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Infinite matrix product states for long-range SU(N) spin models

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Abstract

We construct 1D and 2D long-range SU(N) spin models as parent Hamiltonians associated with infinite matrix product states. The latter are constructed from correlators of primary fields in the SU$(N)_1$ WZW model. Since the resulting groundstates are of Gutzwiller–Jastrow type, our models can be regarded as lattice discretizations of fractional quantum Hall systems. We then focus on two specific types of 1D spin chains with spins located on the unit circle, a uniform and an alternating arrangement. For an equidistant distribution of identical spins we establish an explicit connection to the SU(N) Haldane–Shastry model, thereby proving that the model is critical and described by a SU$(N)_1$ WZW model. In contrast, while turning out to be critical as well, the alternating model can only be treated numerically. Our numerical results rely on a reformulation of the original problem in terms of loop models.

1. Introduction

Long-range spin models such as the Gaudin model [1] or the Haldane–Shastry model [2,3] have attracted the attention of physicists and mathematicians for a long period of time. In its original formulation, the Haldane–Shastry model describes the dynamics of SU(2) spins on a circle with inverse distance square interactions. It received a lot of attention due to its exact solvability and due to the form of its groundstate which is closely related to a bosonic Laughlin wavefunction

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at filling fraction $\nu = 1/2$. The Haldane–Shastry model can be viewed as realizing a 1D analogue of a chiral spin liquid, with spinon excitations satisfying a generalized Pauli exclusion principle and obeying fractional statistics [4]. Many of the remarkable properties of the Haldane–Shastry model have their origin in the existence of an infinite-dimensional Yangian symmetry [5]. The latter also allowed to identify its thermodynamic limit as the SU(2) WZW conformal field theory at level $k = 1$ [6] (see also [7]). The Haldane–Shastry model admits obvious generalizations to symmetry groups such as the unitary series SU(N) [5] or its supersymmetric analog SU($M|N$) [8,9].

For our current work, there are two aspects of the SU(N) Haldane–Shastry model that will be particularly important. First of all, it provides an efficient discretization of the SU(N) WZW conformal field theory at level $k = 1$, where the scaling laws are not affected by logarithmic corrections. Secondly, wavefunctions of the groundstate as well as the excited states exhibit an intimate relation to the physics of fractional quantum Hall (FQH) systems, also for general values of N [10]. There are of course also differences: While the constituents of FQH systems are particles which are moving on a 2D surface, the degrees of freedom in the spin model are pinned to fixed discrete locations on a circle.

The study of fractional quantum Hall systems is frequently based on the following intriguing dichotomy: One single chiral 2D conformal field theory (CFT) describes the two complementary aspects of the physical sample – its bulk and its boundary. It is known since the work of Moore and Read, for instance, that chiral CFT correlators give rise to realistic trial wave functions for the groundstates of gapped chiral 2D states of matter [11]. Remarkably, chiral correlators also encode the anyonic statistics of quasi-particle excitations above the groundstate. Among other insights, this led to the theoretical prediction of quasi-particles with non-abelian statistics for the FQH state at filling fraction $\nu = 5/2$. At the same time, the chiral CFT describing the bulk can be used to model the properties of the 1D gapless theory describing its chiral edge [12]. The intimate relation between bulk and boundary is also visible in entanglement spectra which can be calculated from the groundstate wave function [13].

Recently, the question whether chiral topological states of matter can be engineered systematically received renewed interest. This is partly due to prospects of simulating strongly correlated systems in optical lattices. On the other hand, one also requires efficient ways of capturing the topological properties of strongly correlated systems from a numerical point of view. In 1D, all properties of gapped states are well captured by matrix product states [14]. However, the situation is by far less obvious in 2D. While simple tensor network realizations for non-chiral topological states such as the Kitaev model [15] or the Levin–Wen models [16] have been known for some time, chiral topological phases resisted all attempts to find such representatives. By now, there is considerable evidence that chiral topological phases cannot be described in terms of tensor network states with finite bond dimension, at least if one insists on a gapped parent Hamiltonian with local interactions [17,18].

An interesting approach to the construction of chiral topological phases using "infinite matrix product states" has recently been suggested by Cirac and Sierra [19] and elaborated in more detail by Nielsen, Cirac and Sierra in [20]. The basic idea is to define a spin model in terms of the data of an associated WZW model [21,22] for specified locations of the spins on the complex plane. More precisely, the Hamiltonian is designed as to annihilate a specific set of WZW correlation functions which, in turn, are used to define the groundstate of the spin model. This can be achieved by employing the existence of null fields in the WZW model (see Section 2.1 for details). The resulting state can be interpreted as an infinite matrix product state since the

fields can be regarded as operators on an infinite-dimensional Hilbert space, with the correlator replacing the usual trace.

When carried out for SU(2) spins on a circle, the previous program gives rise to a slight generalization of the Haldane–Shastry model [20]. The match becomes perfect when the spins are distributed equidistantly. Starting from the SU(2) WZW model at level $k = 1$, the authors of [20] thus succeeded in defining a 1D spin model which gave rise to the same WZW model in the thermodynamic limit.[1] By now, this program has also been put into effect for the groups SO(N) and U(1) [25,26] leading to conceptually similar results. In the latter case it was also possible to interpolate 2D Laughlin states at filling fraction $\nu = 1/q$ from the lattice to the continuum.

One of the notable features of the construction presented in [20] is its remarkable flexibility. The spins can be placed at arbitrary positions in the complex plane, including regular arrangements such as various types of 1D or 2D lattices. Moreover, the transformation behavior of the spins can be chosen at will, even independently on each site. It is then a natural question to which extent the usual dichotomy of FQH states applies to this new type of construction. In particular, one would like to know whether the thermodynamic limit of a 2D setup describes a chiral topological state of matter and how its properties – e.g. its anyonic excitations, its edge theory and its groundstate entanglement spectrum – relate to the data of the WZW model initially put in. Turning one's attention to 1D setups, one may – similarly – expect a flow to a 2D CFT but a priori there is no reason why it should be connected to the original WZW model. One may even speculate (and investigate) whether potential chiral topological phases resulting from 2D setups can be engineered systematically by stacking layers of 1D critical chains. A simple general answer to our previous questions cannot be expected. The exploration of individual examples is therefore the method of choice to gain a better idea about the value and the limitations of the general method of infinite matrix product states.

In this paper, we apply the construction of [20] to the case of the SU(N) WZW model at level $k = 1$. The main motivation for this extension is the additional degree of freedom that comes with the extension from SU(2) to a higher rank unitary symmetry. In particular, for $N \geq 3$ the fundamental field is distinct from the anti-fundamental one. With two types of mutually dual representations at our disposal, we can then realize a family of anti-ferromagnetic spin models on bipartite lattices or study the effects of frustration. Moreover, one may expect additional types of spin interactions due to the existence of higher rank invariant tensors for SU(N). Apart from these conceptual points the generalization from $N = 2$ to arbitrary values of N is interesting since it is likely to relate to well-known FQH trial states such as the Halperin state [27] or the non-abelian spin singlet (NASS) state [28]. In both cases there is a close relation to the SU(3) WZW model.

Our paper is organized as follows. In Section 2 we review the basic philosophy of [20] and, focusing on the fundamental and anti-fundamental representation, we implement it for the $\mathrm{SU}(N)_1$ WZW model. Our construction gives rise to families of long-range SU(N) spin models labeled by the types of spins and their location in the complex plane. Generically, the Hamiltonian involves a mixture of two- and three-spin interactions. We then discuss particular choices of spin configurations and the resulting simplifications. Finally, the WZW correlation functions determining the groundstates of our spin models are evaluated using free field representations. Spin models based on a single representation (the "uniform case") are discussed in more detail in

[1] Parent Hamiltonians with similar features discretizing the SU(2) WZW model (even at arbitrary level k) have also been proposed in [23,24], starting from a slightly different perspective.

Section 3. We first rewrite the general Hamiltonian in terms of permutation operators, thereby making the model amenable to an efficient numerical treatment. Afterwards we show that the three-spin interactions decouple from the local dynamics if the spins are located on the circle. In the case of an equidistant distribution we manage to recover the SU(N) Haldane–Shastry model. Apart from providing a complete analytic solution to the model, this observation also allows to identify the thermodynamic limit of the chain as the SU(N) WZW model at level $k = 1$ (plus generalized chemical potentials incorporating the square and the cube of the total spin).

The situation is very different for the mixed spin models which are discussed in Section 4. Now, the Hamiltonian can be rewritten in terms of permutations and generators of a Temperley–Lieb algebra. Moreover, in contrast to the uniform case, we have not been able to come up with any special configuration of spins which allows to decouple the three-spin interactions. As a consequence, the model appears to be intractable using purely analytic methods. In order to obtain an efficient numerical implementation we use that the Temperley–Lieb generators and the permutations generate a diagram algebra known as the walled Brauer algebra. The latter is the basis for a loop model reformulation of the original eigenvalue problem which is described in Section 5. The final part of this section deals with the exact diagonalization of alternating chains with an equidistant distribution of spins on a circle. Pushing the analysis to chain lengths of up to $L = 18$ sites for several values of N, we are able to predict that the chain becomes conformal in the thermodynamic limit. In addition, we identify part of the conformal spectrum and rule out the possibility of an SU(N) WZW model as the critical theory. The concluding Section 6 summarizes our findings and points out potential directions of future research.

2. The construction of long-range SU(N) spin chains

In this section we define SU(N) spin models involving the fundamental and anti-fundamental representation at arbitrary positions on the complex plane. The construction of the Hamiltonian is based on the SU$(N)_1$ WZW model whose relevant properties are reviewed in detail. The groundstates of all models can be evaluated explicitly and are related to wavefunctions of Gutzwiller–Jastrow type as they appear in the physics of fractional quantum Hall systems.

2.1. The basic philosophy

Let us consider the WZW model associated with a Lie group G [21,22]. It defines a 2D conformal field theory with an infinite-dimensional current algebra symmetry which renders the model exactly solvable. According to the philosophy of [19,20] there is a natural way of associating a quantum mechanical lattice model to any (chiral) correlator

$$\psi(z_1, \ldots, z_L) = \langle \psi_1(z_1) \cdots \psi_L(z_L) \rangle, \tag{1}$$

of WZW primary fields $\psi_i(z_i)$. These fields are inserted at arbitrary but fixed positions z_i on the complex plane and they may be of different type (hence the subscript i). Since the chiral WZW model has a global symmetry G, the fields $\psi_i(z_i)$ should be thought of as vector valued. Each of them transforms in an irreducible representation $\mathcal{H}_i$ of G.

The associated lattice model is obtained by interpreting the numbers $z_i \in \mathbb{C}$ as corresponding to the location of spin operators $\mathbf{S}_i$ representing the infinitesimal action of G on the irreducible representation $\mathcal{H}_i$. This allows one to define a quantum spin model on the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L. \tag{2}$$

The connection to the WZW theory is established by demanding that the groundstate of the spin model may actually be expressed in terms of the WZW correlator as

$$|\psi\rangle = \sum_{\{q_i\}} \psi_{q_1 \dots q_L}(z_1, \dots, z_L) |q_1 \cdots q_L\rangle \in \mathcal{H}. \tag{3}$$

Here, the vectors $|q_i\rangle$ form a basis of the Hilbert space $\mathcal{H}_i$ and the correlator carries the dual quantum numbers q_i with respect to the action of the Lie group G. Since the correlator is G-invariant, the state $|\psi\rangle$ necessarily needs to be a singlet under G.

At this stage of the discussion we did not yet specify the Hamiltonian. The latter can be obtained by defining algebraic operators $\mathcal{P}_i$ associated with the sites z_i, such that

$$\mathcal{P}_i \psi(z_1, \dots, z_L) = 0. \tag{4}$$

The state $|\psi\rangle$ is then automatically a zero energy groundstate of the Hamiltonian [19,20]

$$H = \sum_i \mathcal{P}_i^\dagger \mathcal{P}_i, \tag{5}$$

which is hermitean and positive semi-definite. The sum over sites i in (5) is motivated by the desire that the Hamiltonian reflects the symmetries of the arrangement of sites (e.g. translation symmetry). In view of the construction, it is natural to interpret $|\psi\rangle$ as an infinite matrix product state and H as the associated parent Hamiltonian. For general choices of the representations $\mathcal{H}_i$, the state $|\psi\rangle$ needs not be the only groundstate though. The number of groundstates is given by the dimension of the space of conformal blocks to which the chiral correlation function (1) belongs.[2]

In practice, the operators $\mathcal{P}_i$ are constructed employing the existence of null fields $\chi_i(z_i)$ associated with the fields $\psi_i(z_i)$ [20]. The null field $\chi_i(z_i)$ can be thought of as a descendant of the field $\psi_i(z_i)$ which vanishes identically. It can thus be obtained by acting with currents on the field $\psi_i(z_i)$. Then, by definition of the null field and after the use of Ward identities (see (22) below), one obtains a relation of the form

$$0 = \big\langle \psi_1(z_1) \cdots \chi_i(z_i) \cdots \psi_L(z_L) \big\rangle = \mathcal{P}_i \big\langle \psi_1(z_1) \cdots \psi_L(z_L) \big\rangle. \tag{6}$$

The art of constructing the lattice model is thus reduced to the explicit realization of the null fields $\chi_i(z_i)$ and the derivation of the operators $\mathcal{P}_i$. We refrain from presenting further details of the general construction (see [20]) and focus on the particular case of the $\mathrm{SU}(N)_1$ WZW model from now on. We finally wish to stress that the projectors $\mathcal{P}_i$, the Hamiltonian H and the groundstate $|\psi\rangle$ all explicitly depend on the choice of positions z_i and on the choice of representations $\mathcal{H}_i$. This dependence will be suppressed in our notation since these quantities are thought of as being fixed once and for all.

2.2. *The* $\mathrm{SU}(N)_1$ *WZW model and its null vectors*

The basic structure of any WZW model on a Lie group G is a current algebra extending the corresponding Lie algebra $\mathfrak{g}$. Denoting the chiral currents by $J^a(z)$, the resulting symmetry can be compactly expressed in terms of the operator product expansion (OPE)

[2] To put it differently: The correlation function $\psi(z_1, \dots, z_L)$ is not uniquely defined by Eq. (1) but it rather has a certain degree of arbitrariness related to different choices of fusion channels.

$$J^a(z)J^b(w) = \frac{k\kappa^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w}. \tag{7}$$

Here, the matrix κ^{ab} describes a suitably normalized invariant form and the structure constants $f^{ab}{}_c$ of $\mathfrak{g}$ are real valued. The quantity k is a non-negative integer known as the level. The WZW primary fields $\psi_i(w)$ are labeled by irreducible representations of G. In terms of the currents, they are characterized by the OPE

$$J^a(z)\psi_i(w) = \frac{S_i^a \psi_i(w)}{z-w}. \tag{8}$$

In this formula, we think of $\psi_i(w)$ as being vector valued and S_i^a refers to the corresponding representation matrices.

We now specialize all our considerations to the case of SU(N) and level $k = 1$. At this particular level, each of the WZW primary fields is labeled by one of the $N-1$ fundamental weights. While our analysis can in principle be extended to other cases, this paper will only be concerned with primary fields transforming in either the fundamental representation $\mathcal{V}$ or the anti-fundamental representation $\bar{\mathcal{V}}$.[3] In other words, our goal is to construct a long range quantum spin model on mixed Hilbert spaces of the form

$$\mathcal{H} = \mathcal{V}^{L-\ell} \otimes \bar{\mathcal{V}}^{\ell}. \tag{9}$$

In the language of the previous subsection, the set of sites $\mathbb{L} = \{1, \dots, L\} = \mathbb{S} \cup \bar{\mathbb{S}}$ decomposes into two subsets $\mathbb{S}$ and $\bar{\mathbb{S}}$ such that $\mathcal{H}_i = \mathcal{V}$ if $i \in \mathbb{S}$ and $\mathcal{H}_i = \bar{\mathcal{V}}$ if $i \in \bar{\mathbb{S}}$. In what follows, it will be convenient to distinguish the two types of sites by means of a parity map $d_\bullet : \mathbb{L} \to \mathbb{Z}_2$ which satisfies $d_i = 1$ for $i \in \mathbb{S}$ and $d_i = 0$ for $i \in \bar{\mathbb{S}}$. Two physically particularly interesting setups correspond to the *uniform case* with $\ell = 0$ and to the *alternating case* where L is even and $\ell = L/2$. In these two cases the Hilbert spaces are given by

$$\mathcal{H}_{\text{uniform}} = \mathcal{V}^{\otimes L} \quad \text{or} \quad \mathcal{H}_{\text{alternating}} = (\mathcal{V} \otimes \bar{\mathcal{V}})^{\otimes L/2}. \tag{10}$$

In the former case one has $\mathbb{S} = \mathbb{L}$ and $d_i = 1$ while in the latter case we choose $\mathbb{S} = \{1, 3, \dots, L-1\}$ and $\bar{\mathbb{S}} = \{2, 4, \dots, L\}$ together with $d_i = i \bmod 2$ (the map d_i then determines the parity of the site). While our notation suggests a uniform or alternating arrangement along a 1D chain, we are in principle still free to choose an arbitrary arrangement of spins at this level, including various types of 2D setups (possibly even with random locations). We wish to emphasize that the alternating setup is the most natural one for the description of anti-ferromagnetic spin models on bipartite lattices. Indeed, in that case L only needs to be even in order to admit a singlet in the spectrum while it needs to be a multiple of N in the uniform case.

