)(l-k)} \\
 &\leq c2^{(d_1+d_2)m} \sum_{k=0}^l 2^{-\frac{1}{2}(d_1-r)(l-k)} \asymp n,
 \end{aligned}$$

and for  $r > d_1$  we get

$$\begin{aligned} \tilde{n} &= M_m n_{1,m} n_{2,m} + 2(v+1) \sum_{k=m+1}^l M_k (n_{1,k} - n_{1,k-1}) n_{2,k} \\ &\leq c \sum_{k=m}^l M_k n_{1,k} n_{2,k} \\ &\leq c \sum_{k=m}^l (k + \ln n_{1,k}) (r2^k + 1)^{d_1} 2^{d_2 m - \frac{1}{2}(r+d_1)(k-m)} \\ &\leq c \log n \sum_{k=m}^l 2^{d_1 k} 2^{d_2 m - \frac{1}{2}(r+d_1)(k-m)} \\ &\leq c \log n 2^{(d_1+d_2)m} \sum_{k=m}^l 2^{-\frac{1}{2}(r-d_1)(k-m)} \asymp n \log n, \end{aligned}$$

since  $d_1 k + d_2 m - \frac{1}{2}(r+d_1)(k-m) = (d_1+d_2)m - \frac{1}{2}(r-d_1)(k-m)$ . This means for  $r < d_1$  the cost is of order  $n$  and for  $r > d_1$  it is of order  $n \log n$ , such that a rescaling of  $n$  leads to the proposed rate. In the case  $r = d_1$  the cost is  $\mathcal{O}(n(\log n)^2)$ , so together with (56) we get the additional log-factor in the convergence rate. Now the upper bound of Theorem 2.9 is proved.

Note that the proof of the upper bound was carried out in terms of query errors and can easily be translated into an explicit quantum algorithm for parametric integration. The algorithm has the following form and uses the sequences of optimal algorithms  $A_{\text{sum}}(\cdot, n, N)$  and  $A_{\text{int}}(\cdot, n)$  for quantum summation and quantum integration. For a given  $n$  we recall the needed parameters, which are  $m = \lfloor \frac{1}{d_1+d_2}(\log_2 n) + 1 \rfloor$ , starting level  $\tilde{m} := m$  if  $r \geq d_1$  and zero otherwise, final level  $l := \lceil (1 + d_2/r)m \rceil$  if  $r \geq d_1$  and  $\lceil (1 + d_2/d_1)m \rceil - p$  otherwise, where  $p := \lfloor (\log_2 m)/d_1 \rfloor$ . We have  $n_{1,k} = (r2^k + 1)^{d_1}$ ,  $M_k := \lceil 8(k+3)\ln 2 + 8 \ln n_{1,k} \rceil$ , and  $n_{2,k} := \lceil 2^{d_2 m - \frac{1}{2}(r+d_1)(k-m)} \rceil$ , if  $r \geq d_1$  and  $\lceil M_k^{-1} 2^{(d_1+d_2)m - d_1 k - \frac{1}{2}(d_1-r)(l-k)} \rceil$  otherwise. Finally, we have  $N_k := 2^{rkd_2} n_{2,k}^{d_2}$ . Now the algorithm  $A_{\text{parint}}(f, n)$  is the following:

1. Starting level  $\tilde{m}$ : For all  $s \in A_{\tilde{m}}^{(1)}$  do:
  - (a) If  $r \geq d_1$ , compute  $M_{\tilde{m}}$  times  $A_{\tilde{m},s} := A_{\text{int}}(f(s, \cdot), n_{2,\tilde{m}})$  and let  $\xi_{\tilde{m}}(s)$  be the median of these  $M_{\tilde{m}}$  results.
  - (b) If  $r < d_1$ , compute  $M_{\tilde{m}}$  times  $A_{\tilde{m},s} := A_{\text{sum}}(\Gamma_{\tilde{m},s} f, n_{2,\tilde{m}}, N_{\tilde{m}})$  and let  $\xi_{\tilde{m}}(s)$  be the median of these  $M_{\tilde{m}}$  results.
2. Finer levels: For  $k = \tilde{m} + 1, \dots, l$  do:
  - For all  $s \in A_k^{(1)} \setminus A_{k-1}^{(1)}$  do:  $M_k$  times compute  $A_{k,s} := c_1 2^{-rk} A_{\text{sum}}(c_1^{-1} 2^{rk} \Gamma_{k,s} f, n_{2,k}, N_k)$  and let  $\xi_k(s)$  be the median of these  $M_k$  results.

