ELSEVIER

Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE @ DIRECT®

Theoretical Computer Science 301 (2003) 271 – 284

Theoretical Computer Science

www.elsevier.com/locate/tcs

Independent domination in finitely defined classes of graphs

R. Boliac*, V. Lozin

RUTCOR, Rutgers University, 640 Bartholomew Rd. Piscataway, NJ 08854-8003, USA

Received 3 February 2002; received in revised form 4 July 2002; accepted 19 July 2002
Communicated by A. Razborov

Abstract

We study the independent dominating set problem restricted to graph classes defined by finitely many forbidden induced subgraphs. The main result is two sufficient conditions for the problem to be NP-hard in a finitely defined class of graphs. We conjecture that those conditions are also necessary and describe several classes of graphs verifying the conjecture.

Keywords: Graph; Independent dominating set; Computational complexity

1. Introduction

In a simple graph, a subset of vertices S is called *dominating* if every vertex outside S has a neighbor in S. A subset S is called *independent* if no two vertices in S are linked by an edge. We study the problem of finding an independent dominating set (ID-set) of minimum cardinality in a graph. Sometimes it is referred to as the minimum maximal independent set (IS) problem, since every maximal (under inclusion) independent set is obviously dominating. In a sense, independent domination can be viewed as an intermediate problem between two others: the minimum DS and the maximum IS. To justify this point of view, we first point out the relationships between the respective graph parameters:

$$\gamma(G) \leqslant i(G) \leqslant \alpha(G),$$

* Corresponding author.
E-mail addresses: boliac@rutcor.rutgers.edu (R. Boliac), lozin@rutcor.rutgers.edu (V. Lozin).

0304-3975/03/$ - see front matter
PII: S0304-3975(02)00586-8

Table 1

Classes\problems	DS	IDS	IS
Planar graphs	NP-c	NP-c	NP-c
Triangle-free graphs	NP-c	NP-c	NP-c
Bipartite graphs	NP-c	NP-c	P
Claw-free graphs	NP-c	NP-c	P
Split graphs	NP-c	P	P
$2K_2$-free graphs	NP-c	P	P
Co-bipartite graphs	P	P	P
P_4-free graphs	P	P	P

where $\gamma(G)$, $i(G)$, $\alpha(G)$ stand for the size of a minimum dominating, minimum independent dominating and maximum IS in a graph G, respectively. Another observation compares the complexity of the three problems on different classes of graphs. Table 1 compiles several available results on this issue and suggests the idea that DS is the "most difficult" problem among those under the comparison, while the independent domination (IDS) is somewhat between the two others.

There is a vast literature that investigates the complexity of DS and IS on different graph classes. Surprisingly enough, the ID-set problem is not so well studied. Several available results deal mainly with subclasses of perfect graphs (such as chordal [14] or co-comparability graphs [20]) or some other well-structured classes (such as circular-arc graphs [7]). In the present paper, we study the problem on graph classes that can be defined by finitely many forbidden induced subgraphs. The main result is two sufficient conditions for the problem to be NP-hard in a finitely defined class of graphs. However, the question whether those conditions are necessary remains open. Toward answering the question, we consider several graph classes that fail both conditions and present polynomial time algorithms for them.

All graphs we consider are undirected, without loops or multiple edges. Graph G is said to be H-free if G does not contain H as an induced subgraph. The subgraph of G induced by a subset of vertices U is denoted $G[U]$. For a vertex v of G, we denote by $N(v)$ the neighborhood of V (the subset of vertices adjacent to v), and by $N[v]$, the closed neighborhood of v, i.e. $N[v]=N(v)\cup\{v\}$. As usual, K_n is the complete graph on n vertices, $K_{n,m}$ is the complete bipartite graph with parts of size n and m, C_n is a chordless cycle and P_n is a chordless path on n vertices. By mK_2 we denote the disjoint union of m copies of a K_2. Following the tradition, we use special names for some particular graphs: a *claw* is a $K_{1,3}$; a *paw* is the graph obtained from a claw by adding an edge; a *diamond*, denoted K_4-e, is the graph obtained from a K_4 by deleting an edge; a *bull* can be obtained from a P_5 by connecting its non-adjacent vertices of degree 2 by an edge; a *fork* (also called a *chair*) is the result of a single subdivision of an edge in a claw. We refer to the complement of a graph G as to co-G.