The next step towards the construction of the Hamiltonian is the discussion of the desired groundstate correlation function. Let us denote by $\psi(z)$ and $\bar{\psi}(z)$ the vector valued primary fields associated with the representations $\mathcal{V}$ and $\bar{\mathcal{V}}$ and the corresponding representation matrices by $\mathbf{T}$ and $\bar{\mathbf{T}}$. We wish to emphasize that $\bar{\psi}$ does not refer to an anti-chiral field but merely to the dual representation. Since we are only dealing with chiral CFTs there should be no chance of confusion. In terms of the general setup the restriction to two types of fields means

$$\psi_i(z) = \begin{cases} \psi(z), & i \in \mathbb{S} \\ \bar{\psi}(z), & i \in \bar{\mathbb{S}} \end{cases} \quad \text{and} \quad \mathbf{S}_i = \begin{cases} \mathbb{I}^{i-1} \otimes \mathbf{T} \otimes \mathbb{I}^{L-i}, & i \in \mathbb{S}, \\ \mathbb{I}^{i-1} \otimes \bar{\mathbf{T}} \otimes \mathbb{I}^{L-i}, & i \in \bar{\mathbb{S}}. \end{cases} \tag{11}$$

[3] In terms of Dynkin labels one has $\mathcal{V} = (1, 0, \dots, 0)$ and $\bar{\mathcal{V}} = (0, \dots, 0, 1)$.

The representation matrices $\mathbf{T}$ and $\bar{\mathbf{T}}$ are related by transposition as $\bar{T}^a = -(T^a)^T$. The ground-state (3) of the desired spin system will be determined by the correlation functions (1) which, for the uniform and the alternating model, become

$$\psi_{\text{uniform}}(z_1,\dots,z_L) = \langle \psi(z_1)\cdots\psi(z_L)\rangle \quad \text{and}$$
$$\psi_{\text{alternating}}(z_1,\dots,z_L) = \langle \psi(z_1)\bar{\psi}(z_2)\cdots\psi(z_{L-1})\bar{\psi}(z_L)\rangle. \tag{12}$$

Both correlators can be evaluated exactly using a free field representation (see Section 2.7). However, in this section we are merely interested in finding the operators $\mathcal{P}_i$ that annihilate these correlators.

It turns out that both of the fields $\psi(w)$ and $\bar{\psi}(w)$ have null descendants on the first energy level. They are obtained by acting on the fields with the current algebra modes

$$J^a_{-1} = \oint_0 \frac{dz}{2\pi i} z^{-1} J^a(z) \tag{13}$$

and performing a suitable projection. In order to find this projection we first list all candidate fields on the first energy level. Since the current $J^a(z)$ is transforming in the adjoint representation $\mathcal{J}$ (which is selfdual, $\mathcal{J} = \bar{\mathcal{J}}$) the potential fields are those in the tensor products $\mathcal{J}\otimes\mathcal{V}$ and $\mathcal{J}\otimes\bar{\mathcal{V}}$, respectively. The decomposition of these tensor products can be established using Young tableaux techniques and it reads[4]

$$\mathcal{J}\otimes\mathcal{V} = \mathcal{V}\oplus\mathcal{N}\oplus\Lambda \quad \text{and} \quad \mathcal{J}\otimes\bar{\mathcal{V}} = \bar{\mathcal{V}}\oplus\bar{\mathcal{N}}\oplus\bar{\Lambda}. \tag{14}$$

As can be inferred from a comparison of conformal dimensions ($h_{\mathcal{N}} = h_{\mathcal{V}} + 1$) or from an explicit construction (cf. [29]), the relevant null fields $\chi(w)$ and $\bar{\chi}(w)$ are associated with the representations $\mathcal{N}$ and $\bar{\mathcal{N}}$, respectively.

Let us now focus our attention onto a fixed single site i, with an insertion of the field $\psi_i(z)$ of type either $\psi(z)$ (for $i\in\mathbb{S}$) or $\bar{\psi}(z)$ (for $i\in\bar{\mathbb{S}}$). It remains to construct the projector $\mathcal{P}_i$ onto the space of null states which can be either of the form $\mathcal{N}\subset\mathcal{J}\otimes\mathcal{V}$ or $\bar{\mathcal{N}}\subset\mathcal{J}\otimes\bar{\mathcal{V}}$. These projections can easily be realized using the action of the quadratic Casimir operator C_i on the relevant tensor product and its known eigenvalues $C_\bullet$ on the irreducible representations in its decomposition. These eigenvalues are identical for dual representations. In both cases one hence obtains

$$\mathcal{P}_i = \frac{(C_i - C_{\mathcal{V}})(C_i - C_{\Lambda})}{(C_{\mathcal{N}} - C_{\mathcal{V}})(C_{\mathcal{N}} - C_{\Lambda})}, \tag{15}$$

as is obvious from restricting C_i to any of the irreducible components appearing in (14). In order to rewrite this expression in terms of spin operators we introduce matrices $\mathbf{t}$ for the adjoint representation and write $C_i = (\mathbf{S}_i + \mathbf{t})^2$.[5] The Casimir eigenvalues $C_{\mathcal{V}}$, $C_{\mathcal{N}}$ and C_{Λ} needed to evaluate the projector (15) are summarized in Table 1. After some elementary algebra one then ends up with

$$\mathcal{P}_i = \frac{1}{2(N+1)}\big[(\mathbf{S}_i\cdot\mathbf{t})^2 + (N+1)\mathbf{S}_i\cdot\mathbf{t} + N\big]. \tag{16}$$

[4] In terms of Dynkin labels one has $\mathcal{J} = (1,0,\dots,0,1)$, $\Lambda = (0,1,0,\dots,1)$ and $\mathcal{N} = (2,0,\dots,0,1)$. The entries are swapped for the dual representations.

[5] Scalar products of spin operators are defined as $\mathbf{t}_1\cdot\mathbf{t}_2 = t_1^a\kappa_{ab}t_2^b$. The square $\mathbf{t}^2$ is an abbreviation for $\mathbf{t}\cdot\mathbf{t}$.

Table 1
Representations and their Casimir eigenvalues. The duals are obtained by swapping the entries of the tuple. They have the same Casimir eigenvalues.

Symbol	Arises in	Dynkin label	Name	Casimir eigenvalue	Interpretation
0	$\mathcal{V}\otimes\bar{\mathcal{V}}$	$(0,\dots,0)$	Trivial	0	
$\mathcal{V}$	$\mathcal{J}\otimes\mathcal{V}$	$(1,0,\dots,0)$	Fundamental	$\frac{1}{N}(N^2-1)$	Physical site
$\mathcal{J}$	$\mathcal{V}\otimes\bar{\mathcal{V}}$	$(1,0,\dots,0,1)$	Adjoint	$2N$	Current modes
$\mathcal{N}$	$\mathcal{J}\otimes\mathcal{V}$	$(2,0,\dots,0,1)$		$\frac{1}{N}(N+1)(3N-1)$	Null field
Λ	$\mathcal{J}\otimes\mathcal{V}$	$(0,1,0,\dots,0,1)$		$\frac{1}{N}(N-1)(3N+1)$	
Ξ	$\mathcal{V}\otimes\mathcal{V}$	$(2,0,\dots,0)$		$\frac{2}{N}(N-1)(N+2)$	
Υ	$\mathcal{V}\otimes\mathcal{V}$	$(0,1,0,\dots,0)$		$\frac{2}{N}(N-2)(N+1)$	

The expression for the projector (16) can be simplified by noting that the unit matrix $\mathbb{I}$ together with the N^2-1 spin matrices $\mathbf{S}_i$ span the full space of $N\times N$ matrices available on site i. As a consequence, bilinears in $\mathbf{S}_i$ can be reduced in degree using the identity

$$S_i^a S_i^b = \frac{1}{2}\big[S_i^a, S_i^b\big] + \frac{1}{2}\big\{S_i^a, S_i^b\big\} = \frac{i}{2} f^{ab}{}_c S_i^c - \frac{1}{2}(-1)^{d_i} d^{ab}{}_c S_i^c + \frac{1}{N}\kappa^{ab}\mathbb{I}. \tag{17}$$

In order to distinguish between the two types of representation matrices $\mathbf{T}$ and $\bar{\mathbf{T}}$ which could enter here we used the parity map $d_\bullet : \mathbb{L}\to\mathbb{Z}_2$ which was introduced below (9). Eq. (17) can be read as the defining relation of the completely symmetric rank-three tensor $d^{ab}{}_c$ which, moreover, is traceless. More details about the definition and the properties of the tensors f, d and κ can be found in Appendix A. Using the product formula (17) and the explicit matrices $(t^a)^b{}_c = -i f^{ab}{}_c$ for the adjoint representation it is now possible to derive an "irreducible" formula for the projector (16). Employing the formulas listed in Appendix A it is straightforward even though slightly lengthy to verify that

$$[\mathbf{t}\cdot\mathbf{S}_i]^a{}_b = -i f^a{}_{bc} S_i^c \quad\text{and}\quad \big[(\mathbf{t}\cdot\mathbf{S}_i)^2\big]^a{}_b = \frac{iN}{2} f^a{}_{bc} S_i^c - \frac{N}{2}(-1)^{d_i} d^a{}_{bc} S_i^c + 2\delta_b^a\mathbb{I}. \tag{18}$$

Adding up all contributions with the correct coefficients, we find

$$\mathcal{P}_i^a{}_b = -\frac{i}{4}\frac{N+2}{N+1} f^a{}_{bc} S_i^c - (-1)^{d_i}\frac{N}{4(N+1)} d^a{}_{bc} S_i^c + \frac{N+2}{2(N+1)}\delta_b^a\mathbb{I}. \tag{19}$$

We note that $\mathcal{P}_i$ is a hermitean operator-valued matrix which satisfies the projector property $\mathcal{P}_i^2 = \mathcal{P}_i$.

2.3. Derivation of the quantum spin Hamiltonians

The projectors in the previous subsection may be used to construct operators that annihilate correlation functions of the form (1). The starting point is the chiral correlator

$$0 = \big\langle \psi_1(z_1)\cdots\chi_i(z_i)\cdots\psi_L(z_L)\big\rangle, \tag{20}$$

which is obtained from (1) by replacing the field $\psi_i(z_i)$ on site z_i by its associated null field $\chi_i(z_i)$. The null field $\chi_i(z_i)$ can be identified with the fields in the subspace $\mathcal{N}\subset\mathcal{J}\otimes\mathcal{V}$ (or $\bar{\mathcal{N}}\subset\mathcal{J}\otimes\bar{\mathcal{V}}$), where the tensor product is spanned by fields of the form $J^a_{-1}\psi_i(z_i)$. Formally, this

amounts to replacing the matrices t^a by the operators J^a_{-1}. Using the projector (19), the previous equation may then be rewritten as

$$0 = \mathcal{P}^a_{i\ b}\big\langle\psi_1(z_1)\cdots\big[J^b_{-1}\psi_i(z_i)\big]\cdots\psi_L(z_L)\big\rangle. \tag{21}$$

We wish to stress that, while implicit, the operator $\mathcal{P}_i$ still contains the spin operator $\mathbf{S}_i$ acting on the field $\psi_i(z_i)$ on site i. Next we employ the affine Ward identity

$$\big\langle\psi_1(z_1)\cdots\big[J^b_{-1}\psi_i(z_i)\big]\cdots\psi_L(z_L)\big\rangle = \sum_{j(\neq i)}\frac{S^b_j}{z_i - z_j}\big\langle\psi_1(z_1)\cdots\psi_i(z_i)\cdots\psi_L(z_L)\big\rangle \tag{22}$$

in order to move the action of J^b_{-1} to the other fields in the chiral correlation function. Here and below the sum is not performed over the indices appearing in parentheses. It originates from combining equation (13) for the modes of the current with the definition (8) of primary fields. As a result, we can re-interpret the trivial equation (20) as the following algebraic condition on the original correlator:

$$\mathcal{P}^a_i\big(\{z_l\}\big)\big\langle\psi_1(z_1)\cdots\psi_L(z_L)\big\rangle = 0 \quad \text{with } \mathcal{P}^a_i\big(\{z_l\}\big) = \sum_{j(\neq i)}\frac{\mathcal{P}^a_{i\ b}S^b_j}{z_i - z_j}. \tag{23}$$

We note in passing that these operators somewhat resemble the Gaudin Hamiltonians [1]. It should be stressed, however, that the operators (23) still carry an adjoint index instead of merely implementing an SU(N)-invariant spin–spin coupling.

In order to build an SU(N)-invariant operator from $\mathcal{P}^a_i(\{z_l\})$ we need to form bilinears and contract the index a. However, depending on the choice of parameters z_i this may not result in a hermitean operator. This situation may be cured by conjugating one of the two operators before performing the contraction [20]. This procedure results in a family of Hamiltonians

$$H\big(\{z_l\}\big) = \sum_k \mathcal{P}^\dagger_{k,a}\big(\{z_l\}\big)\mathcal{P}^a_k\big(\{z_l\}\big) = \sum_k\sum_{i,j(\neq k)}\frac{S^a_i\mathcal{P}_{k,ab}S^b_j}{(\bar z_k - \bar z_i)(z_k - z_j)}, \tag{24}$$

parametrized by the fixed but arbitrary positions $z_i \in \mathbb{C}$ of the spins. During the substitution we used the property that the matrices $\mathcal{P}_i$ of Eq. (19) are hermitean projectors. By construction, the resulting Hamiltonians (24) are hermitean, SU(N)-invariant and positive semi-definite. In addition, they annihilate the wave function $\langle\psi_1(z_1)\cdots\psi_L(z_L)\rangle$ (assuming that the latter is non-trivial). For some purposes, it will be convenient to replace the operators $\mathcal{P}^a_i(\{z_l\})$ by expressions with a slightly modified dependence on the coordinates z_i. The precise details and the motivation for this substitution will be explained in Section 2.4.

In the case under investigation (with level $k = 1$), the fusion of all WZW primary fields is abelian and described by the group $\mathbb{Z}_N$. The representations $\mathcal{V}$ and $\bar{\mathcal{V}}$ correspond to the charges 1 and -1 (modulo N), respectively. The space of conformal blocks corresponding to the general mixed setup described in (9) is thus one-dimensional if $L - 2\ell \equiv 0 \bmod N$ and trivial otherwise. For the uniform spin model on the Hilbert space $\mathcal{V}^{\otimes L}$ we therefore expect a unique zero energy groundstate if L is a multiple of N.[6] In contrast, for the alternating case there is always a unique zero energy groundstate.

[6] For different L the Hamiltonian of course still exists but its groundstate(s) have neither zero energy nor are they given in terms of the chiral correlators (1).

2.4. Modifications of the Hamiltonian

It was shown in [20] in the case of SU(2) that the Hamiltonian (24) is closely related to a Haldane–Shastry Hamiltonian provided one replaces the operators $\mathcal{P}_i^a(\{z_l\})$ by new operators of the form

$$\mathcal{C}_i^a\big(\{w_l\}\big) = \sum_{j(\neq i)} w_{ij} \mathcal{P}_i^a{}_b S_j^b \tag{25}$$

and chooses the special values $w_{ij} = (z_i + z_j)/(z_i - z_j)$ for the parameters. We now briefly discuss in which sense such a substitution is also possible for SU(N) and of what form such modifications may generally be.

The possible alterations we have an mind are based on the following observation. Every singlet wave function $|\psi\rangle$ satisfies the equation (for arbitrary but fixed i)

$$0 = \sum_{j(\neq i)} \mathcal{P}_i^a{}_b S_j^b |\psi\rangle. \tag{26}$$

Once more it is straightforward to verify this equation using the relations summarized in Appendix A. Any singlet solution to $\mathcal{P}_i^a(\{z_l\})|\psi\rangle = 0$ will thus also be a solution to $\mathcal{C}_i^a(\{w_l\})|\psi\rangle = 0$ for any

$$w_{ij} = \frac{f(z_i)}{z_i - z_j} + g(z_i), \tag{27}$$

and an arbitrary choice of $f(z)$ and $g(z)$.[7] The particular setup discussed in [20] is based on the choice $f(z_i) = 2z_i$ and $g(z_i) = -1$.

The Hamiltonians we shall consider in this article are all of the form

$$H = \sum_k \mathcal{C}_{k,a}^\dagger\big(\{z_l\}\big)\mathcal{C}_k^a\big(\{z_l\}\big) = \sum_k \sum_{i,j(\neq k)} \bar{w}_{ki} w_{kj} S_i^a \mathcal{P}_{k,ab} S_j^b, \tag{28}$$

where $\mathcal{C}_k^a(\{z_l\})$ is defined in (25) with parameters w_{ij} as defined in (27). A priori, it is not clear whether the transition from $w_{ij} = 1/(z_i - z_j)$ to a more general setup is modifying the basic physical properties in the thermodynamic limit. This may concern a potential criticality of a 1D system or the statistics of anyonic excitations in a potential gapped chiral 2D topological phase. Let us, however, stress that any singlet groundstate of the modified Hamiltonian is still unique (as a zero energy *singlet*) since the procedure above can of course always be reversed.[8]

2.5. Simplification of the general Hamiltonian

Before discussing particular setups we would like to simplify the general Hamiltonian of the form (28). Plugging in the concrete expression (19) for $\mathcal{P}_{ab}$ we find

[7] The functions f and g can, in principle, also be chosen differently for each value of i.