- For  $s \in A_{k-1}^{(1)}$  do:  $\zeta_k(s) := 0$ .

3. **Final approximation:**  $A_{\text{parint}}(f, n) := P_{\tilde{m}}^{(1)} \xi_{\tilde{m}} + \sum_{k=\tilde{m}+1}^l (P_k^{(1)} - P_{k-1}^{(1)}) \xi_k$ .

In step 2, the function  $\Gamma_{k,s} f \in L_{\infty}^{N_k}$  is scaled by  $c_1^{-1} 2^{rk}$  and the result then rescaled to make sure that the algorithm  $A_{\text{sum}}$  is applied to a function with  $L_{\infty}^{N_k}$ -norm smaller or equal to one.

#### 4. Lower bound

In this section, we first cite a general result for lower bounds on the quantity  $e_n^q(S, F)$  and then we apply this result to the case of parametric integration.

Let  $D$  and  $K$  be nonempty sets, let  $L \in \mathbb{N}$  and let to each

$$u = (u_0, \dots, u_{L-1}) \in \{0, 1\}^L$$

an  $f_u \in \mathcal{F}(D, K)$  be assigned such that the following is satisfied:

*Condition (I):* There are functions  $g_0, g_1 \in \mathcal{F}(D, K)$  and a decomposition  $D = \bigcup_{l=0}^{L-1} D_l$  with  $D_l \cap D_{l'} = \emptyset$  ( $l \neq l'$ ) such that for  $t \in D_l$

$$f_u(t) = \begin{cases} g_0(t) & \text{if } u_l = 0, \\ g_1(t) & \text{if } u_l = 1. \end{cases}$$

Next we define the function  $\rho(L, l, l')$  for  $L \in \mathbb{N}, 0 \leq l \neq l' \leq L$  by

$$\rho(L, l, l') = \sqrt{\frac{L}{|l - l'|} + \frac{\min_{j=l, l'} \sqrt{j(L-j)}}{|l - l'|}}.$$

Note that  $j(L-j)$  is minimized iff  $|L/2 - j|$  is maximized. For  $u \in \{0, 1\}^L$  we set  $|u| = \sum_{l=0}^{L-1} u_l$ . Then we have the following.

**Lemma 4.1.** *There is a constant  $c_0 > 0$  such that the following holds: Let  $D, K$  be nonempty sets, let  $F \subset \mathcal{F}(D, K)$  be a set of functions,  $G$  a normed space,  $S: F \rightarrow G$  a function, and  $L \in \mathbb{N}$ . Suppose  $(f_u)_{u \in \{0,1\}^L} \subseteq \mathcal{F}(D, K)$  is a system of functions satisfying condition (I). Let finally  $0 \leq l \neq l' \leq L$  and assume that*

$$f_u \in F \quad \text{whenever} \quad |u| \in \{l, l'\}.$$

Then

$$e_n^q(S, F) \geq \frac{1}{2} \min\{\|S(f_u) - S(f_{u'})\| : |u| = l, |u'| = l'\}$$

for all  $n$  with

$$n \leq c_0 \rho(L, l, l').$$

A proof of Lemma 4.1 can be found in Heinrich [4]. With the help of this lemma we can now prove the lower bound for parametric integration:

Let  $c_0$  be the constant from Lemma 4.1, and let  $d = d_1 + d_2$ . For  $n \in \mathbb{N}$  we choose an even number  $m \in \mathbb{N}$  such that

$$L = m^d \geq \sqrt{4c_0^{-2}n^2 + 4}. \tag{57}$$

Let

$$i = j_1 m^{d-1} + j_2 m^{d-2} + \dots + j_{d-1} m + j_d$$

for  $i = 0, \dots, L - 1$ . Let

$$D_i = \prod_{l=1}^d \left[ \frac{j_l}{m}, \frac{j_l + 1}{m} \right], \quad D = \bigcup_{i=0}^{L-1} D_i.$$

Define the functions  $\psi_i \in C^\infty(D)$  by

$$\begin{aligned} \psi_i(s_1, \dots, s_{d_1}, t_{d_1+1}, \dots, t_{d_1+d_2}) &= \psi_i^{(1)}(s_1, \dots, s_{d_1}) \cdot \psi_i^{(2)}(t_{d_1+1}, \dots, t_{d_1+d_2}) \\ &= \prod_{l=1}^{d_1} \eta(ms_l - j_l) \cdot \prod_{l=d_1+1}^{d_1+d_2} \eta(mt_l - j_l) \end{aligned}$$

with

$$\eta(x) = \begin{cases} e^{-\frac{1}{x(1-x)}}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

For  $k = \sum_{l=1}^{d_1+d_2} k_l$  it holds that

$$\begin{aligned} \frac{\partial \psi_i(s, t)}{\partial s_1^{k_1} \partial s_2^{k_2} \dots \partial t_{d_1+d_2}^{k_{d_1+d_2}}} &= \prod_{l=1}^{d_1} \frac{d^{k_l}}{ds_l^{k_l}} \eta(ms_l - j_l) \cdot \prod_{l=d_1+1}^{d_1+d_2} \frac{d^{k_l}}{dt_l^{k_l}} \eta(mt_l - j_l) \\ &= \prod_{l=1}^{d_1} \eta^{(k_l)}(ms_l - j_l) m^{k_l} \cdot \prod_{l=d_1+1}^{d_1+d_2} \eta^{(k_l)}(mt_l - j_l) m^{k_l} \\ &= m^k \prod_{l=1}^{d_1} \eta^{(k_l)}(ms_l - j_l) \cdot \prod_{l=d_1+1}^{d_1+d_2} \eta^{(k_l)}(mt_l - j_l). \end{aligned}$$

From this we get

$$\begin{aligned} \|\psi_i\|_{C^r(D)} &= \max_{\sum_{l=1}^{d_1+d_2} k_l \leq r} \sup_{(s,t) \in D} \left| \frac{\partial^k \psi_i(s, t)}{\partial s_1^{k_1} \partial s_2^{k_2} \dots \partial t_{d_1+d_2}^{k_{d_1+d_2}}} \right| \\ &= \max_{\sum_{l=1}^{d_1+d_2} k_l \leq r} m^k \prod_{l=1}^{d_1} \sup_{s_l \in [0,1]} |\eta^{(k_l)}(s_l)| \prod_{l=d_1+1}^{d_1+d_2} \sup_{t_l \in [0,1]} |\eta^{(k_l)}(t_l)| \\ &\leq \gamma m^r, \end{aligned}$$

where

$$\gamma = \max_{\sum_{l=1}^{d_1+d_2} k_l \leq r} \prod_{l=1}^{d_1} \sup_{s_l \in [0,1]} |\eta^{(k_l)}(s_l)| \cdot \prod_{l=d_1+1}^{d_1+d_2} \sup_{t_l \in [0,1]} |\eta^{(k_l)}(t_l)|.$$

Therefore, setting  $\hat{\psi}_i := \frac{1}{\gamma m^r} \psi_i$ , we have  $f_u := \sum_{i=0}^{L-1} u_i \hat{\psi}_i \in F$  for all  $u_i \in \{0, 1\}, i = 0, \dots, L - 1$ . Since the  $\psi_i$  have disjoint support,  $f_u$  satisfies condition (I). Let  $\sigma_0 = \int_0^1 \eta(x) dx > 0$ . It is easy to show that

$$\int_D \psi_i(s_1, \dots, t_{d_1+d_2}) ds_1 \dots dt_{d_1+d_2} = \frac{\sigma_0^d}{m^d}.$$