Throughout the paper, "independent dominating set" is abbreviated as "ID-set".

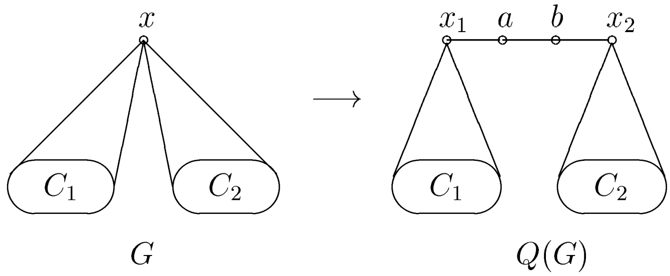

Fig. 1. Transformation Q.

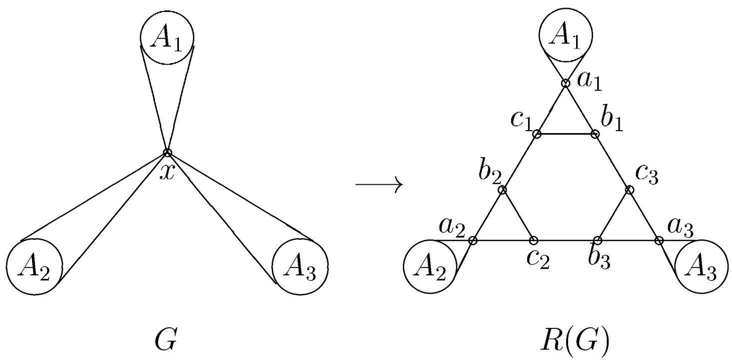

Fig. 2. Transformation R.

2. NP-hardness

The results of this section are based on the following graph transformations.

Transformation P: In a graph G, let (x_1, x_2) be an edge. If $N(x_1) - N[x_2]$ or $N(x_2) - N[x_1]$ is a clique, then replace (x_1, x_2) with a chordless path $P_5 = (x_1, a, b, c, x_2)$. Informally, transformation P can be viewed as a triple subdivision of the edge (x_1, x_2). The resulting graph will be denoted $P(G) = P(G, (x_1, x_2))$.

Transformation Q: In a graph G, let x be a vertex whose neighborhood admits a partition into two cliques C_1 and C_2. Replace x with a path $P_4 = (x_1, a, b, x_2)$ and connect x_j to each vertex in C_j for $j = 1, 2$ (see Fig. 1). The resulting graph will be denoted $Q(G) = Q(G, x)$.

Transformation R: Let x be a vertex of degree at least 3 in a graph G. Partition the neighborhood of x into three subsets A_1, A_2, A_3 in an arbitrary way. Replace x with the subgraph induced by vertices $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ depicted in Fig. 2 and connect a_j to every vertex in A_j for $j = 1, 2, 3$. The resulting graph will be denoted $R(G) = R(G, x, A_1, A_2, A_3)$.

Lemma 1. $i(P(G)) = i(G) + 1$.

Proof. Consider a minimum ID-set I in G. If I contains x_1, then $I' = I \cup \{c\}$ is an ID-set in $P(G,(x_1,x_2))$. The case $x_2 \in I$ is symmetric. If $I \cap \{x_1,x_2\} = \emptyset$, then obviously I contains both a vertex adjacent to x_1 and a vertex adjacent to x_2 else I is not dominating. But then $I' = I \cup \{b\}$ is an ID-set in $P(G)$. Hence $i(P(G)) \leqslant i(G) + 1$.

Conversely, let I' be a minimum ID-set in $P(G)$. Clearly, $I' \cap \{a,b,c\} \neq \emptyset$, otherwise I' is not dominating. Assume first $b \in I'$ and let $N(x_2) - N[x_1]$ be a clique. If $x_1 \notin I'$, then $I' - \{b\}$ is an ID-set in G. If $x_1 \in I'$, then we can assume without loss of generality that $x_2 \notin I'$. Indeed, if $x_2 \in I'$, then there must be a vertex z in $N(x_2) - N[x_1]$ such that x_2 is the only neighbor of z in I' else $(I' - \{b,x_2\}) \cup \{c\}$ is a smaller ID-set in $P(G)$, contradicting the assumption. But now $(I' - \{x_2\}) \cup \{z\}$ is an ID-set of the same size as I'. Thus we assume $x_2 \notin I'$ and hence $I = I' - \{b\}$ is an ID-set in G.