[8] We note in passing that an undesired effect of passing from $w_{ij} = 1/(z_i - z_j)$ to $w_{ij} = (z_i + z_j)/(z_i - z_j)$ becomes visible for $L = 2$ and $z_2 = -z_1$. In that case $w_{12} = 0$ and the Hamiltonian (28) vanishes identically.

$$H = \sum_k \sum_{i,j(\neq k)} \bar{w}_{ki} w_{kj} \left\{ -\frac{i}{4} \frac{N+2}{N+1} f_{abc} S_i^a S_j^b S_k^c - \frac{N(-1)^{d_k}}{4(N+1)} d_{abc} S_i^a S_j^b S_k^c + \frac{N+2}{2(N+1)} \mathbf{S}_i \cdot \mathbf{S}_j \right\}. \tag{29}$$

In order to simplify this expression further, we split the sum into contributions with $i = j$ and others with $i \neq j$. Whenever $i = j$ we can use the identities (17) and (122) in order to reduce the degree employing the relations

$$f_{abc} S_i^a S_i^b S_k^c = iN \mathbf{S}_i \cdot \mathbf{S}_k \quad \text{and} \quad d_{abc} S_i^a S_i^b S_k^c = -(-1)^{d_i} \frac{N^2-4}{N} \mathbf{S}_i \cdot \mathbf{S}_k. \tag{30}$$

After some simple algebra this results in[9]

$$\begin{aligned} H = {} & \sum_{i \neq k} |w_{ki}|^2 \left\{ \frac{N+2}{4(N+1)} \left[N + (-1)^{d_i + d_k} (N-2) \right] \mathbf{S}_i \cdot \mathbf{S}_k + \frac{(N+2)(N-1)}{2N} \right\} \\ & + \sum_{i \neq j \neq k} \bar{w}_{ki} w_{kj} \left\{ -\frac{i}{4} \frac{N+2}{N+1} f_{abc} S_i^a S_j^b S_k^c - \frac{(-1)^{d_i} N}{4(N+1)} d_{abc} S_i^a S_j^b S_k^c \right. \\ & \left. + \frac{N+2}{2(N+1)} \mathbf{S}_i \cdot \mathbf{S}_j \right\}. \end{aligned} \tag{31}$$

Splitting the contributions according to the order of the interaction, this can be rewritten as

$$\begin{aligned} H = {} & \frac{(N+2)(N-1)}{2N} \sum_{i \neq j} |w_{ij}|^2 \\ & + \frac{N+2}{4(N+1)} \sum_{i \neq j} \left\{ \left[N + (-1)^{d_i + d_j} (N-2) \right] |w_{ji}|^2 + 2 \sum_{k(\neq i,j)} \bar{w}_{ki} w_{kj} \right\} \mathbf{S}_i \cdot \mathbf{S}_j \\ & + \frac{1}{4(N+1)} \sum_{i \neq j \neq k} \bar{w}_{ki} w_{kj} \left\{ -i(N+2) f_{abc} S_i^a S_j^b S_k^c - N(-1)^{d_k} d_{abc} S_i^a S_j^b S_k^c \right\}. \end{aligned} \tag{32}$$

Using the symmetry properties of the tensors f_{abc} and d_{abc} we can achieve a final simplification. Restricting the summation to $i < j < k$ and adding the missing permutations by hand, we can express the Hamiltonian in terms of the two quantities

$$\begin{aligned} \Omega^T_{ijk} &= (-1)^{d_i} [\bar{w}_{ij} w_{ik} + \bar{w}_{ik} w_{ij}] + (-1)^{d_j} [\bar{w}_{jk} w_{ji} + \bar{w}_{ji} w_{jk}] \\ &\quad + (-1)^{d_k} [\bar{w}_{ki} w_{kj} + \bar{w}_{kj} w_{ki}] \\ &= 2 \operatorname{Re} \left[(-1)^{d_i} \bar{w}_{ij} w_{ik} + (-1)^{d_j} \bar{w}_{jk} w_{ji} + (-1)^{d_k} \bar{w}_{ki} w_{kj} \right], \\ \Omega^A_{ijk} &= -i (\bar{w}_{ij} w_{ik} - \bar{w}_{ik} w_{ij} - \bar{w}_{ji} w_{jk} - \bar{w}_{kj} w_{ki} + \bar{w}_{jk} w_{ji} + \bar{w}_{ki} w_{kj}) \\ &= 2 \operatorname{Im} (\bar{w}_{ij} w_{ik} + \bar{w}_{jk} w_{ji} + \bar{w}_{ki} w_{kj}). \end{aligned} \tag{33}$$

The superscripts stand for "twisted (symmetrized)" and "anti-symmetrized", respectively. With these definitions one immediately finds

[9] The summation range in the second sum is an abuse of notation. What is meant is that all indices are different.

$$H = \frac{(N+2)(N-1)}{2N} \sum_{i\neq j} |w_{ij}|^2 + \frac{N+2}{4(N+1)} \sum_{i\neq j} \left\{ \left[N + (-1)^{d_i+d_j}(N-2) \right] |w_{ji}|^2 + 2 \sum_{k(\neq i,j)} \bar{w}_{ki} w_{kj} \right\} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{4(N+1)} \sum_{i<j<k} \left\{ (N+2) \Omega^A_{kij} f_{abc} S^a_i S^b_j S^c_k - N \Omega^T_{kij} d_{abc} S^a_i S^b_j S^c_k \right\}. \tag{34}$$

The resulting Hamiltonian involves long-ranged two-spin and three-spin interactions coupling every site with every other site. As it stands it is still valid for general choices of w_{kl} and parameters z_k (see (27)), including 2D spin systems. For this reason, there is hardly any hope for succeeding with an analytical treatment beyond writing down the exact groundstate (see Section 2.7). In contrast, drastic simplifications can be expected in case the quantities Ω^T_{ijk} and Ω^A_{ijk} both vanish or are at least constant (i.e. independent of the indices ijk). This precisely occurs for specific types of 1D setups which will now be discussed in more detail. In that case, the three-spin couplings can be rewritten in terms of the *total* spin, i.e. they basically decouple from the local dynamics.

2.6. Discussion of special setups

The Hamiltonians derived in the previous subsection make sense for arbitrary parameters $z_i \in \mathbb{C}$ on the complex plane. Mostly, we will however be interested in quite particular spin locations which lead to considerable simplifications of the Hamiltonians. After briefly discussing general aspects of the freedom of choice we will present a few concrete and physically relevant examples that will be used in subsequent sections.

2.6.1. General aspects

As we have seen in Section 2.5, the two main parameters governing the complexity of the Hamiltonians are the quantities Ω^T_{ijk} and Ω^A_{ijk} that have been defined in Eq. (33). A general Hamiltonian of the form (34) will always involve three-spin interactions. These three-spin interactions can, however, be rewritten in terms of the *total spin* in case the two quantities Ω^T_{ijk} and Ω^A_{ijk} defined in (33) are constant (or even vanishing). This leads to drastic simplifications and allows to relate the Hamiltonians (34) to more familiar quantum systems such as the Haldane–Shastry model for specific choices of parameters. We shall present some physically relevant examples in the subsequent sections.

Before diving into concrete models we wish to summarize a few general properties of the two assignments we shall mainly be concerned with. These are

$$\text{(a)} \quad w_{ij} = \frac{1}{z_i - z_j} \quad \text{and} \quad \text{(b)} \quad w_{ij} = \frac{z_i + z_j}{z_i - z_j}. \tag{35}$$

In both cases one has the property $w_{ij} = -w_{ji}$. In addition, there are a number of non-trivial identities which, however, depend on the particular case under consideration. In particular, case (a) leads to

$$\Delta_{ijk} = w_{ij} w_{ik} + w_{jk} w_{ji} + w_{ki} w_{kj} = 0 \quad \text{(case (a))}. \tag{36}$$

Similarly, case (b) has the immediate but important consequence

$$\Delta_{ijk} = w_{ij}w_{ik} + w_{jk}w_{ji} + w_{ki}w_{kj} = 1 \quad \text{(case (b))}. \tag{37}$$

It may be shown that the parametrization (b) of w_{kl} in terms of the variables z_i is the unique solution to this equation [5]. In case (b) it is, moreover, possible to simplify squares in view of the relation

$$w_{kl}^2 = 1 + 4\frac{z_k z_l}{z_{kl}^2} \quad \text{with } z_{kl} = z_k - z_l \quad \text{(for case (b))}. \tag{38}$$

The importance of the quantity Δ_{ijk} and the relations (36) and (37) stems from the fact that Ω^T_{ijk} essentially reduces to Δ_{ijk} (up to the signs) for either purely real or purely imaginary values of w_{ij} (such that $\bar{w}_{ij}$ can be replaced by w_{ij} up to a sign). In both of these cases one, in addition, has $\Omega^A_{ijk} = 0$, a relation which even holds for mixed spin models. We thus expect significant simplifications of the Hamiltonian (34) for both of the cases (35) provided the parameters w_{ij} satisfy these extra conditions.

We conclude the general discussion with an analysis of conditions which enforce all w_{kl} to be either real or imaginary for the two specific choices listed in (35). We start with case (a) where we find

$$\mathrm{Re}(w_{kl}) = \frac{\mathrm{Re}(z_k - z_l)}{|z_k - z_l|^2} \quad \text{and} \quad \mathrm{Im}(w_{kl}) = -\frac{\mathrm{Im}(z_k - z_l)}{|z_k - z_l|^2} \quad \text{(case (a))}. \tag{39}$$

As a consequence we expect simplifications in case a) if the spins are positioned along a horizontal or a vertical line. We note that the distance of these lines from the origin does not matter since it will cancel out when passing from z_k to w_{kl}. In case (b) of (35) one similarly obtains

$$\mathrm{Re}(w_{kl}) = \frac{|z_k|^2 - |z_l|^2}{|z_k - z_l|^2} \quad \text{and} \quad \mathrm{Im}(w_{kl}) = -2\frac{\mathrm{Im}(z_k \bar{z}_l)}{|z_k - z_l|^2} \quad \text{(case (b))}. \tag{40}$$

The general solution to $\mathrm{Re}(w_{kl}) = 0$ is thus $z_k = re^{i\theta_k}$, for arbitrary (real) values of θ_k, i.e. the spins need to be located on a circle. Note that the resulting value of w_{kl} does not depend on the choice of radius r. The general solution to $\mathrm{Im}(w_{kl}) = 0$ requires all the z_k to have the same phase (up to π), i.e. they should all be located on the same line through the origin. After having addressed potential simplifications in some detail we are now going to discuss particular setups in which they are realized.

2.6.2. *Spins on the circle*

As was found in [20] for SU(2) and motivated more generally in Section 2.6.1, drastic simplifications occur if the spins are located on a circle

$$z_k = re^{i\theta_k} \quad \text{together with the choice} \quad w_{kl} = \frac{z_k + z_l}{z_k - z_l} = -i\cot\frac{1}{2}(\theta_k - \theta_l). \tag{41}$$

In this case one obtains $\bar{w}_{kl} = -w_{kl}$ and hence $\Omega^A_{ijk} = 0$. We note a possible relation to the trigonometric Haldane–Shastry model in view of the relation

$$\frac{z_k z_l}{z_{kl}^2} = -\frac{1}{4\sin^2\frac{1}{2}(\theta_k - \theta_l)}. \tag{42}$$

Also the equation $\Delta_{ijk} = 1$ (see (37)) has a number of consequences. Using the antisymmetry $w_{kl} = -w_{lk}$, one for instance finds

$$\sum_{k(\neq i,j)} w_{ki}w_{kj} = L - 2 + 2w_{ij}^2 - w_{ij}(\xi_i - \xi_j) \quad \text{with} \quad \xi_i = \sum_{k(\neq i)} w_{ik}. \tag{43}$$

Note that the convention for ξ_i used here is different from the convention used for c_i in [20].

2.6.3. Equidistant distribution of spins on the circle

For physical applications the most important choice of spin locations is the equidistant distribution on the circle. In the language of Section 2.6.2 this corresponds to $\theta_k = \frac{2\pi}{L}k$ such that one has

$$z_k = re^{\frac{2i\pi}{L}k} \quad \text{and} \quad w_{kl} = -i\cot\frac{\pi}{L}(k-l). \tag{44}$$

One of the technical advantages of the equidistant distribution is the fact that certain summations can now be carried out explicitly. For instance one finds

$$\xi_i = \sum_{k(\neq i)} w_{ik} = 0 \quad \text{and (for even } L) \quad \sum_{k(\neq i)} (-1)^k w_{ik} = 0. \tag{45}$$

Other important sums which can be evaluated using these insights are

$$\sum_{i\neq j} |w_{ij}|^2 = \frac{1}{3}L(L-1)(L-2) \quad \text{and} \quad \sum_{i\neq j\neq k} \bar{w}_{ki} w_{kj} = -\frac{1}{3}L(L-1)(L-2). \tag{46}$$

2.6.4. Spins on the real line

A case that was not studied in [20] but which appears to be of similar interest is the case of real values $z_k = x_k \in \mathbb{R}$ together with the unmodified choice a) of w_{kl} (see (35)),

$$w_{kl} = \frac{1}{z_k - z_l} = \frac{1}{x_k - x_l}. \tag{47}$$

This case leads to a significant simplification of the system since now $\Omega^T_{ijk} = \Omega^A_{ijk} = \Delta_{ijk} = 0$ in the uniform case. For the Haldane–Shastry model, one may regard this setup as the classical limit of the chain on the circle, see [8] for a more detailed discussion of this point.

2.6.5. Hyperbolic case

Just for completeness we also briefly describe the hyperbolic case where the spin locations are chosen to be on the real line with

$$z_k = re^{\omega_k} \quad \text{together with the choice} \quad w_{kl} = \frac{z_k + z_l}{z_k - z_l} = \coth\frac{1}{2}(\omega_k - \omega_l). \tag{48}$$

In this equation, the ω_k are meant to be arbitrary real parameters. Formally, the assignment (48) corresponds to the choice $\theta_k = -i\omega_k$ in the discussion of Section 2.6.2. Correspondingly, Eq. (42) now gets replaced by

$$\frac{z_k z_l}{z_{kl}^2} = \frac{1}{4\sinh^2\frac{1}{2}(\omega_k - \omega_l)}. \tag{49}$$

Even though of limited physical interest we decided to include this case since the associated Haldane–Shastry model exhibits a Yangian symmetry [5]. For the Yangian symmetry to be present one needs to work with a uniform setup and spin locations $\omega_k = \alpha k$ for some arbitrary constant $\alpha \in \mathbb{R}$. In particular, this requires an infinite number of sites right from the very beginning.

2.7. Groundstate wavefunctions

We have seen in Section 2.2 that the groundstates of the Hamiltonians (34) are given in terms of WZW correlators (12). We will now calculate these correlators using the vertex operator realization of $\mathrm{SU}(N)_1$. In the alternating case we shall also employ a free fermion construction. The resulting wave functions are always of Gutzwiller–Jastrow type.

2.7.1. Vertex operator construction

The correlation functions (12) entering the groundstate of our physical system can be evaluated explicitly, thanks to the fact that the $\mathrm{SU}(N)_1$ WZW model is equivalent to a free field theory. Indeed, it simply corresponds to a system of $N-1$ free bosons which are compactified on the root lattice of $\mathfrak{su}(N)$. Accordingly, the WZW currents and also the WZW primary fields can be expressed in terms of these free fields. The $N-1$ Cartan operators just correspond to derivatives of the $N-1$ free bosons. On the other hand, vertex operators are required to represent root operators and primary fields. While the corresponding vertex operators can easily be identified on the basis of their conformal dimension, there is a certain subtlety regarding cocycle phase factors which are need to ensure the correct statistics of fields. Since in our approach relative phases have a drastic influence on the state (1), it is important to get these phases right.

Let us start with defining a multi-component chiral bosonic field $\varphi^i(z)$ using the OPE

$$\varphi^i(z)\varphi^j(w) = -\delta^{ij}\ln(z-w). \tag{50}$$

The derivatives $H^i(z) = i\partial\varphi^i(z)$ generate a $\mathrm{U}(1)^{N-1}$ current algebra

$$H^i(z)H^j(w) = \frac{\delta^{ij}}{(z-w)^2}. \tag{51}$$

The associated primary fields are vertex operators

$$V_\mu(z) = :e^{i\mu\cdot\varphi(z)}: \tag{52}$$

which are labeled by $(N-1)$-tuples μ and which have the conformal dimension $h_\mu = \frac{1}{2}\mu^2$. For our purposes it will be useful to identify the tuple μ with weights of $\mathfrak{su}(N)$.

We recognize that the vertex operators $V_\alpha(z)$ associated with the roots α have conformal dimension $h=1$ due to $\alpha^2 = 2$. They may be used to extend the free boson chiral algebra to $\mathrm{SU}(N)_1$. The concrete expression for the root generators is

$$E^\alpha(z) = c_\alpha V_\alpha(z) \tag{53}$$

where c_α is a $\mathbb{Z}_2$-valued cocycle ensuring the correct statistics of the currents (see e.g. [29]). In our context, more important than the currents are the WZW primary fields associated with the fundamental and the anti-fundamental representation. The latter are known to have conformal dimension $h_{\mathcal{V}} = h_{\bar{\mathcal{V}}} = \frac{N-1}{2N}$. They are realized in terms of vertex operators $V_\mu(z)$ where μ is any weight of the corresponding representations. Indeed, the length of the corresponding weights is given by $\mu^2 = \frac{N-1}{N}$, in accordance with our previous claim about the conformal dimension. The fundamental WZW primaries admit a representation as

$$\psi_q(z) = c_{\mu(q)}V_{\mu(q)}(z) \quad \text{and} \quad \bar{\psi}_q(z) = c_{\bar{\mu}(q)}V_{\bar{\mu}(q)}(z) \tag{54}$$

with another cocycle c_μ (see [30,29]). The cocycle depends on the indices q through their respective weight $\mu(q)$ or $\bar{\mu}(q)$ in the fundamental or anti-fundamental representation. The latter may be written as

$$\mu(q) = \omega_1 - \sum_{r=1}^{q-1} \alpha_r \quad \text{and} \quad \bar{\mu}(q) = \omega_{N-1} - \sum_{r=1}^{q-1} \alpha_{N-r}. \tag{55}$$

In this formula, α_i denote the simple roots of $\mathfrak{su}(N)$ while ω_1 and ω_{N-1} refer to the highest weights of the fundamental and anti-fundamental representations $\mathcal{V}$ and $\bar{\mathcal{V}}$.