Let now  $l := L/2 - 1$  and  $l' := l + 1 = L/2$ . Then with (57) it follows that

$$\rho(L, l, l') > \min_{j=\{l, l'\}} \sqrt{j(L-j)} = \sqrt{\left(\frac{L}{2} - 1\right)\left(\frac{L}{2} + 1\right)} = \sqrt{\frac{L^2}{4} - 1} \geq c_0^{-1} n.$$

Since

$$\begin{aligned} & \min\{\|S(f_u) - S(f_{u'})\|_{C(D_1)} \mid |u| = l, |u'| = l'\} \\ &= \|S(\hat{\psi}_0)\|_{C(D_1)} = \frac{1}{\gamma m^r} \frac{\sigma_0^{d_2}}{m^{d_2}} \|\psi_0^{(2)}(t_{d_1+1}, \dots, t_{d_1+d_2})\|_{C(D_1)} \\ &= e^{-4d_1} \sigma_0^{d_2} \gamma^{-1} m^{-(r+d_2)} = C(m^{d_1+d_2})^{-\frac{r+d_2}{d_1+d_2}} \\ &= C' n^{-\left(\frac{r}{d_1+d_2} + \frac{d_2}{d_1+d_2}\right)}, \end{aligned}$$

from Lemma 4.1 we get a general lower bound for parametric integration and the lower bound of Theorem 2.9 for  $r \geq d_1$ .

We get a lower bound of  $n^{-r/d_1}$  for approximation by simply choosing  $d_2 = 0$ , and since parametric integration can never have a better convergence rate than approximation, this gives us the lower bound of Theorem 2.9 for the case  $r < d_1$ .

### 5. Comments

We have solved the problem of the quantum complexity of parametric integration for the class  $C^r(D)$  by providing upper and lower bounds, where the rates match up to a logarithmic factor. Now we compare our results to the known results for deterministic and Monte Carlo methods stated in Section 2.4. We again assume  $r \geq 1$ . Then the optimal quantum rates are always better than the optimal deterministic rate, except for the case  $d_2 = 0$ , where we have equal rates, and the following table provides a comparison of the deterministic, Monte Carlo and quantum case (without log-factors).

	$e_n^{\det}(S, F)$	$e_n^{\text{mc}}(S, F)$	$e_n^q(S, F)$
$r \geq d_1$	$n^{-r/(d_1+d_2)}$	$n^{-(r+d_2/2)/(d_1+d_2)}$	$n^{-(r+d_2)/(d_1+d_2)}$
$d_1/2 \leq r < d_1$	$n^{-r/(d_1+d_2)}$	$n^{-(r+d_2/2)/(d_1+d_2)}$	$n^{-r/d_1}$
$r < d_1/2$	$n^{-r/(d_1+d_2)}$	$n^{-r/d_1}$	$n^{-r/d_1}$

We have to distinguish three different situations, depending on the relation of the problem parameters  $r$  and  $d_1$ . For  $r < d_1/2$  the quantum rate provides an improvement over the deterministic rate, but it is as fast as the Monte Carlo rate. For this parameter constellation both the Monte Carlo and the quantum algorithm achieve the optimal rate of approximation which is the same in both cases.

When we have  $d_1/2 \leq r < d_1$ , then the quantum rate is still the optimal rate of approximation, but the Monte Carlo rate is slower, which leads to the superiority of the quantum rate for this situation.

For the case  $r \geq d_1$ , we still have a better performance of the quantum algorithm as compared to Monte Carlo.

Summarizing the discussion we can say that the quantum rate is always better than the deterministic rate, it is always at least as good as the Monte Carlo rate, and for  $r \geq d_1/2$  it is better than the optimal rate of Monte Carlo algorithms.

Let us finally consider our problem in the bit model. Referring to [4,6], we find that for our algorithm the number of qubits needed is  $\mathcal{O}(\log n)$ , the number of quantum gates is  $\mathcal{O}(n \log n)$  and the number of measurements is  $\mathcal{O}((\log n)^3 \log \log n)$ .

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