Now let $b \notin I'$. If $a \in I'$ and $c \notin I'$, then x_2 belongs to I' due to its maximality and hence $I' - \{a\}$ is an ID-set in G. If both $a \in I'$ and $c \in I'$, then exactly one of the following three sets is an ID-set in G: $I' - \{a,c\}$ or $(I' - \{a,c\}) \cup \{x_1\}$ or $(I' - \{a,c\}) \cup \{x_2\}$. Therefore, $i(G) \leqslant i(P(G)) - 1$. □

Lemma 2. $i(Q(G)) = i(G) + 1$.

Proof. Let I be a minimum ID-set in G. If $x \in I$, then $I' = (I - \{x\}) \cup \{x_1,x_2\}$ is an ID-set in $Q(G,x)$. If $I \cap C_1 \neq \emptyset$, then $I' = I \cup \{b\}$ is an ID-set in $Q(G,x)$. Consequently, $i(Q(G)) \leqslant i(G) + 1$.

Now let I' be a minimum ID-set in $Q(G,x)$. Obviously, $1 \leqslant |I' \cap \{x_1,a,b,x_2\}| \leqslant 2$. If no vertex in $C_1 \cup C_2$ belongs to I', then the $P_4 = (x_1,a,b,x_2)$ contains exactly two vertices in I'. In that case, replacing those vertices with x results in an ID-set in G. Assume next that C_1 contains a vertex in I'. Then obviously $x_1 \notin I'$. Moreover, since I' is of minimum size, we may suppose that $x_2 \notin I'$ as well. Indeed, if $x_2 \in I'$, then $a \in I'$ else a has no neighbors in I'. Besides, there must be a vertex $z \in C_2$ such that x_2 is the only neighbor of z in I' else $(I' - \{a,x_2\}) \cup \{b\}$ is a smaller ID-set in $Q(G,x)$, contradicting the assumption. But then $(I' - \{x_2\}) \cup \{z\}$ is an ID-set in $Q(G,x)$ of the same size as I' but without x_2. Thus, assuming that $x_2 \notin I'$, we conclude that exactly one vertex of the P_4 belongs to I'. Deleting this vertex from I' produces an ID-set in G. Therefore, $i(G) \leqslant i(Q(G)) - 1$. □

Lemma 3. $i(R(G)) = i(G) + 2$.

Proof. Let I be a minimum ID-set in G. If I contains vertex x, then $I' = (I - \{x\}) \cup \{a_1,a_2,a_3\}$ is an ID-set in $R(G)$. If I contains a vertex $y \in A_1$, then $I' = I \cup \{b_2,c_3\}$ is an ID-set in $R(G)$. Up to the symmetry this exhausts all possibilities and hence $i(R(G)) \leqslant i(G) + 2$.

Conversely, let I' be a minimum ID-set in graph $R(G)$. We will show that G contains an ID-set I of cardinality $|I| \leqslant |I'| - 2$. Denote $T_j = \{a_j,b_j,c_j\}$ $(j = 1,2,3)$, and $I'' = I' - V(G)$. Clearly I'' contains at most one vertex in each T_j.

If $|I''| = 3$, then either $I = I' - I''$ or $I = (I' - I'') \cup \{x\}$ is the desired set. If $|I''| < 3$, then $|I''| = 2$. Indeed, if $I'' \cap T_1 = \emptyset$, then both $b_2 \in I'$ and $c_3 \in I'$ else vertices b_1, c_1 have no neighbors in I'. In that case $I = I' - I''$ is the set sought for. Thus $i(G) \leqslant i(R(G)) - 2$, which completes the proof of the lemma. □

Proposition 1. *The ID-set problem is* NP-*hard in the class of* $(K_{1,3}, K_4, K_4 - e)$-*free graphs with maximum degree* 3.