The primary fields $\psi_q(z)$ and $\bar{\psi}_q(z)$ define correlation functions of the form (12) and thereby the desired quantum state

$$|\psi\rangle = \sum_{\{q_i\}} \psi_{q_1 \cdots q_L}(z_1, \ldots, z_L) |q_1 \cdots q_L\rangle. \tag{56}$$

Up to a coordinate independent sign stemming from the cocycles c_μ, all relevant correlation functions can easily be calculated using the free field expression

$$\left\langle V_{\mu_1}(z_1) \cdots V_{\mu_L}(z_L) \right\rangle = \delta_{\mu,0} \prod_{i<j} (z_i - z_j)^{\mu_i \cdot \mu_j}. \tag{57}$$

The Kronecker delta $\delta_{\mu,0}$ with $\mu = \sum_i \mu_i$ results from charge conservation. In the uniform case, it forces all correlation functions to vanish except if L is a multiple of N. In the alternating case, it is sufficient for L to be even. We shall now present two alternative ways for the explicit construction of the states (56).

2.7.2. *Determination of the sign factors*

Let us focus on the uniform case first where all fields transform in the fundamental representation. The relevant correlation function then reads

$$\psi_{q_1 \cdots q_L}(z_1, \ldots, z_L) = \delta_{\mu(q),0} e^{if(\{q_i\})} \prod_{i<j} (z_i - z_j)^{\mu(q_i) \cdot \mu(q_j)}, \tag{58}$$

and the non-trivial task consists in determining the sign factor $e^{if(\{q_i\})}$. There are at least three distinct ways of accomplishing this. First of all, the sign can be determined through a detailed analysis of the cocycles entering the definition (54) of the WZW primaries. Alternatively, the relative signs are fixed by the invariance of the state (56) under the global action of SU(N). Here we shall follow an even simpler route which has been suggested in [31]. It employs the fact that singlet wavefunctions of the type (58) previously arose in the context of the SU(N) Haldane–Shastry model [10]. There is just one slight difference to our setup: we are interested in general locations of the spins while the coordinates z_k are distributed uniformly on the circle for the Haldane–Shastry model, $z_k = e^{2\pi i k/L}$. Since the sign factors in (58) do not depend on these coordinates they may nevertheless be obtained by means of a simple comparison.

The groundstate of the SU(N) Haldane–Shastry model is described in terms of the wave function [10]

$$\psi_G\left(n_i^{(a)}\right) = e^{-i\pi \sum_{i,a} n_i^{(a)}} \prod_{a,i<j} D\left(n_i^{(a)} - n_j^{(a)}\right)^2 \prod_{a<b,i,j} D\left(n_i^{(a)} - n_j^{(b)}\right) \tag{59}$$

with $D(x-y) = \sin(\pi(x-y)/L)$. Here $a = 2, 3, \ldots, N$ and $n_i^{(a)}$ is the position of the i-th site with spin a. The state is obtained from a Gutzwiller projection and hence a singlet by construction. Assuming neutrality of the configurations and noting that $\mu(q)^2 = (N-1)/N$ for every $q = 1, \ldots, N$, one gets

$$\begin{aligned}\prod_{n<m}(z_n-z_m)^{\mu(q_n)\cdot\mu(q_m)} &= \prod_{n<m}\left[2ie^{i\pi(n+m)/L}D(n-m)\right]^{\mu(q_n)\cdot\mu(q_m)}\\ &= e^{\frac{1}{2}(\sum_{n,m}-\sum_{n,m}\delta_{mn})(\log(2i)+\frac{i\pi}{L}(n+m))\mu(q_n)\cdot\mu(q_m)}\prod_{n<m}D(n-m)^{\mu(q_n)\cdot\mu(q_m)}\\ &= e^{-\frac{N-1}{2N}(\log(2i)L+i\pi(L+1))}\prod_{n<m}D(n-m)^{\mu(q_n)\cdot\mu(q_m)} \equiv C_L\prod_{n<m}D(n-m)^{\mu(q_n)\cdot\mu(q_m)}.\end{aligned} \tag{60}$$

Relating this expression to the Gutzwiller wavefunction (59) is a simple exercise. Note first the following fact:

$$\big(\mu(p)-\omega_1\big)\cdot\big(\mu(q)-\omega_1\big) = \sum_{r=1}^{p-1}\sum_{s=1}^{q-1}A_{rs} = \begin{cases}0 & \text{if } q=1 \text{ or } p=1,\\ 2 & \text{if } q=p\neq 1,\\ 1 & \text{otherwise,}\end{cases} \tag{61}$$

where A_{rs} is the Cartan matrix of $\mathfrak{su}(N)$. This allows us to rewrite ψ_G as

$$\psi_G\big(n_i^{(a)}\big) = \delta_{\mu(q),0}\,e^{-i\pi\sum_{i,a}n_i^{(a)}}\prod_{n<m}\left[(-1)^{\theta(q_n-q_m)}D(n-m)\right]^{(\mu(q_n)-\omega_1)\cdot(\mu(q_m)-\omega_1)}. \tag{62}$$

The step function satisfies $\theta(p-q)=1$ if $p>q$ and 0 otherwise. It arises from the condition $a<b$ in Eq. (59), meaning that if $q_n>q_m$, that term has an additional minus sign. Note that if $N=2$ the exponent is even, so that this extra sign is not present. Still assuming charge neutrality we now further note that

$$\begin{aligned}&\prod_{n<m}D(n-m)^{-\omega_1\cdot(\mu(q_n)+\mu(q_m))}\\ &\quad=\prod_{n>m}(-1)^{-\omega_1\cdot\mu(q_n)}\prod_{n\neq m}D(n-m)^{-\omega_1\cdot\mu(q_n)}\\ &\quad=\prod_n(-1)^{-\omega_1\cdot\mu(q_n)(n-1)}\prod_n\left[(-1)^{L-n}2^{1-L}L\right]^{-\omega_1\cdot\mu(q_n)}=1.\end{aligned} \tag{63}$$

As a consequence, the wavefunction simplifies to

$$\begin{aligned}\psi_G\big(n_i^{(a)}\big) = \tilde{C}_L\delta_{\mathbf{q},0}e^{-i\pi\sum_{i,a}n_i^{(a)}}&\prod_{n<m}(-1)^{\theta(q_n-q_m)(\mu(q_n)-\omega_1)\cdot(\mu(q_m)-\omega_1)}\\ &\times\prod_{n<m}D(n-m)^{\mu(q_n)\cdot\mu(q_m)},\end{aligned} \tag{64}$$

where we introduced the constant $\tilde{C}_L=\prod_{n<m}D(n-m)^{\omega_1^2}$ and $\omega_1^2=(N-1)/N$. The sign factor may now be fixed by demanding that the previous expression equals the chiral correlator (58) when $z_n=e^{2\pi in/L}$. We then find

$$\begin{aligned}e^{if(\{q_i\})} &= e^{-i\pi\sum_{i,a}n_i^{(a)}}\prod_{n<m}(-1)^{\theta(q_n-q_m)(\mu(q_n)-\omega_1)\cdot(\mu(q_m)-\omega_1)}\\ &=\prod_n e^{-i\frac{\pi}{2}n(\mu(q_n)-\omega_1)^2}\prod_{n<m}(-1)^{\theta(q_n-q_m)(\mu(q_n)-\omega_1)\cdot(\mu(q_m)-\omega_1)}.\end{aligned} \tag{65}$$

For the alternating setup involving correlation functions of both the fundamental and the anti-fundamental field the previous trick is not applicable and one would need to understand the cocycle properties in more detail. We shall employ a shortcut in that case, employing free fermions instead of free bosons.

2.7.3. Free fermion construction

In the case of the alternating spin model there is an alternative perspective on the derivation of the groundstate wavefunction which we find worth mentioning. Namely, the fundamental representation $\mathcal{V}$ of $\mathrm{SU}(N)$ can be interpreted as an N-dimensional representation of $\mathrm{U}(N)$ on which its $\mathrm{U}(1)$ subgroup acts trivially. This has to be distinguished from the fundamental representation $\mathcal{V}_Q$ of $\mathrm{U}(N)$ which carries a non-trivial $\mathrm{U}(1)$ charge Q. Similar arguments apply to the anti-fundamental representation $\bar{\mathcal{V}}_Q$ which carries a $\mathrm{U}(1)$ charge $-Q$. Since the $\mathrm{U}(1)$ charges simply add up in tensor product, one has the identity $\mathcal{V}\otimes\bar{\mathcal{V}} = \mathcal{V}_Q\otimes\bar{\mathcal{V}}_Q$ where both sides can now be regarded as representations of $\mathrm{SU}(N)$.[10]

The previous arguments can be lifted to the level of WZW theories. The great advantage of this re-interpretation is that the $\mathrm{U}(N)_1$ WZW model admits a representation in terms of N complex fermions with non-trivial OPE[11]

$$\Psi_p(z)\bar{\Psi}^q(w) = \frac{\delta_p^q}{z-w} \quad (\text{with } p,q = 1,\dots,N). \tag{66}$$

The currents $J_p{}^q(z) = :\Psi_p\bar{\Psi}^q:(z)$ are simply bilinears in these fermions. More importantly, it is easy to verify that the fields $\Psi_p(z)$ and $\bar{\Psi}^q(z)$ are WZW primary fields with $h = 1/2$ and that they correspond to the fundamental and anti-fundamental representation of $\mathrm{U}(N)$, respectively.

From the perspective of the $\mathrm{SU}(N)_1$ WZW model one can reconstruct the $\mathrm{U}(N)_1$ WZW model by extending it by a free field $\varphi(z)$ generating the extra $\mathrm{U}(1)$. The associated $\mathrm{U}(1)$ charges $\pm Q$ are carried by vertex operators $V_\pm(z)$ of this bosonic field. In this language one can then realize the fundamental $\mathrm{U}(N)$ fields $\Psi_p(z)$ and $\bar{\Psi}^q(z)$ in terms of the fundamental $\mathrm{SU}(N)$ fields $\psi_p(z)$ and $\bar{\psi}^q(z)$ as

$$\Psi_q(z) = V_+\psi_q(z) \quad \text{and} \quad \bar{\Psi}^q(z) = V_-\bar{\psi}^q(z). \tag{67}$$

The primary fields ψ_q and $\bar{\psi}_q$ of the $\mathrm{SU}(N)_1$ WZW model have conformal dimension $h = \frac{N-1}{2N}$. In order to make up for the desired $h = 1/2$, the difference needs to be carried by the vertex operator. This forces the latter to have the form

$$V_\pm(z) = :e^{\pm i\varphi/\sqrt{N}}:, \tag{68}$$

with conformal dimension $h_\pm = 1/2N$.

With the identification (67) one can now easily determine the desired correlation functions of fundamental and anti-fundamental fields. They are given by

$$\psi_{q_1\dots q_L}(z_1,\dots,z_L) = \langle\psi_{q_1}(z_1)\bar{\psi}_{q_2}(z_2)\cdots\rangle = \frac{\langle\Psi_{q_1}(z_1)\bar{\Psi}_{q_2}(z_2)\cdots\rangle}{\langle V_+(z_1)V_-(z_2)\cdots\rangle}. \tag{69}$$

The two correlation functions entering this expression can be calculated using Wick's Theorem for free fields. One obtains

$$\left\langle\Psi_{q_1}(z_1)\bar{\Psi}^{q_2}(z_2)\cdots\Psi_{q_{L-1}}(z_{L-1})\bar{\Psi}^{q_L}(z_L)\right\rangle = \operatorname*{Det}_{1\le i,j\le L/2}\left(\frac{\delta^{q_{2j}}_{q_{2i-1}}}{z_{2i-1}-z_{2j}}\right) \tag{70}$$

$$\left\langle V_+(z_1)V_-(z_2)\cdots V_+(z_{L-1})V_-(z_L)\right\rangle = \prod_{1\le i<j\le L}(z_i-z_j)^{(-1)^{i+j}/N}. \tag{71}$$

[10] Note that we need the alternation in order to eventually reach a representation with vanishing $\mathrm{U}(1)$ charge, thereby allowing us to descend from the group $\mathrm{U}(N)$ to its quotient $\mathrm{SU}(N)$.

[11] We stress once more that the bar is used to denote the dual representation. All fields considered here are holomorphic.

The advantage of this representation of the correlation function is the absence of any cocycles which obscure the correct sign factors.

3. Discussion of the uniform spin models

This section will be used to illuminate the structure of the Hamiltonians (34) in the uniform case. For 1D models with spins located on a circle we will recover a slight modification of the SU(N) Haldane–Shastry model. For the equidistant case this allows to come up with a complete analytic solution for the spectrum. As a byproduct we find that the thermodynamic limit of the spin chain is described by an SU$(N)_1$ WZW model.

3.1. Simplification of the Hamiltonian

The uniform spin model is defined in terms of the partition $\mathbb{S} = \mathbb{L}$ and $\bar{\mathbb{S}} = \emptyset$. The degree map will be chosen such that $(-1)^{d_i} = -1$. The general Hamiltonian (34) simplifies accordingly and becomes

$$H = \frac{(N+2)(N-1)}{2N} \sum_{i \neq j} |w_{ij}|^2 + \frac{N+2}{2(N+1)} \sum_{i \neq j} \Big\{ (N-1)|w_{ji}|^2 + \sum_{k(\neq i,j)} \bar{w}_{ki} w_{kj} \Big\} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{4(N+1)} \sum_{i<j<k} \big\{ (N+2) f_{abc} \Omega^A_{kij} - N d_{abc} \Omega^T_{kij} \big\} S^a_i S^b_j S^c_k. \tag{72}$$

Apart from the simplified expression for Ω^T_{ijk} there is otherwise nothing we can achieve on this level of generality. Further simplifications, however, can be realized if we restrict our attention to special choices of the positions z_i and of the associated parameters w_{ij}.

The form (72) of the Hamiltonian is not particularly suitable for a numerical treatment, in particular for larger values of N, since it involves rather complicated sums over the spin indices. On the other hand, we know that all contributions correspond to SU(N)-invariant operators on the tensor products $\mathcal{V} \otimes \mathcal{V}$ and $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$ of two and three physical sites, respectively. Fortunately, these operators are exhausted by the identity operator $\mathbb{I}$, the two-site permutations $\mathbb{P}_{ij}$ and the cyclic permutations $\mathbb{T}_{ijk}$. In terms of these operators the numerical implementation becomes much more efficient. Eventually, the complexity of the diagonalization problem even becomes independent of the value of N, see Section 5 for a more detailed discussion of this point. The labor to find explicit expressions for the individual terms entering (72) is the only prize we have to pay.

For the transpositions the story is not too difficult, given the known Casimir eigenvalues in the decomposition of $\mathcal{V} \otimes \mathcal{V} = \Xi \oplus \Upsilon$, see Table 1. Indeed, one may easily verify the relation

$$\mathbb{P}_{ij} = \mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{N}. \tag{73}$$

These operators provide a unitary representation of the permutation group, i.e. they satisfy $\mathbb{P}^\dagger_{ij} = \mathbb{P}^{-1}_{ij}$ as well as

$$\mathbb{P}_{ij}\mathbb{P}_{jk}\mathbb{P}_{ij} = \mathbb{P}_{jk}\mathbb{P}_{ij}\mathbb{P}_{jk} \quad (i \neq k) \quad \text{and} \quad \mathbb{P}^2_{ij} = \mathbb{I}. \tag{74}$$

These relations are also depicted in Fig. 3. It is only slightly more cumbersome to work out the expression for the cyclic permutations on three sites since the latter can be expressed as a product

of two transpositions, $\mathbb{T}_{ijk} = \mathbb{P}_{ij}\mathbb{P}_{jk}$ and $\mathbb{T}^{-1}_{ijk} = \mathbb{P}_{jk}\mathbb{P}_{ij} = \mathbb{T}^{\dagger}_{ijk}$. After some straightforward algebra one then finds

$$\begin{aligned}\mathbb{T}_{ijk} &= \mathbb{P}_{ij}\mathbb{P}_{jk} \\ &= -\frac{i}{2} f_{abc} S_i^a S_j^b S_k^c + \frac{1}{2} d_{abc} S_i^a S_j^b S_k^c + \frac{1}{N}[\mathbf{S}_i \cdot \mathbf{S}_j + \mathbf{S}_j \cdot \mathbf{S}_k + \mathbf{S}_i \cdot \mathbf{S}_k] + \frac{1}{N^2}. \end{aligned} \tag{75}$$

For our purposes we need to invert these relations and solve for the two cubic invariants involving the invariant rank-three tensors f and d. After some simple manipulations we find

$$d_{abc} S_i^a S_j^b S_k^c = \mathbb{T}_{ijk} + \mathbb{T}^{\dagger}_{ijk} - \frac{2}{N}[\mathbb{P}_{ij} + \mathbb{P}_{jk} + \mathbb{P}_{ik}] + \frac{4}{N^2}, \tag{76a}$$

$$f_{abc} S_i^a S_j^b S_k^c = i\big(\mathbb{T}_{ijk} - \mathbb{T}^{\dagger}_{ijk}\big). \tag{76b}$$

The most convenient starting point for a replacement of the spin operators in terms of permutations seems to be (32). After a lengthy but straightforward calculation one then recovers a Hamiltonian of the form

$$H = g\mathbb{I} + \sum_{i<j} (g_{ij} + g_{ji})\mathbb{P}_{ij} + \sum_{i<j<k} \big(g_{ijk}\mathbb{T}_{ijk} + \bar{g}_{ijk}\mathbb{T}^{\dagger}_{ijk}\big), \tag{77}$$

where the individual constants are given by

$$g = \frac{(N-1)(N+2)}{2(N+1)} \sum_{i\neq j} |w_{ij}|^2 - \frac{1}{2(N+1)} \sum_{i\neq j\neq k} \bar{w}_{ki} w_{kj}, \tag{78a}$$

$$g_{ij} = \frac{N}{2} |w_{ij}|^2 + \frac{1}{2} \sum_{k(\neq i,j)} \bar{w}_{ki} w_{kj} - \frac{1}{N+1} \mathrm{Re}\Big[\bar{w}_{ji} \sum_{k(\neq j)} w_{jk}\Big], \tag{78b}$$

$$g_{ijk} = \frac{1}{2}\Big\{[\bar{w}_{ki} w_{kj} + \bar{w}_{jk} w_{ji} + \bar{w}_{ij} w_{ik}] - \frac{1}{(N+1)}[\bar{w}_{kj} w_{ki} + \bar{w}_{ji} w_{jk} + \bar{w}_{ik} w_{ij}]\Big\}. \tag{78c}$$

We believe that the expression (77), together with the decomposition of the Hilbert space as a representation of the symmetric group (employing the so-called Schur–Weyl duality), provides the computationally most efficient way of implementing the uniform spin model numerically, both in 1D and 2D.

3.2. The Hamiltonian for spins on a circle

We will now focus our attention to the 1D arrangement of spins on the unit circle with $w_{kl} = (z_k + z_l)/(z_k - z_l)$, see Section 2.6.2 for a concise definition of the setup. This choice implies a considerable number of non-trivial identities which allow us to simplify the Hamiltonian (72) and, in particular, to basically eliminate the three-spin couplings. First of all, one gets rid of complex conjugations in view of $\bar{w}_{kl} = -w_{kl}$. More importantly, the anti-symmetric three-spin coupling drops out due to $\Omega^A_{ijk} = 0$. Finally, the symmetric three-spin coupling simplifies considerably due to $\Omega^T_{ijk} = 2$. After employing these simplifications, the original Hamiltonian (72) may be rewritten as

$$H = -\frac{(N+2)(N-1)}{2N} \sum_{i\neq j} w_{ij}^2 - \frac{N+2}{2(N+1)} H^{(2)} - \frac{N}{2(N+1)} H^{(3)}, \tag{79}$$

where we used the abbreviations

$$H^{(2)} = \sum_{i\neq j}\Big\{(N-1)w_{ij}^2 + \sum_{k(\neq i,j)} w_{ki}w_{kj}\Big\}\mathbf{S}_i\cdot\mathbf{S}_j \quad \text{and} \quad H^{(3)} = \sum_{i<j<k} d_{abc}S_i^a S_j^b S_k^c. \tag{80}$$

In the next step we will consider the individual terms one by one. In order to simplify the quadratic term we shall use identity (43). We then find

$$H^{(2)} = \sum_{i\neq j}\big\{L-2+(N+1)w_{ij}^2 - w_{ij}(\xi_i-\xi_j)\big\}\mathbf{S}_i\cdot\mathbf{S}_j. \tag{81}$$

Finally, we employ (38) and after a number of simplifications this leads to

$$H^{(2)} = \sum_{i\neq j}\Big\{L-2+(N+1)+4(N+1)\frac{z_iz_j}{z_{ij}^2} - w_{ij}(\xi_i-\xi_j)\Big\}\mathbf{S}_i\cdot\mathbf{S}_j \tag{82}$$

$$= \sum_{i\neq j}\Big\{4(N+1)\frac{z_iz_j}{z_{ij}^2} - w_{ij}(\xi_i-\xi_j)\Big\}\mathbf{S}_i\cdot\mathbf{S}_j + (L+N-1)\sum_{i\neq j}\mathbf{S}_i\cdot\mathbf{S}_j. \tag{83}$$

The last part can be converted into an expression involving the total spin $\mathbf{S}=\sum_j\mathbf{S}_j$ using

$$\sum_{i\neq j}\mathbf{S}_i\cdot\mathbf{S}_j = \sum_{i,j}\mathbf{S}_i\cdot\mathbf{S}_j - \sum_j\mathbf{S}_j^2 = \mathbf{S}^2 - \frac{L(N^2-1)}{N}. \tag{84}$$

Summing up all contributions we are left with

$$H^{(2)} = 4(N+1)\sum_{i\neq j}\Big\{\frac{z_iz_j}{z_{ij}^2} - \frac{w_{ij}(\xi_i-\xi_j)}{4(N+1)}\Big\}\mathbf{S}_i\cdot\mathbf{S}_j + (L+N-1)\mathbf{S}^2 - \frac{L(N^2-1)(L+N-1)}{N}. \tag{85}$$

Next we turn our attention to the three-spin coupling. Our goal is to rewrite $H^{(3)}$ such that it again only involves the total spin $\mathbf{S}$. This can be achieved by restoring the summation over the full range of indices and enforcing the absence of the diagonal parts, and it leads to

$$H^{(3)} = \frac{1}{6}d_{abc}\sum_{i,j,k}(1-\delta_{ij})(1-\delta_{ik}-\delta_{jk})S_i^aS_j^bS_k^c = \frac{1}{6}d_{abc}\sum_{i,j,k}\big[1-(\delta_{ij}+\delta_{ik}+\delta_{jk})+\delta_{ij}(\delta_{ik}+\delta_{jk})\big]S_i^aS_j^bS_k^c. \tag{86}$$

The individual contributions can be evaluated step by step, resulting first of all in

$$H_1^{(3)} = \frac{1}{6}d_{abc}\sum_{i,j,k}S_i^aS_j^bS_k^c = \frac{1}{6}d_{abc}S^aS^bS^c = \frac{1}{6}\mathbf{S}^3. \tag{87}$$

On the right-hand side we defined the symbol $\mathbf{S}^3$ as the cubic invariant for the total spin which is obtained using the completely symmetric tensor. Then, splitting the summation into the diagonal part and the off-diagonal parts and using (30) one obtains

$$H_2^{(3)} = -\frac{N}{3}d_{abc}\sum_{i,j,k}(\delta_{ij}+\delta_{ik}+\delta_{jk})S_i^aS_j^bS_k^c = -\big(N^2-4\big)\mathbf{S}^2. \tag{88}$$

Finally, using the Casimir eigenvalues $\mathbf{S}_i^2 = (N^2-1)/N$ one recovers the expression

$$H_3^{(3)} = \frac{N}{3} d_{abc} \sum_{i,j,k} \delta_{ij}(\delta_{ik}+\delta_{jk}) S_i^a S_j^b S_k^c = \frac{2L}{3N}\big(N^2-4\big)\big(N^2-1\big). \tag{89}$$

Putting all the previous calculations together and reordering the terms one obtains the Hamiltonian

$$\begin{aligned} H = {} & -2(N+2)\sum_{i\neq j}\left\{\frac{z_i z_j}{z_{ij}^2} - \frac{w_{ij}(\xi_i-\xi_j)}{4(N+1)}\right\}\mathbf{S}_i\cdot\mathbf{S}_j - \frac{(N+2)(N+2L)}{4(N+1)}\mathbf{S}^2 \\ & - \frac{N}{12(N+1)}\mathbf{S}^3 + \frac{L(N-1)(N+2)(3L+2N-1)}{6N} \\ & - \frac{(N+2)(N-1)}{2N}\sum_{i\neq j} w_{ij}^2. \end{aligned} \tag{90}$$

Using relation (42), we identify the first term as a modification of the SU(N) Haldane–Shastry model [5]. Since the coefficients ξ_i vanish if the spins are positioned equidistantly on the unit circle, the model above includes the original Haldane–Shastry model as a special case.

One can think of the Hamiltonian (90) as a modification of the SU(N) Haldane–Shastry model. Its two-spin interaction has an altered (and actually rather intricate) distance dependence and the remaining terms correspond to the addition of two generalized chemical potentials. Indeed, while the usual chemical potential couples to the conserved particle number of a system of bosonic or fermionic particles, the coupling here favors spin configurations according to their conserved total Casimir eigenvalues. From this perspective, the Hamiltonian (90) should be regarded as a special instance of the family

$$H(\lambda_i) = H_{\text{mod HS}} + \lambda_2\mathbf{S}^2 + \lambda_3\mathbf{S}^3 + \cdots + \lambda_N\mathbf{S}^N. \tag{91}$$

When writing this Hamiltonian we used that SU(N) has $N-1$ independent Casimir operators which are described by symmetric tensors of rank $2,\ldots,N$. It should be noted that the additional terms in (91) turn a finite-size scaling analysis into a rather complicated issue, even if the first term $H_{\text{mod HS}}$ has a clean thermodynamic limit. In Section 5 we will comment more on these subtleties.

3.3. The Hamiltonian for equidistant spins on a circle

In the case of an equidistant distribution of spins on the circle one has further simplifications such as (see (46))

$$\xi_i = 0 \quad \text{and} \quad \sum_{i\neq j} w_{ij}^2 = -\frac{1}{3}L(L-1)(L-2). \tag{92}$$

In that case, the Hamiltonian essentially reduces to the Haldane–Shastry form and may be written as

$$\begin{aligned} H = {} & (N+2)\sum_{k<l}\frac{\mathbf{S}_k\cdot\mathbf{S}_l}{\sin^2\frac{\pi}{L}(k-l)} - \frac{(N+2)(N+2L)}{4(N+1)}\mathbf{S}^2 - \frac{N}{12(N+1)}\mathbf{S}^3 \\ & + \frac{L(N+2)(N-1)(L^2+2N+1)}{6N}. \end{aligned} \tag{93}$$

This quantum spin system is exactly solvable since the underlying SU(N) Haldane–Shastry model can be treated analytically due to its Yangian symmetry [5,32]. We note that the latter is not present and that originally degenerate multiplets are split when the chemical potentials λ_2 and λ_3 are added. However, this does not affect the statement that the model is exactly solvable. The thermodynamic limit of the first term in the Hamiltonian (93) is well known to be critical and described by a SU(N)$_1$ WZW model [5,33], the starting point of our construction. The additional terms do not affect this conclusion but they modify the resulting WZW spectrum. Since the SU(N) Haldane–Shastry model has already been studied thoroughly in the past, we refrain from entering a more detailed discussion here.

4. Discussion of the mixed spin models

The structure of the Hamiltonians (34) will be discussed for mixed spin models, involving both the fundamental and the anti-fundamental representation. Unfortunately, a reduction to two-spin couplings is not possible in this case, not even for an alternating chain of equidistant spins. However, we comment on possible simplifications in terms of generators of the walled Brauer algebra.

4.1. Simplification of the Hamiltonian

In contrast to the discussion in Section 3 we now deal with the general situation where the physical spins may either transform in the fundamental or the anti-fundamental representation of SU(N). Accordingly, the set of sites $\mathbb{L}$ splits into two subsets $\mathbb{S}$ and $\bar{\mathbb{S}}$ and there is a non-trivial grade map $d_\bullet : \mathbb{L} \to \mathbb{Z}_2$ encoding this decomposition (see our discussion around Eq. (9)).

As in Section 3, the two-spin interactions entering (34) are still described in terms of SU(N)-invariant operators. However, on a mixed Hilbert space $\mathcal{V} \otimes \bar{\mathcal{V}}$ the latter can of course not be implemented in terms of a permutation. Instead, the natural invariant operator (besides the identity) is the projection onto the singlet which, up to normalization, can be expressed in terms of the spin–spin coupling as

$$\mathbb{E}_{ij} = -\mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{N}. \tag{94}$$

The algebra of invariant operators is then generated by the operators $\mathbb{P}_{ij}$ (for $i, j \in \mathbb{S}$ or $i, j \in \bar{\mathbb{S}}$) and $\mathbb{E}_{ij}$ (for $i \in \mathbb{S}$ and $j \in \bar{\mathbb{S}}$ or vice versa). While the former obey the relations (74) of the permutation group, the latter satisfy the Temperley–Lieb relations

$$\mathbb{E}_{ij}\mathbb{E}_{jk}\mathbb{E}_{ij} = \mathbb{E}_{ij} \quad \text{and} \quad \mathbb{E}_{ij}^2 = N\mathbb{E}_{ij} \tag{95}$$

with loop fugacity $N = \dim(\mathcal{V})$. Of course, there are also non-trivial relations between the operators $\mathbb{P}_{ij}$ and $\mathbb{E}_{kl}$. If these relations are taken into account one is led to a representation of the so-called walled Brauer algebra, see Section 5 for a more detailed explanation of this structure.

Our ultimate goal is to rewrite the Hamiltonian (34) in terms of invariant operators. It is obvious that the notation becomes too cumbersome when sticking to the symbols $\mathbb{P}_{ij}$ and $\mathbb{E}_{ij}$ since we always need to distinguish the different types of indices. Instead we will introduce a unified notation and define the invariant two site operator (for $i \neq j$)

$$\mathbb{Q}_{ij} = (-1)^{d_i + d_j}\mathbf{S}_i \cdot \mathbf{S}_j + \frac{1}{N}. \tag{96}$$

We note that this operator is hermitean, i.e. $\mathbb{Q}_{ij} = \mathbb{Q}_{ij}^\dagger$. Moreover, it is symmetric in its indices, $\mathbb{Q}_{ij} = \mathbb{Q}_{ji}$. Depending on the nature of the indices of $\mathbb{Q}_{ij}$ we either recover the usual permutation or the projection onto the singlet.

Just as in Section 3, our considerations easily generalize to three-spin interactions. As for the permutations, the invariant operators on three sites are either acting on two sites only or they are a product of two two-site operators. As it turns out, just two of these product operators are independent and they read

$$\mathbb{H}_{ijk} = \mathbb{Q}_{ij}\mathbb{Q}_{jk} \quad \text{and} \quad \mathbb{H}_{ijk}^\dagger = \mathbb{Q}_{jk}\mathbb{Q}_{ij} = \mathbb{H}_{kji}. \tag{97}$$

Permutations of the indices result in the same two operators but the precise outcome depends on the degree of all three labels involved. The defining relation (96) for the operators $\mathbb{Q}$ in terms of the spin matrices imply the representation

$$\begin{aligned}\mathbb{H}_{ijk} = {} & -\frac{i}{2}(-1)^{d_i+d_k} f_{abc} S_i^a S_j^b S_k^c - \frac{1}{2}(-1)^{d_i+d_j+d_k} d_{abc} S_i^a S_j^b S_k^c \\ & + \frac{1}{N}\big[(-1)^{d_i+d_j}\,\mathbf{S}_i\cdot\mathbf{S}_j + (-1)^{d_j+d_k}\mathbf{S}_j\cdot\mathbf{S}_k + (-1)^{d_i+d_k}\mathbf{S}_i\cdot\mathbf{S}_k\big] + \frac{1}{N^2},\end{aligned} \tag{98}$$

which in turn allows to express the invariant operators in terms of spins. After some elementary algebra one finds the inversion formulas

$$d_{abc} S_i^a S_j^b S_k^c = -(-1)^{d_i+d_j+d_k}\left[\big(\mathbb{H}_{ijk} + \mathbb{H}_{ijk}^\dagger\big) - \frac{2}{N}[\mathbb{Q}_{ij} + \mathbb{Q}_{jk} + \mathbb{Q}_{ik}] + \frac{4}{N^2}\right] \tag{99a}$$

$$f_{abc} S_i^a S_j^b S_k^c = i(-1)^{d_i+d_k}\big(\mathbb{H}_{ijk} - \mathbb{H}_{ijk}^\dagger\big). \tag{99b}$$

We are now in a position to express the general Hamiltonian (34) in terms of the invariant operators $\mathbb{Q}$ and $\mathbb{H}$.

After a simple but lengthy computation we find the Hamiltonian

$$H = g\mathbb{I} + \sum_{i<j}(g_{ij} + g_{ji})\mathbb{Q}_{ij} + \sum_{i<j<k}(g_{ijk} + g_{kij} + g_{jki} + g_{jik} + g_{kji} + g_{ikj})\mathbb{H}_{ijk}, \tag{100}$$

where the individual constants are given by

$$\begin{aligned} g = {} & \frac{N+2}{4(N+1)}\sum_{i\neq j}\big\{2N - \big[1 + (-1)^{d_i+d_j}\big]\big\}|w_{ij}|^2 \\ & - \frac{1}{2(N+1)}\sum_{i\neq j}(-1)^{d_i+d_j}\sum_{k(\neq i,j)}\bar{w}_{ki}w_{kj}, \end{aligned} \tag{101a}$$

$$\begin{aligned} g_{ij} = {} & \frac{N}{4(N+1)}\big[N + (N+2)(-1)^{d_i+d_j}\big]|w_{ij}|^2 + \frac{1}{2}(-1)^{d_i+d_j}\sum_{k(\neq i,j)}\bar{w}_{ki}w_{kj} \\ & - \frac{1}{N+1}\,\mathrm{Re}\left[(-1)^{d_i}w_{ji}\sum_{k(\neq j)}(-1)^{d_k}\bar{w}_{jk}\right], \end{aligned} \tag{101b}$$

$$\begin{aligned} g_{ijk} = {} & \frac{1}{4(N+1)}\big\{\bar{w}_{ki}w_{kj}(-1)^{d_i}\big[(N+2)(-1)^{d_k} + (-1)^{d_j}N\big] \\ & - \bar{w}_{ik}w_{ij}(-1)^{d_k}\big[(N+2)(-1)^{d_i} - (-1)^{d_j}N\big]\big\}. \end{aligned} \tag{101c}$$

It is possible to verify that this Hamiltonian reduces to (77) in the case of a uniform chain.