Proof. The proof is given in several steps. We first apply transformation R to reduce the problem to graphs with vertex degree at most 4. To this end, consider a graph G and a vertex x in G with neighborhood $N(x) = \{x_1, x_2, \ldots, x_l\}$, where $l > 4$. Apply transformation R with respect to x with $A_1 = \{x_1\}$, $A_2 = \{x_2, x_3\}$ and $A_3 = N(x) - (A_1 \cup A_2)$. In the graph $R(G, x, A_1, A_2, A_3)$ all the new vertices, except a_3, have degree at most 4, and the degree of a_3 is exactly one less than that of x. Thus, repeatedly applying R as described above, one can transform G into a graph G' with maximum degree 4.

In the second step, we transform G' in such a way that the neighborhood of every 4-degree vertex would induce a $2K_2$. Assume first that a 4-degree vertex y has an edge (z_1, z_2) in the neighborhood. Then apply transformation R with respect to y with $|A_1| = |A_2| = 1$, and $A_3 = \{z_1, z_2\}$. In the transformed graph, the neighborhood of a_3 induces a $2K_2$ and every other new vertex has degree 3. If the neighborhood of y is edgeless, then application of R with $|A_1| = |A_2| = 1$, and $|A_3| = 2$ creates a new vertex a_3 of degree 4 with an edge in the neighborhood.

In the next step, we reduce the problem to graphs of degree at most three by applying transformation Q to every vertex of degree 4.

Finally, given a graph with maximum degree 3, we apply R to each vertex of degree 3. In the resulting graph G^* the maximum degree is at most 3 as well, and moreover, every vertex of degree 3 is the center of a paw. Therefore, G^* is $(K_{1,3}, K_4, K_4 - e)$-free. It is not hard to see that the total time complexity for transforming an arbitrary graph into a $(K_{1,3}, K_4, K_4 - e)$-free graph with vertex degree at most 3 is bounded by a polynomial in the size of the input graph. Together with Lemmas 2 and 3, this proves the proposition. □

Denote by $S_{i,j,k}$ and $T_{i,j,k}$ the graphs depicted in Figs. 3(a) and (b), respectively. In this notation, $S_{1,1,1}$ is a claw, $S_{1,1,2}$ is a fork, $T_{1,1,1}$ is a triangle, $T_{1,1,2}$ is a paw, $T_{1,2,2}$ is a bull. Moreover, we let some of the indices i, j, k equal 0. In particular, $S_{0,j,k} = P_{j+k+1}$ and $T_{0,j,k} = P_{j+k}$.

Denote by S the class of graphs whose every connected component is of the form $S_{i,j,k}$, and by T the class of graphs whose every connected component is of the form $T_{i,j,k}$.

Theorem 1. *Let X be a class of graphs defined by a finite set F of forbidden induced subgraphs. If $F \cap S = \emptyset$, then the ID-set problem is* NP-*hard for bipartite graphs in X.*

Proof. Let k be an integer greater than the number of vertices in a largest graph in F. We will show that any $(K_{1,3}, K_4, K_4 - e)$-free graph G with maximum degree 3 can

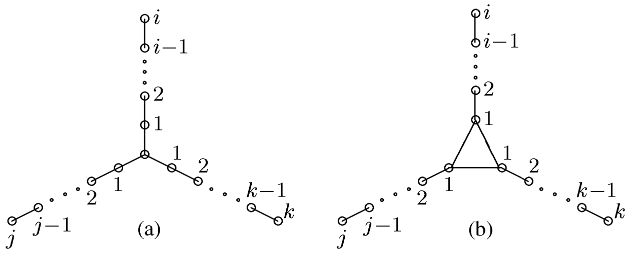

Fig. 3. Graphs $S_{i,j,k}$ (a) and $T_{i,j,k}$ (b).

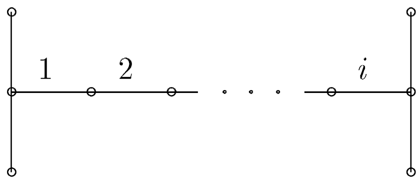

Fig. 4. Graph H_i.

be transformed by transformation P into a bipartite graph in X with maximum degree 3 in polynomial time.

It is not hard to see that transformation P can be applied to any edge of a $(K_{1,3}, K_4, K_4 - e)$-free graph G with maximum degree three. Denote by $\bar{P}$ the transformation consisting in applying P to each edge of the graph. Under $\bar{P}$ the length of every induced cycle increases by a factor of four, and therefore $\bar{P}(G)$ is a bipartite graph. Moreover, it is obvious that graph $\bar{P}(G)$ is again of degree at most 3 and every edge of this graph admits transformation P. Applying $\bar{P}$ sufficiently many times, we transform G into a bipartite graph G' of degree at most 3 containing no induced cycles C_i with $i \leqslant k$ and no graphs of form H_i (see Fig. 4) with $i \leqslant k$.