4.2. The Hamiltonian for equidistant spins on a circle

The Hamiltonian above may be simplified by assuming special positions for the spins. As before, the most convenient setup corresponds to equidistant spins on the circle, see Section 2.6.2. Unlike in the uniform case, however, the three-spin couplings cannot be simplified now since Ω^T_{ijk} fails to be constant (i.e. independent of the indices). Nevertheless using Eqs. (45)–(46) the coupling constants (101) can be rewritten in the simplified form

$$g = \frac{L(L-2)[L(4N^2+7N+2)-4(N^2+N-1)]}{24(N+1)}, \tag{102a}$$

$$g_{ij} = -\frac{(-1)^{d_i+d_j}}{4(N+1)}\big[\big(N^2\big(1+(-1)^{d_i+d_j}\big)+6N+4\big)w_{ij}^2+2(N+1)(L-2)\big] = g_{ji}, \tag{102b}$$

$$g_{ijk} = \frac{1}{4(N+1)}\big\{-w_{ki}w_{kj}(-1)^{d_i}\big[(N+2)(-1)^{d_k}+(-1)^{d_j}N\big] + w_{ik}w_{ij}(-1)^{d_k}\big[(N+2)(-1)^{d_i}-(-1)^{d_j}N\big]\big\}. \tag{102c}$$

The advantage of using these formulas is that they do not involve sums over the sites anymore, and thus can be efficiently evaluated numerically.

5. Loop formulation and numerical implementation

As was discussed in Section 4, the Hamiltonians for the mixed SU(N) spin models can be expressed in terms of generators of the walled Brauer algebra. We will now adopt a more abstract point of view and interpret the system from the perspective of loop models. This permits an efficient numerical implementation whose complexity is independent of the parameter N. For the alternating chain with equidistant spins on the unit circle we find evidence that the thermodynamic limit is described by a conformal field theory and we establish some properties of the latter.

5.1. Definition of the loop model

Since the dimensions of the SU(N) representations $\mathcal{V}$ and $\bar{\mathcal{V}}$ are given by N, the dimension of the total Hilbert space (9) grows as $\dim\mathcal{H} = N^L$ where L is the number of spins. Even for small values of $N \geq 3$ the full implementation of the Hamiltonian quickly exceeds the available memory on computers. In order to avoid this complication we are seeking for a formulation of the diagonalization problem where the complexity is independent of N but rather only depends on the number of spins L. This is achieved by relating our setup to loop models where $N = \dim\mathcal{V}$ can be interpreted as the fugacity of the loops.

In imaginary time, the exponential of the quantum spin Hamiltonian defines an evolution of the spin configuration along the longitudinal axis of a cylinder. It is convenient to interpret the N different internal states of each spin as different types of particles. This allows one to illustrate the time evolution in terms of world-lines of these particles. Depending on whether the spin transforms in the fundamental or in the anti-fundamental representation we will either think of particles or their anti-particles and we will keep track of this difference by giving the corresponding world-lines opposite orientations. Let us now recall that the Hamiltonian (100) can be

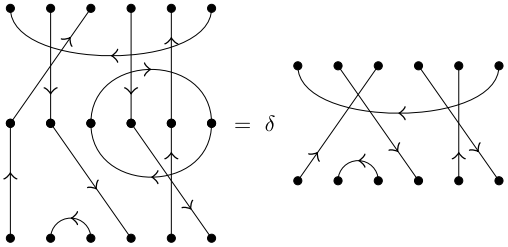

Fig. 1. An example for the multiplication of diagrams.

expressed in terms of either permutations or projections onto a singlet. In the world-line picture, these two operations correspond to the permutation of two (anti-)particles or to the pairwise annihilation of a particle and its associated anti-particle, followed by the creation of a mixed state involving all particle species. The latter process can be visualized diagrammatically by arcs connecting the strands horizontally, so that the particle type is conserved along a line. The operator $\mathbb{P}_{ij}$ instead simply permutes the particles and admits a graphical representation as a crossing of the strands at sites i and j. During the dynamics loops may be formed, and each loop (contractible or not) carries a weight $\mathrm{tr}_{\mathcal{V}}(\mathbb{I}) = \dim \mathcal{V} = N$. All processes just described must respect the orientation of world-lines.

Given the previous correspondence, the study of the spin chain can now be approached graphically by studying the long-range model of crossing loops with weight N. We remark, however, that the transition from the spin chain to the loop formulation involves some subtleties. Indeed, for N not sufficiently large (in a sense to be made precise below), some observables in the loop model may not have a counterpart in the spin chain. This leads to the fact that (disregarding SU(N)-related degeneracies) the spectrum of the loop model contains additional eigenvalues compared to that of the spin chain, as will be discussed in detail in Section 5.3. Although the geometrical loop formulation can be employed for a general setup with arbitrary positions of $\mathcal{V}$ and $\bar{\mathcal{V}}$, we will assume L even in the following and focus on the alternating case $(\mathcal{V} \otimes \bar{\mathcal{V}})^{L/2}$.

5.2. The walled Brauer algebra

We consider now the loop model and discuss abstractly the properties of the algebra of diagrams associated to the elementary interactions $\mathbb{E}_{ij}$ and $\mathbb{P}_{ij}$. A diagram is a set of L top nodes and L bottom nodes, numbered from left to right, so that each node is connected to precisely one other by a line. We call the lines connecting bottom to top nodes "through lines". The diagrams relevant for our analysis have some constraints. As before we assign alternating orientations to the lines, and consider only diagrams whose connectivities respect the orientation. The linear span over $\mathbb{C}$ of these diagrams is turned into an algebra by specifying a product $D_1 \cdot D_2$, which is given by the diagram obtained by placing D_1 on top of D_2 and replacing all loops formed with a fixed weight $\delta \in \mathbb{C}$. In Fig. 1 this multiplication law for diagrams is illustrated in a specific example. The algebra so formed is called the walled Brauer algebra $\mathrm{WB}_L(\delta)$. Clearly the relation with the spin chains we would like to study comes about when we specify $\delta = N$, but it is useful to regard δ as an arbitrary complex number for the moment.

We now summarize some properties of $\mathrm{WB}_L(\delta)$ that we need below for the discussion of the spectrum of the spin chain. We denote by E_{ij}, P_{ij} the abstract diagrams corresponding to the action of $\mathbb{E}_{ij}, \mathbb{P}_{ij}$ in the loop formulation of the spin model, see Fig. 2.

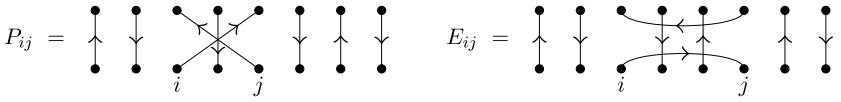

Fig. 2. The elements P_{ij} and E_{ij} of the walled Brauer algebra.

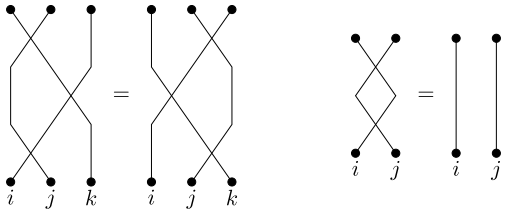

Fig. 3. The relations (74) of the permutation group in the loop formulation. The lines are assumed to have the same orientation.

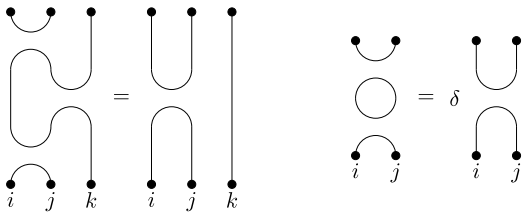

Fig. 4. The relations (95) in the loop formulation. In the SU(N) spin models, the loop fugacity δ is given by $N = \dim \mathcal{V}$. The lines are assumed to have alternating orientations.

The walled Brauer algebra $\mathrm{WB}_L(\delta)$ can be presented as a set of generators and relations. As generators it is sufficient to take the permutations $P_{i,i+2}$ with $i = 1, \ldots, L-2$ together with E_{12}. On products of these generators one then imposes the natural relations which ensure that diagrams with the same connectivity are identified and that loops have weight δ. (Note that the remaining elements E_{ij} can be obtained by multiplying E_{12} from the left and the right by the permutation exchanging $(1,2)$ with either (i,j) or (j,i), depending on their parity.) The diagrammatic form of the relations (74) and (95) is depicted in Figs. 3 and 4.

We stress furthermore that by flipping the arrows on all the odd nodes one obtains diagrams belonging to (the group algebra of) the symmetric group $\mathfrak{S}_L$. In particular this shows that the dimension of $\mathrm{WB}_L(\delta)$ equals that of $\mathfrak{S}_L$, namely $L!$, independently of δ.

In the loop formulation the Hamiltonian is expressed in terms of diagrams of the walled Brauer algebra. We now discuss the problem of diagonalizing such an operator. It will be convenient to reduce the dimension of the space of states of our problem by looking at sub-sectors labeled by some quantum numbers determining individual representations of the algebra at hand. The walled Brauer algebra is a finite-dimensional algebra and as such all its irreducible representations can be realized by acting with the algebra on itself (this is called the regular representation). This means that we can restrict ourselves to studying the action of the walled Brauer algebra on diagrams. Our next goal is to find subspaces on which this action is closed. For this purpose we introduce the notation $D = X_{v,w,\sigma}$ for a diagram D, where v is the configuration of the $(L-K)/2$ northern arcs, w that of the $(L-K)/2$ bottom arcs, and $\sigma \in \mathfrak{S}_{K/2} \times \mathfrak{S}_{K/2} \subset \mathfrak{S}_K$ (the two factors refer to the two orientations) is a permutation specifying how the nodes not

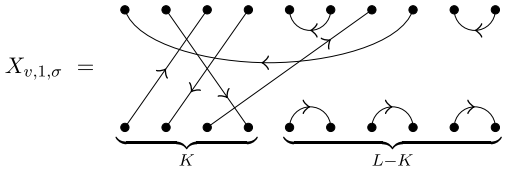

Fig. 5. Illustration of an element $X_{v,1,\sigma}$. The index 1 stands for the configuration chosen at the bottom, v corresponds to the configuration of arcs at the top together with the choice of nodes where the arcs lie, and σ encodes the permutation of through lines.

occupied by arcs are connected: $\sigma(i) = j$ indicates that the ith bottom node is connected to node j.

One can easily convince oneself that the number of through lines can only be lowered under the action of the algebra but never increased. It is then reasonable to work in a basis of diagrams which is ordered in an increasing fashion according to the number of through lines. In such a basis, any Hamiltonian based on the walled Brauer algebra will have a block upper-triangular structure. To compute the eigenvalues of the Hamiltonian it is then sufficient to restrict to the blocks by acting on diagrams with a *fixed* number of through lines. This reduces the calculational effort and can be implemented in practice by setting the action on a state to zero if the number of through lines is reduced. Furthermore, the multiplication rule of the walled Brauer algebra implies that the action of the Hamiltonian on a given diagram modifies only the connectivity of its top row of nodes. As a consequence, the eigenvalues have a huge degeneracy. The latter can be removed by restricting oneself to diagrams where the configurations on the bottom are frozen to a given one.[12] For definiteness, fixed the number K of through lines (which is always even for L even), we choose the leftmost K nodes at the bottom to be connected with the top, and the remaining $L-K$ nodes to have the arc configuration connecting node $K+i$ with $K+i+1$ for $i = 0, 2, \ldots, L-K-2$. Using the notation introduced above, such elements are denoted by $X_{v,1,\sigma}$ where 1 refers to the fixed configuration of bottom arcs chosen. For a fixed arc connectivity v with $(L-K)/2$ arcs, the diagrams $X_{v,1,\sigma}$ differ by the permutation $\sigma \in \mathfrak{S}_{K/2} \times \mathfrak{S}_{K/2}$ encoding how the through lines with the same orientation are permuted, see Fig. 5 for an example.

The Hamiltonian can be further block diagonalized by projecting onto subspaces which transform according to irreducible representations of $\mathfrak{S}_{K/2} \times \mathfrak{S}_{K/2}$. We recall that the irreducible representations S^λ of $\mathfrak{S}_m$, the so-called Specht modules [34], are labeled by partitions of m, denoted by $\lambda \vdash m$. A partition $\lambda \vdash m$ is a non-increasing sequence of positive integers which sum up to m: $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} \geq 1$ and $\lambda_1 + \cdots + \lambda_{\ell(\lambda)} = m$. $\ell(\lambda)$ is called the length of the partition. A convenient way of depicting the partition λ is in terms of Young tableaux. In our case, we have two identical copies of the permutation group and hence the irreducible representations $S^{\mu,\nu} = S^\mu \times S^\nu$ of $\mathfrak{S}_{K/2} \times \mathfrak{S}_{K/2}$ are indexed by a pair of partitions $(\mu, \nu) \vdash (K/2, K/2)$. The resulting representation of the walled Brauer algebra will be denoted by $\mathcal{W}_L(\mu, \nu)$.

We now present an explicit construction of the space $\mathcal{W}_L(\mu, \nu)$ in terms of a suitable projection on the set of all diagrams [35]. Denote by I_L^K the space spanned by the diagrams $X_{v,1,\mathrm{id}}$

[12] The degeneracy results from the fact that the Brauer algebra admits an action by left multiplication and another one by right multiplication. Our way of concatenating diagrams in the regular representation singles out the left multiplication. The (irrelevant) right multiplication may then be used to freeze the bottom configuration.

where v is any allowed arc configuration of the top row with *exactly* $(L-K)/2$ arcs and id is the identity permutation (no crossings between through lines). $\mathcal{W}_L(\mu,\nu)$ is given by the tensor product of I_L^K with $S^{\mu,\nu}$, and its basis elements are of the form $X_{v,1,\mathrm{id}}\otimes x$, where x runs through a basis of $S^{\mu,\nu}$. (The construction of a basis of the irreducible representation of the symmetric group is standard, see e.g. [34]. Since it will not be explicitly needed later on, we omit the details here.) The action of a diagram on this basis is given by concatenation from above on $X_{v,1,\mathrm{id}}$. The result is set to zero if the number of through lines is reduced, since this would modify the pattern at the bottom nodes. Furthermore, a permutation $\sigma\in\mathfrak{S}_{K/2}\times\mathfrak{S}_{K/2}$ of the through lines would also produce an element outside of I_L^K since the permutation id is replaced by something else. However, such a permutation of through lines will be absorbed by acting on the irreducible representation $S^{\mu,\nu}$ instead.

The spaces $\mathcal{W}_L(\mu,\nu)$ with $(\mu,\nu)\vdash(K/2,K/2)$ and $K=0,2,\dots,L$ are the essential building blocks on which we want to diagonalize our Hamiltonian. Note that the dimension of I_L^K corresponds to all possible ways of choosing arc configurations at the top nodes with exactly $(L-K)/2$ arcs. It is determined by $\binom{L/2}{(L-K)/2}\binom{L/2}{(L-K)/2}((L-K)/2)!$, where the last factor comes from the possible ways of pairing $(L-K)/2$ objects with $(L-K)/2$ other objects. Then due to the tensor product structure we have

$$\begin{aligned} d_L^{\mu,\nu} &:= \dim\bigl(\mathcal{W}_L(\mu,\nu)\bigr) = \binom{L/2}{(L-K)/2}^2 \bigl((L-K)/2\bigr)!\dim\bigl(S^\mu\bigr)\dim\bigl(S^\nu\bigr) \\ &= \frac{(L/2)!(L/2)!}{(K/2)!(K/2)!((L-K)/2)!}\dim\bigl(S^\mu\bigr)\dim\bigl(S^\nu\bigr). \end{aligned} \tag{103}$$

In particular, one obtains the dimension $(L/2)!$ for $K=0$ and $L(L/2)!/2$ for $K=2$.

We now briefly comment on the properties of the representations $\mathcal{W}_L(\mu,\nu)$. It has been proven in [35, Thm. 6.3] that $\mathrm{WB}_L(\delta)$ is semisimple when $\delta\notin\mathbb{Z}$ or $|\delta|\geq L-1$. For these values of δ all representations are fully reducible and the representations $\mathcal{W}_L(\mu,\nu)$ form a complete set of irreducible representations. Moreover, denoting the matrix algebra of $d\times d$ matrices (over $\mathbb{C}$) by $\mathcal{M}_d$, one has the decomposition (as algebras and bimodules)

$$\mathrm{WB}_L(\delta)\cong\bigoplus_{K=0,2}^{L}\bigoplus_{\mu,\nu\vdash K/2}\mathcal{M}_{d_L^{\mu,\nu}}, \tag{104}$$

where the subscript indicates that the first summation runs in steps of two. It is useful to think of the $\mathcal{M}_{d_L^{\mu,\nu}}$ as the space of linear maps on $\mathcal{W}_L(\mu,\nu)$. The decomposition (104) is supported by the following comparison of dimensions:

$$\begin{aligned} &\sum_{K=0,2}^{L}\sum_{\mu,\nu\vdash K/2}\dim(M_{d_L^{\mu,\nu}}) \\ &\quad=\sum_{K=0,2}^{L}\sum_{\mu,\nu\vdash K/2}\left(\frac{(L/2)!(L/2)!}{(K/2)!(K/2)!((L-K)/2)!}\dim\bigl(S^\mu\bigr)\dim\bigl(S^\nu\bigr)\right)^2 \\ &\quad=\sum_{K=0,2}^{L}\left(\frac{(L/2)!(L/2)!}{(K/2)!(K/2)!((L-K)/2)!}\right)^2\left(\sum_{\mu\vdash K/2}\bigl(\dim\bigl(S^\mu\bigr)\bigr)^2\right)^2 \\ &\quad=\sum_{K=0,2}^{L}\binom{L/2}{K/2}^2\bigl((L/2)!\bigr)^2=L!, \end{aligned} \tag{105}$$

where for the third equality we used that $m! = \dim(\mathfrak{S}_m) = \sum_{\mu\vdash m}(\dim(S^\mu))^2$. If δ instead is a small integer, Eq. (104) ceases to be true (as a decomposition of algebras and bimodules) and the representation theory of $\mathrm{WB}_L(\delta)$ gets much more complicated. For the purpose of computing the eigenvalues of the Hamiltonian, entering such details is not necessary, and in the following we will simply restrict the numerical diagonalization to the spaces $\mathcal{W}_L(\mu,\nu)$.

We conclude this section by summarizing what we have done and what we have gained. The spin chains have been mapped onto a geometrical model of crossing loops with long-range interactions. The number N of the $\mathrm{SU}(N)$ spin chains enters in the loop model as a parameter. Since the dimension of the space of states $\mathcal{W}_L(\mu,\nu)$ does not depend on N, our reformulation allows us to efficiently investigate the $\mathrm{SU}(N)$ spin chains for N arbitrary large, a task which is not feasible when diagonalizing the spin chain directly. Since our presentation silently skipped over a few subtleties, we will devote the next section to a precise discussion of how to reconstruct the spectrum of the spin chain from that of the loop model.

5.3. Relation with the spin chains

So far we have discussed the motivation for a loop reformulation of our spin chains and reviewed some of the algebraic properties of the loop model. In this section, we will now comment on the precise relation between the energy spectrum in the loop model as compared to that of the spin chain. The algebraic considerations which follow are based on [36].

In the following it will be convenient to view our models from the perspective of $\mathrm{GL}(N)$ instead of $\mathrm{SU}(N)$. This is justified since the Hamiltonian for the alternating chain commutes with the generators of $\mathrm{GL}(N)$ which span the Lie algebra $\mathfrak{gl}_N$. The Hamiltonian can be regarded as an element of the centralizer algebra $Z(\mathfrak{gl}_N)$, the algebra of all linear operators on $(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L/2}$ that commute with the action of $\mathfrak{gl}_N$. As we shall discuss below, the algebra $Z(\mathfrak{gl}_N)$ is closely related to the walled Brauer algebra. In fact, in the "stable" (but rather unphysical) regime where $N \geq L$ one has $Z(\mathfrak{gl}_N) \cong \mathrm{WB}_L(N)$ and a Hilbert space decomposition of the form [36][13]

$$(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L/2}\big|_{\mathfrak{gl}_N\otimes\mathrm{WB}_L(N)} \cong \bigoplus_{K=0,2}^{L} \bigoplus_{\mu,\nu\vdash K/2} V\big([\mu,\nu]_N\big)\otimes\mathcal{W}_L^{\mu,\nu} \quad (\text{for } N\geq L). \tag{106}$$

The symbol $V([\mu,\nu]_N)$ refers to the $\mathfrak{gl}_N$-representation corresponding to the highest weight $[\mu,\nu]_N$. It is obtained from the two partitions $\mu=(\mu_1,\ldots,\mu_{\ell(\mu)})$ and $\nu=(\nu_1,\ldots,\nu_{\ell(\nu)})$ by setting[14]

$$[\mu,\nu]_N := \big[\mu_1,\mu_2,\ldots,\mu_{\ell(\mu)},0^{N-\ell(\mu)-\ell(\nu)},-\nu_{\ell(\nu)},-\nu_{\ell(\nu)-1},\ldots,-\nu_1\big]. \tag{107}$$

The condition $N\geq L$ ensures that $\ell(\mu)+\ell(\nu)\leq N$ and that this assignment is well defined.

For the task of finding the spin chain spectrum we now focus on the action of $\mathrm{WB}_L(N)$. Denoting by $\tilde{z}_{\mu,\nu}=\dim V([\mu,\nu]_N)$ the degeneracy associated with the $\mathfrak{gl}_N$-symmetry, the relevant information in the decomposition (106) is

[13] We stress that the bound for the equivalence of $Z(\mathfrak{gl}_N)$ and $\mathrm{WB}_L(N)$ is different from $N\geq L-1$, the range for the semisimplicity of $\mathrm{WB}_L(N)$ mentioned above [35, Thm. 6.3].

[14] Here we chose the $\mathfrak{gl}_N$ Cartan subalgebra as the diagonal matrices with a single unit element. The corresponding weight of $\mathrm{SU}(N)$ is therefore given by $(\mu_1-\mu_2,\mu_2-\mu_3,\ldots)$.

$$(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L/2}\big|_{\mathrm{WB}_L(N)} \cong \bigoplus_{K=0,2}^{L} \bigoplus_{\mu,\nu\vdash K/2} \tilde{z}_{\mu,\nu}\mathcal{W}_L^{\mu,\nu} \quad (\text{for } N\geq L). \tag{108}$$

Indeed, the Hamiltonian can be regarded as an element of $\mathrm{WB}_L(N)$ and hence it is sufficient to consider the diagonalization problem on the invariant subspaces $\mathcal{W}_L^{\mu,\nu}$ entering the decomposition (108). This establishes the connection to the loop model. It is then obvious how the spectrum of the spin chain can be reconstructed from that of the loop model, at least as long as $N\geq L$.

The situation is more complicated in the regime $N<L$ which is relevant for the thermodynamic limit of the chain. In this case we need to understand the decomposition of the Hilbert space $(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L/2}$ with respect to $\mathfrak{gl}_N\otimes Z(\mathfrak{gl}_N)$. Let us introduce a map ϕ which represents the action of the walled Brauer algebra $\mathrm{WB}_L(N)$ on the Hilbert space of the spin chain. This map ϕ sends E_{ij} to $\mathbb{E}_{ij}$ and P_{ij} to $\mathbb{P}_{ij}$ and it constitutes an algebra homomorphism, i.e. it satisfies $\phi(D_1D_2)=\phi(D_1)\phi(D_2)$ for all $D_1,D_2\in\mathrm{WB}_L(N)$. As already observed, $\mathbb{E}_{ij}$ and $\mathbb{P}_{ij}$ commute with $\mathfrak{gl}_N$, so that ϕ can be regarded as a map from $\mathrm{WB}_L(N)$ to $Z(\mathfrak{gl}_N)$. It can be shown [36, Thm. 5.8] that the image of the walled Brauer algebra exhausts the centralizer, $\phi(\mathrm{WB}_L(N))\cong Z(\mathfrak{gl}_N)$ (this is true for any N) and that ϕ is an isomorphism for $N\geq L$, so that $Z(\mathfrak{gl}_N)\cong\mathrm{WB}_L(N)$ in that case. For $N<L$ on the other hand, the commutant $Z(\mathfrak{gl}_N)$ exhibits more relations as compared to $\mathrm{WB}_L(N)$. Indeed, in this parameter range every simple basis element $e_{a_1}\otimes e^{a_2}\otimes\cdots e_{a_{L-1}}\otimes e^{a_L}$ of $(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L/2}$ has at least two subscripts which are identical, and attempts to antisymmetrize them will result in zero. (Here e_a and e^a stand, respectively, for a basis of $\mathcal{V}$ and its dual.) This means that the map ϕ representing the walled Brauer algebra has a non-trivial kernel, so that the representation is not faithful.

Let us now study the implications of the previous statements. For general values of L and N, the relevant decomposition of the Hilbert space reads

$$(\mathcal{V}\otimes\bar{\mathcal{V}})^{\otimes L/2}\big|_{\mathfrak{gl}_N\otimes Z(\mathfrak{gl}_N)} \cong \bigoplus_{K=0,2}^{L} {\bigoplus_{\mu,\nu\vdash K/2}}' V\big([\mu,\nu]_N\big)\otimes\mathcal{Z}_L^{\mu,\nu}, \tag{109}$$

where $\mathcal{Z}_L^{\mu,\nu}$ are certain representations of $Z(\mathfrak{gl}_N)$. According to Eq. (107) it is required to restrict the summation to pairs of partitions satisfying $\ell(\mu)+\ell(\nu)\leq N$ in order to ensure the existence of a bona fide weight $[\mu,\nu]_N$. This is indicated by the prime. Due to the existence of the homomorphism $\phi:\mathrm{WB}_L(N)\to Z(\mathfrak{gl}_N)$, the spaces $\mathcal{Z}_L^{\mu,\nu}$ can also be regarded as representations of $\mathrm{WB}_L(N)$. For $N\geq L$ one has $Z(\mathfrak{gl}_N)\cong\mathrm{WB}_L(N)$ and $\mathcal{Z}_L^{\mu,\nu}\cong\mathcal{W}_L^{\mu,\nu}$. On the other hand, it may occur that the dimension $z_{\mu,\nu}$ of $\dim\mathcal{Z}_L^{\mu,\nu}$ is strictly smaller than the dimension $d_L^{\mu,\nu}$ of the representation $\mathcal{W}_L^{\mu,\nu}$. A precise condition for this to happen has been given in [36, Thm. 2.14]. It states that $z_{\mu,\nu}\leq d_L^{\mu,\nu}$ for all $(\mu,\nu)\vdash(K/2,K/2)$, and $z_{\mu,\nu}=d_L^{\mu,\nu}$ if and only if

$$N\geq\ell(\mu)+\ell(\nu)+(L-K)/2. \tag{110}$$

In these cases one encounters a mismatch between the spectrum of the spin model and the spectrum of the loop model since the latter is obtained by diagonalizing the Hamiltonian on the larger spaces $\mathcal{W}_L^{\mu,\nu}$. In practice, certain energy eigenvalues simply have to be discarded (we comment on this below).

It is instructive to illustrate this result with a concrete example. Let us set $L=4$ and consider the decomposition of the Hilbert space for

$$N \geq 4: \quad (\mathcal{V} \otimes \bar{\mathcal{V}})^2 \cong 2V\big([0^N]\big) \oplus 4V\big([1, 0^{N-2}, -1]\big) \oplus V\big([2, 0^{N-2}, -2]\big) \oplus V\big([2, 0^{N-3}, -1^2]\big) \oplus V\big([1^2, 0^{N-3}, -2]\big) \oplus V\big([1^2, 0^{N-4}, -1^2]\big). \tag{111}$$

Here, the first term corresponds to $K = 0$, the next to $K = 2$ and the last four to $K = 4$. For smaller values of N one instead has

$$\begin{aligned} N = 3: \quad & (\mathcal{V} \otimes \bar{\mathcal{V}})^2 \cong 2V\big([0^3]\big) \oplus 4V\big([1, 0, -1]\big) \oplus V\big([2, 0, -2]\big) \oplus V\big([2, -1^2]\big) \oplus V\big([1^2, -2]\big), \\ N = 2: \quad & (\mathcal{V} \otimes \bar{\mathcal{V}})^2 \cong 2V\big([0^2]\big) \oplus 3V\big([1, -1]\big) \oplus V\big([2, -2]\big). \end{aligned} \tag{112}$$

We see in particular that the multiplicity of $V([1, -1])$ is reduced when $N = 2$. This can be quickly checked by computing the dimension of each term on the r.h.s. of Eq. (112). The space $V([1, -1])$ can be identified with the SU(2) representation of spin 1 and $V([2, -2])$ with that of spin 2, so that the dimension is $2 \times 1 + 3 \times 3 + 1 \times 5 = 16$, coinciding indeed with the dimension of the Hilbert space. How the multiplicity is reduced in the general case if condition (110) is not met has also been described in [36], but the algorithm to compute it is quite complex and we do not describe it here.

Let us finally briefly comment on a second source for a mismatch between the spectrum of the spin model and that of the loop model. In the decomposition of the spin Hilbert space we encountered a restriction to pairs of partitions satisfying $\ell(\mu) + \ell(\nu) \leq N$. This constraint has no counterpart in the loop model. Of course, the resulting additional eigenvalues are under complete control and can easily be eliminated in the process of computing the spectrum.

Altogether we now got a fairly complete picture of how the loop model can be used for studying the spin chain. As we argued above, the spectra are absolutely identical when $N \geq L$. More importantly, even for general values of L and N we expect a faithful representation of the spectrum in all sectors satisfying condition (110). We have checked these statements numerically. In particular, we have verified for $N = 2, 3, 4, 5, 6$ and $L = 4, 6$ that the lowest eigenvalues which will be relevant in Section 5.4 are present in the SU(N) spin chains. We remark that the missing eigenvalues of the loop model arise in a supersymmetric generalization of the spin chains at hand, with symmetry GL($N + M|M$) and $M \geq 0$. This fact has been discussed for short-range spin chains in [37,38]. The consequences for our long-range models will be addressed in a future publication [39].

Up to now we have only considered the on-site symmetry $\mathfrak{gl}_N$ and its commutant $Z(\mathfrak{gl}_N)$, the image of the walled Brauer algebra $\mathrm{WB}_L(N)$. For the diagonalization problem it is, however, also useful to keep track of other conserved charges that commute with the Hamiltonian. In particular, the dimension of the blocks of the Hamiltonian can be further reduced by exploiting its translational symmetry. For an equidistant arrangement of spins on the circle the Hamiltonian obviously commutes with the operator u that implements the shift $\mathbf{S}_i \mapsto \mathbf{S}_{i+1}$. However, since the translation by a single site exchanges the roles of $\mathcal{V}$ and $\bar{\mathcal{V}}$ it is more appropriate in our context to work with the translation by two sites. Indeed, in contrast to u the operator u^2 admits a natural interpretation as an element of the walled Brauer algebra WB_L. Its corresponding diagram is depicted in Fig. 6.

The eigenvalues of u^2 are of the form $e^{4\pi i s/L}$, where the integer s is defined modulo $L/2$ and coincides with the momentum. The reduced Hamiltonian $H_s \equiv \Pi_s H = H \Pi_s$ acting on the eigenspace of momentum s is defined in terms of the projector

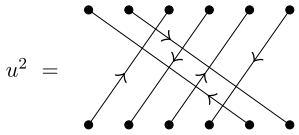

Fig. 6. The diagram u^2 that translates two sites to the right for a system of size $L = 6$. (Recall that periodic boundary conditions are imposed.)

$$\Pi_s = \frac{2}{L} \sum_{t=0}^{L/2-1} e^{-4\pi i t s/L} u^{2t} \quad \text{which satisfies } \Pi_s^2 = \Pi_s. \tag{113}$$

The eigenvalues of the Hamiltonian will then be labeled by the representations of the walled Brauer algebra and by their momentum.

Before concluding this section, we give a brief historical note on the walled Brauer algebra. This algebra was introduced in the mathematical literature precisely to study the problem of decomposing the space formed out of mixed products of fundamental and dual representations of the general linear group. One says that the general linear group and the walled Brauer algebra are in Schur–Weyl duality on these mixed tensor products, since their actions mutually centralize each other. This generalizes the well known Schur–Weyl duality between the symmetric group and the general linear group which applies when both are acting on the tensor product of fundamental representations only. The latter corresponds to the setup of the uniform chain discussed in Section 3. The walled Brauer algebra is a subalgebra of the Brauer algebra [40]. The latter is in Schur–Weyl duality with the orthogonal group for the action in the tensor product of fundamental representations. The name "walled" comes from the fact that one usually considers diagrams where the first $L/2$ lines on the left are directed upwards and the remaining $L/2$ lines on the right are directed downwards. With this convention up and down lines are separated by a domain wall which can only be crossed by horizontal arcs. Our setting is simply obtained by rearranging the order of lines.

5.4. Numerical study

In this section we will discuss the low energy properties of the spin chains and loop models. Before entering the discussion for general N, it is useful to recall what happens for $N = 2$. In that case, the fundamental and the anti-fundamental representation are equivalent and the symmetric rank-three tensor d vanishes identically. This means that one is dealing with a uniform chain and our spin chain Hamiltonian is related to that of the Haldane–Shastry model,

$$H_{\mathrm{HS}} = \sum_{i<j} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{\sin^2(\frac{\pi}{L}(i-j))} \quad \text{by } H(N=2) = 4\, H_{\mathrm{HS}} - \frac{2}{3}(L+1)\,\mathbf{S}^2 + \frac{L}{3}\big(L^2+5\big), \tag{114}$$

see our discussion in Section 3.3. The Haldane–Shastry model is exactly solvable [2,3]. In the continuum limit, the energies E_n^{HS} of the low-lying states are given by the scaling dimensions Δ_n of the SU(2) WZW theory at level $k = 1$: $(E_n^{\mathrm{HS}} - E_0^{\mathrm{HS}})/(2L) = \Delta_n + O(1/L)$. (The unusual power of L comes from the dependence of the coupling on the length.) We remark that although the eigenstates of the Hamiltonian $H(N = 2)$ are in correspondence with the fields in the WZW model, the universal part of the energy of those states is shifted due to the presence of the global Casimir operator $\mathbf{S}^2$.

Table 2
The coefficient c_0 obtained from fitting Eq. (116) for system sizes $L \in \{6, 8, \ldots, 16\}$. The error is the standard error from the fit.