Now we show that $G' \in X$. Assume by contradiction that G' does not belong to X. Then it must contain an induced subgraph $A \in F$. First, we note that A is a cycle-free graph. Indeed, due to the above observation, A is C_i-free for $i \leqslant k$. In addition, A is C_i-free for $i > k$, since $|V(A)| < k$ due to the choice of k. Hence A is a forest. Similar arguments show us that A contains no graphs of form H_i. Thus, every connected component of A has at most one vertex of degree 3 and hence $A \in S$, contradicting the assumption. As a consequence, G' belongs to X. The time needed to transform G into G' is obviously bounded by a polynomial in the size of G. Together with Lemma 1 and Proposition 1 this yields the conclusion. □

Theorem 2. *Let X be a class of graphs defined by a finite set F of forbidden induced subgraphs. If $F \cap T = \emptyset$, then the ID-set problem is* NP-*hard in the class X.*

Proof. The proof is similar to that of Theorem 1, so we restrict ourselves to a sketch. We start with a $(K_{1,3}, K_4, K_4 - e)$-free graph G with maximum degree 3 and apply

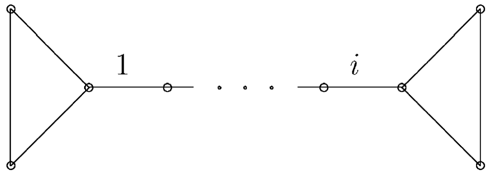

Fig. 5. Graph Δ_i.

transformation P to those edges of G that belong to no triangle. Obviously, such an application preserves the initial properties of the graph. Applying the transformation sufficiently many times we produce a graph G' in which any induced cycle of length more than 3 and any graph of form Δ_i (see Fig. 5) is large enough, so that no induced subgraph of G' belongs to F.

Hence, the resulting graph G' belongs to X, which means that the ID-set problem is NP-hard in X. □

3. Polynomially solvable cases

Theorems 1 and 2 provide two sufficient conditions for the ID-set problem to be NP-hard in a finitely defined class of graphs. A natural question is whether those conditions are necessary as well. If the answer is affirmative, the only way to prove it is to develop polynomial time algorithms for graph classes that fail both conditions. This problem seems to be much more difficult. In this section, we review several general graph techniques that might be useful in solving the problem and illustrate their application to particular graph classes. Most approaches permit to solve the problem even in the case of weighted graphs (weighted independent domination).

A simple idea to solve the problem is to generate all maximal ISs. For a graph G with n vertices and m edges, this can be done in time $O(nmN)$, where N is the number of maximal ISs in G [27]. In case that N is bounded by a polynomial in the size of the graph, this idea leads to a polynomial algorithm to find an ID-set of minimum weight. This is the case for mK_2-free graphs with arbitrary fixed m, which has been proven independently by several researchers [1,2,25].

Another important notion that provides polynomial time solutions to weighted independent domination in a large family of graph classes is the clique-width of a graph. This notion was introduced in [8] and is defined as the minimum number of labels needed to represent the graph by an algebraic expression over a set of certain graph operations. As proved in [9], many NP-hard problems become tractable when restricted to graphs with bounded clique-width, provided that an algebraic expression representing the graph can be constructed in polynomial time. This is true for problems expressible in a monadic second-order logic with quantification over subsets of vertices but not edges. Paper [9] lists several such problems, including minimum dominating and maximum IS problems. As an immediate consequence, one may conclude that independent domination fits the formalism. Among graphs of bounded clique-width are trees, cographs (P_4-free graphs), distance hereditary graphs [18], $S_{1,2,3}$-free bipartite graphs [21]. Moreover, for any graph in the listed classes, an algebraic expression with bounded number of labels that represents the graph can be constructed in polynomial time. Examples of graphs with unbounded clique-width are split graphs [22] and bipartite permutation graphs [6]. As a result, the clique-width is not bounded for $2K_2$-free graphs (contain split graphs) and general bipartite graphs. Moreover, it has been shown in [10] that the clique-width is bounded for graphs in a certain class if and only if it is bounded for their complementary graphs. Consequently, co-bipartite graphs are not of bounded clique-width, in general.