N	c_0
2	0
3	0.00448 ± 0.00419
4	0.00669 ± 0.00760
5	0.00460 ± 0.00880
6	0.00150 ± 0.00958
7	0.00221 ± 0.01263
8	0.00142 ± 0.01486
9	0.00030 ± 0.01695
10	0.00050 ± 0.01980

We now return to the case of general N and use a numerical implementation of the loop model to determine the scaling properties of the energy gaps. If the low energy theory describing our model is a conformal field theory, one expects that

$$\frac{E_n - E_0}{2L} = v_s \Delta_n + O(1/L), \tag{115}$$

where $\Delta_n = h + \bar{h}$ is the scaling dimension of the associated state. Here, we introduced the speed of sound v_s to account for a possible numerical normalization factor which is independent of the energy level n (it may depend on N though). Furthermore, as usual we identify the momentum s of the state with the conformal spin: $s = h - \bar{h}$. Note that since the ground state energy of our model is zero, the gaps coincide with the energy of the excited states. In a CFT, the ground state energy scales with the length, with a prefactor proportional to the central charge of the theory. Unfortunately, we do not know how to shift our ground state energy in order to extract the central charge, and we will therefore focus on the spectrum of excited states only. We also recall that if instead our Hamiltonian is gapped, the energy difference between the first excited state and the ground state would behave as $(E_1 - E_0)/(2L) \sim L$.

In the following we will use the notation $E_{K,s,\ell}$ for the energy of the ℓ-th excited state in the sector with K non-contractible strings and momentum s. The first excited state in the loop model occurs in the sector with $(K, s) = (2, 0)$. In order to verify the absence of a gap for our Hamiltonian it is thus sufficient to study the scaling of the gap $E_{2,0,0} - E_{0,0,0}$. Specifically, we fit our data against the following function of L:

$$\frac{E_{2,0,0} - E_{0,0,0}}{2L} = c_0 L + c_1 + \frac{c_2}{L}. \tag{116}$$

By definition of the gap one has $c_0 \geq 0$. Table 2 summarizes the resulting values of c_0 determined in this way. We studied the values $N \in \{2, 3, \ldots, 10\}$, and in all these cases we found $c_0 < 10^{-2}$, which gives strong support to the hypothesis that the system is gapless and conformal for any $N \in \mathbb{Z}_{\geq 2}$. In particular, c_0 turns out to be exactly zero for $N = 2$, since finite size corrections are practically absent in the Haldane–Shastry model. In what follows we will build on the conformal hypothesis to extract some conformal dimensions of the theory.

We first address the problem of determining the speed of sound v_s entering the scaling of the gaps (115). This is crucial in order to be able to extract the spectrum in the correct conformal units. We use the following argument. The gaps of our model (at least in the range $N \in \mathbb{Z}_{\geq 2}$ we

are interested in) are positive, and the ground state is identified in the CFT with the identity field, for which $h = \bar{h} = 0$. The CFT state with $(h, \bar{h}) = (2, 0)$ is the holomorphic stress tensor which is always present in a CFT. If the spectrum is positive, then it *always* corresponds to the lowest state with conformal spin $s = 2$. Moreover, it is an SU(N) singlet and should hence appear in the sector $K = 0$ of the loop model. Indeed, our numerics confirm that the lowest state with momentum $s = 2$ occurs in the $K = 0$ sector of the loop model, and we therefore expect that

$$\frac{E_{0,2,0} - E_{0,0,0}}{2L} = 2v_s + O(1/L). \tag{117}$$

Determining these gaps will enable us to determine v_s. The speed of sound measured as a function of N is reported in Table 3. We note that for $N = 2, 3, 4$ the speed of sound is $N + 2$ within the errors bars, while for larger N it deviates from this value. At the same time the uncertainties also increase with N, showing that these deviations may be due to finite size corrections.[15] As a consistency check, we also determined v_s by looking at the gaps $(E_{0,1,0} - E_{0,0,1})/(2L)$ between the first excited state with $(K, s) = (0, 1)$ and the second excited state with $(K, s) = (0, 0)$. This energy difference is exactly v_s if the state corresponding to $E_{0,1,0}$ is a descendant at level one of that corresponding to $E_{0,0,1}$. These states have lower energy than $E_{0,2,0}$ and they are less sensitive to finite size corrections. However, the speed of sound extracted from $(E_{0,1,0} - E_{0,0,1})/(2L)$ does not deviate significantly from the one presented in Table 3. In the following we will therefore continue to use the values v_s from the latter table. Our findings can also be viewed as a confirmation that the state $E_{0,1,0}$ indeed corresponds to a descendant of $E_{0,0,1}$.

Having determined the speed of sound we can now estimate the lowest conformal dimensions of the CFT describing our model. As remarked at the beginning of this section for the $N = 2$ case, we expect that in general the universal part of our Hamiltonian will be the sum of a CFT Hamiltonian plus non-local terms which shift the CFT conformal dimensions extracted from finite size scaling. Due to the symmetry of our spin chain Hamiltonians these non-local terms have to correspond to global SU(N) Casimir operators. The general form of the resulting theory for $L \to \infty$ is then

$$H = H_{\mathrm{CFT}} + \lambda_1 + \lambda_2 \mathbf{S}^2 + \lambda_3 \mathbf{S}^3 + \cdots + \lambda_N \mathbf{S}^N, \tag{118}$$

see also Eq. (91). The aforementioned shifts will not be present in energy differences of states carrying the same representation. In particular, this is the case for the gaps in the singlet sector, where all Casimir invariants act trivially. In the following we shall hence focus on the subspace of singlets, corresponding to the sector $K = 0$ of the loop model (absence of non-contractible lines), and determine the scaling of the first excited state. The latter has momentum $s = 0$, and we denote its energy by $E_{0,0,1}$. The results for the extracted dimensions are presented in Table 4. We find that the measured values of $\Delta_{0,0,1}$ are well described by the function $(N + 4)/(6N)$. In Fig. 7 we plot $\Delta_{0,0,1}$ (black points) against this function (solid curve) and the smallest positive scaling dimension of a SU$(N)_1$ WZW singlet field, namely $(N - 1)/N$ (dashed curve). We see that the dimensions extracted are not consistent with those predicted by the SU$(N)_1$ WZW model for $N > 2$. We cannot exclude the possibility that the scaling dimensions are not exactly described by $(N + 4)/(6N)$ since, as remarked above, the results for larger N (> 4) are less reliable due to finite size effects. Irrespective of their exact (but unknown) values, we note a clear tendency in our data: the measured dimensions decrease with N while those of the WZW model

[15] We note that normalizing energies as above, the speed of sound for the uniform spin chain of Eq. (93) is precisely $N + 2$, as known from the solution of the Haldane–Shastry model [8].

Table 3
The rescaled gap $(E_{0,2,0} - E_{0,0,0})/(4L)$ as a function of N and L. The row at $L = \infty$ is the extrapolated value of the speed of sound v_s obtained by fitting the data with a polynomial in $1/L$ of degree 2. The error is the standard error from the fit.

L \ N	2	3	4
6	2	2.60417	3.2
8	2.5	3.27199	4.03
10	2.8	3.65978	4.50023
12	3	3.90832	4.79233
14	3.14286	4.07846	4.98525
16	3.25	4.2005	5.11822
16	3.33333	4.29114	5.21265
∞	4	5.0144 ± 0.01613	5.96207 ± 0.03390

L \ N	5	6	7
6	3.79167	4.38095	4.96875
8	4.7811	5.52829	6.27304
10	5.33125	6.15699	6.97948
12	5.66463	6.53046	7.39236
14	5.87838	6.76405	7.64524
16	6.02061	6.91475	7.80397
16	6.11744	7.01335	7.90404
∞	6.8801 ± 0.05210	7.78294 ± 0.07036	8.67716 ± 0.08858

L \ N	8	9	10
6	5.55556	6.14167	6.72727
8	7.0162	7.75823	8.49945
10	7.79984	8.61872	9.43654
12	8.25169	9.10926	9.96557
14	8.52353	9.39984	10.2747
16	8.69006	9.57403	10.4565
18	8.79142	9.67659	10.5602
∞	9.56609 ± 0.10672	10.4516 ± 0.12479	11.3347 ± 0.14280

increase. This finding provides a strong indication that the CFT describing the alternating spin chains for $N > 2$ is different from the $\mathrm{SU}(N)_1$ WZW model. The identification of this theory can be tackled using the methods presented in this section. Our approach even allows us to study the more general setup of loop models with an arbitrary value of the fugacity δ. We relegate a detailed study of the resulting CFTs to another publication [39], where we will present a more general point of view based on supersymmetric spin chains.

6. Conclusions and outlook

In this article we have constructed several families of long-range $\mathrm{SU}(N)$ spin models in 1D and 2D. They all arise as parent Hamiltonians for infinite matrix product states based on the $\mathrm{SU}(N)$ WZW model at level $k = 1$. The whole construction is based on a given groundstate, which is known exactly and can be expressed in terms of chiral correlation functions of WZW primary fields. At level $k = 1$ the latter can be evaluated explicitly using a free field realization.

Table 4
The scaling dimension $\Delta_{0,0,1}$ of the first excited state in the sector $K = 0$ as a function of N. The values are extracted from fitting data for systems with $L \in \{6, 8, \ldots, 18\}$ with a polynomial in $1/L$ of degree two, and using the numerically determined speed of sound.

N	$\Delta_{0,0,1}$
2	0.5
3	0.38956 ± 0.00201
4	0.33345 ± 0.00296
5	0.29971 ± 0.00349
6	0.27723 ± 0.00381
7	0.26121 ± 0.00402
8	0.24921 ± 0.00417
9	0.23990 ± 0.00428
10	0.23246 ± 0.00436

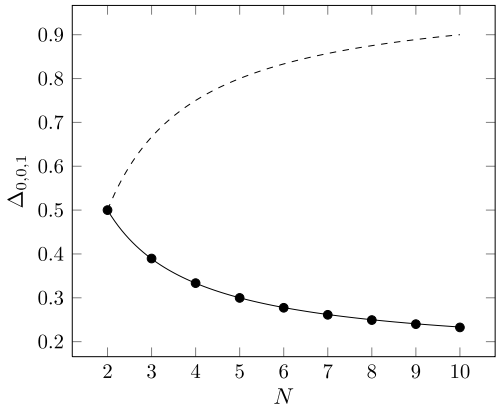

Fig. 7. The black points are the scaling dimension $\Delta_{0,0,1}$ of the first excited state in the sector $K = 0$ as a function of N. The lower solid curve is a fit of the data by $(N+4)/(6N)$, while the upper dashed curve depicts the function $(N-1)/N$, the scaling dimension of the fundamental field in the $\mathrm{SU}(N)_1$ WZW model.

For simplicity we restricted our attention to spin models involving the fundamental and the anti-fundamental representation of $\mathrm{SU}(N)$ but the generalization to other representations should be straightforward.

The models we constructed give rise to a 2D conformal field theory if the spins are placed equidistantly on a circle. If only the fundamental representation is involved, the Hamiltonian essentially reduces to the $\mathrm{SU}(N)$ Haldane–Shastry model up to the addition of non-local chemical potentials corresponding to global $\mathrm{SU}(N)$ Casimir operators. In this case, the model admits an exact analytic solution and it flows to the $\mathrm{SU}(N)_1$ WZW model in the thermodynamic limit. The case of an alternating spin chain turns out to be more complicated. Our numerical analysis provides strong evidence that this long-range spin chain is critical as well. However, our results on the conformal spectrum rule out that the critical point is described by a $\mathrm{SU}(N)_1$ WZW model. Most of our analysis is based on a reformulation of the original problem in terms of loop

models. These are not only providing the computationally most efficient representation of the Hamiltonian (both in 1D and 2D) but they are also interesting in their own right.

With regard to the physical interpretation of our Hamiltonians it will be crucial to achieve a better understanding of different types of 2D setups and to relate them to the physics of fractional quantum Hall states and chiral spin liquids. For the case of the $U(1)_k$ WZW model a connection to Laughlin states could be established in [31,26]. A similar analysis for SU(3) should result in a connection with the Halperin [27] or variants of the non-abelian spin singlet (NASS) states [28,41].

An interesting technical problem regards the determination of spin–spin correlation functions for the models we have constructed. These could be used to substantiate any claim on the gapless or gapped nature of the resulting phases. In the case of SU(2) it was possible to derive a recursion relation for two-point functions which could then be solved systematically, both for the finite and the infinite equidistant chain [20]. Similar recursion relations can be derived for $SU(N)$. However, due to the existence of the non-trivial tensor d_{abc} they now only relate two-point functions to three-point functions instead of giving an equation for the two-point function itself. As a consequence, the recursion relations can only be used to verify existing proposals but not to provide a solution from first principles. In view of existing conjectures about the dynamical spin–spin correlators in the $SU(N)$ Haldane–Shastry model [8,42] the study of these recursion relations might nevertheless be an avenue worth pursuing.

In our opinion, the most pressing open question concerns the nature of the critical theory arising from the alternating $SU(N)$ spin chain on the circle discussed in Section 4.2. In the context of our reformulation in terms of loop models it is natural to revisit this question from a more general perspective. First of all, it is natural to regard the symmetry group $SU(N)$ as a special instance of the family $SU(N+M|M)$ of special unitary supergroups. This alternative point of view has the advantage that the spectrum of the loop model and that of the spin chain match precisely for sufficiently large values of M. Moreover, in the loop formulation the number N can be regarded as a continuous parameter and it will be interesting to explore the different regimes where critical behavior can be expected. For instance, thanks to a mapping onto the N^2 states Potts model, it is known that standard loop models without crossings and with nearest neighbor interactions cease to be critical for fugacities $N > 2$. Our numerical results in Section 5 indicate that this bound is not relevant for our types of long-range crossing loop models. We plan to return to these issues in a forthcoming publication [39].

Taking into account the results of this paper, infinite matrix product states based on WZW models have now been constructed for the symmetry groups U(1), $SU(N)$ and $SO(N)$ [20,26,25]. The only remaining groups of classical type are the symplectic groups $SP(2N)$. This case is currently under investigation and we hope to report on it in the near future.

7. Note added in proof

During the preparation of this manuscript, we learned that related results have been obtained by Hong-Hao Tu, Anne Nielsen and German Sierra [43].

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Appendix A. Some basic facts on SU(N)

Since many of the algebraic expressions in the main text involve the invariant tensors of SU(N) we find it useful to summarize a few of the most important formulas for them. In what follows, the symbol T^a refers to the spin matrices in the fundamental representation $\mathcal{V}$ in an arbitrary basis. The first object of interest is the metric which is defined by

$$\kappa^{ab} = \mathrm{tr}_{\mathcal{V}}\big(T^a T^b\big). \tag{119}$$

Throughout the text, the metric κ^{ab} and its inverse κ_{ab} are used to raise and lower indices. The structure constants $f^{ab}{}_c$ and $d^{ab}{}_c$ may then be introduced via the identity

$$T^a T^b = \frac{1}{2}\big[T^a, T^b\big] + \frac{1}{2}\big\{T^a, T^b\big\} = \frac{i}{2} f^{ab}{}_c T^c + \frac{1}{2} d^{ab}{}_c T^c + \frac{1}{N}\kappa^{ab}\mathbb{I}, \tag{120}$$

where the first term corresponds to the antisymmetric part and the remaining ones to the symmetric part. An alternative way of introducing these tensors is

$$f^{abc} = -i\,\mathrm{tr}_{\mathcal{V}}\big(\big[T^a, T^b\big]T^c\big) \quad \text{and} \quad d^{abc} = \mathrm{tr}_{\mathcal{V}}\big(\big\{T^a, T^b\big\}T^c\big). \tag{121}$$

By construction, these tensors are completely (anti)-symmetric, respectively. Both of them are traceless, i.e. $\kappa_{ab} f^{abc} = \kappa_{ab} d^{abc} = 0$. The tensor d^{abc} vanishes for SU(2) but it is non-trivial for all integers $N \geq 3$. It remains to summarize a few identities involving two or three of these tensors,

$$f^{ac}{}_d f^{bd}{}_c = -2N\kappa^{ab}, \qquad f^a{}_{ec} f^e{}_{bd} f^{cd}{}_g = -N f^a{}_{bg}, \tag{122}$$

$$d^{ac}{}_d d^{bd}{}_c = \frac{2(N^2-4)}{N}\kappa^{ab}, \qquad f^a{}_{ec} f^e{}_{bd} d^{cd}{}_g = -N d^a{}_{bg}. \tag{123}$$

More relations of a similar type can be found in [44, p. 92] and references therein.

For the discussion of the alternating chain it is also necessary to have some information on the anti-fundamental representation $\bar{\mathcal{V}}$. The corresponding representation matrices $\bar{T}^a$ are related to those in the fundamental representation by transposition, $\bar{T}^a = -(T^a)^T$. This definition together with (120) then immediately implies a product formula of the form

$$\bar{T}^a \bar{T}^b = \frac{1}{2}\big[\bar{T}^a, \bar{T}^b\big] + \frac{1}{2}\big\{\bar{T}^a, \bar{T}^b\big\} = \frac{i}{2} f^{ab}{}_c \bar{T}^c - \frac{1}{2} d^{ab}{}_c \bar{T}^c + \frac{1}{N}\kappa^{ab}\mathbb{I}. \tag{124}$$

We note that there is an important sign difference as compared to the analogous expression (120) for the fundamental representation $\mathcal{V}$. In the unified language of Section 2.2, the two relations (120) and (124) may be compactly expressed as (17).

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