3.1. Modular decomposition

The notion of clique-width was developed as a natural generalization of the concept of modular decomposition [23]. The latter approach was applied repeatedly to solve the maximum weight IS problem in special classes of graphs (see e.g. [3,12,15–17]). In fact, this approach is helpful for the ID-set problem as well. Given a class of graphs, modular decomposition reduces both problems to prime graphs in that class, defined as follows. Let M be a subset of vertices in a graph G. We say that a vertex $x \notin M$ distinguishes M if x has both a neighbor and a non-neighbor in M. A *module* in the graph is a proper subset of vertices in M indistinguishable to the vertices outside of M. A module M is called *trivial* if $|M| = 1$. A graph whose every module is trivial is called *prime*.

A remarkable property of maximal modules is that if G and co-G are both connected, then maximal modules of G are disjoint and they can be found in polynomial time (see e.g. [23]). This property permits to reduce both problems from graph G to a graph G^* obtained from G by contracting each maximal module to a single vertex. We describe this reduction more formally in the recursive procedure SET(G) below, where $w(S)$ denotes the weight of set S, i.e. the sum of weights of its vertices.

Procedure SET(G)
Input: a weighted graph G
Output: a maximal IS S in G with minimum (maximum) weight.

1. If $|V(G)| = 1$, set $S = V(G)$ and go to 7.
2. If G is disconnected, partition it into connected components $M_1, \dots, M_k$.
3. If co-G is disconnected, partition G into co-components $M_1, \dots, M_k$.
4. If G and co-G are connected, partition G into maximal modules $M_1, \dots, M_k$.
5. Construct a weighted graph G^* by contracting each M_j ($j = 1, \dots, k$) to a single vertex and assigning to that vertex weight $w(\mathrm{SET}(G[M_j]))$.
6. Find in G^* a maximal IS S^* with minimum (maximum) weight, and set $S = \bigcup_{j \in S^*} \mathrm{SET}(G[M_j])$.
7. Return S and STOP.

Procedure SET shows that weighted independent domination in a class of graphs X is polynomially equivalent to the same problem on prime graphs in X. For certain classes of graphs, this leads to polynomial time algorithms. As an example, we consider the class of (P_5,co-A,co-domino)-free graphs, where a *domino* is the graph obtained from a cycle C_6 by connecting two vertices at distance 3. An A is the graph produced from a domino by disconnecting a pair of 2-degree vertices.

Theorem 3. *Weighted independent domination is polynomially solvable in the class of* (P_5, *co-A*, *co-domino*)*-free graphs.*

Proof. It has been proven in [19] that any prime graph containing an induced $2K_2$ contains also either a P_5 or co-A or co-domino as induced subgraphs. Therefore, any prime (P_5,co-A,co-domino)-free graph is $2K_2$-free. As mentioned above, for $2K_2$-free graphs, an ID-set of minimum weight can be found in polynomial time with exhaustive search of all maximal ISs. Hence the problem is polynomially solvable for (P_5,co-A,co-domino)-free graphs by means of modular decomposition. □

Remark 1. Note that (P_5,co-A,co-domino)-free graphs include both P_4-free and $2K_2$-free graphs and hence are not of bounded clique-width.

Our next theorem deals with two subclasses of bull-free graphs: (bull, P_5)-free and (bull, chair)-free graphs. Observe that independent domination is NP-hard in the class of bull-free graphs (by Theorem 1) as well as in the class of chair-free graphs (by Theorem 2).

Theorem 4. *Weighted independent domination is polynomially solvable for* (*bull*, P_5)*-free and* (*bull*, *chair*)*-free graphs.*

Proof. For both classes, we use the result of De Simone [12], who studied the class of graphs in which every prime graph is either bipartite or co-diamond-free or an odd hole (i.e. an induced cycle C_k with odd $k \geqslant 5$). Specifically, she has characterized this class by a list of forbidden induced subgraphs. It is a trivial task to verify that every forbidden graph in the list containing no bull as an induced subgraph contains either a chair or a P_5. Therefore, both (bull, chair)-free graphs and (bull, P_5)-free graphs are subclasses of the class studied by De Simone. As a consequence, every prime graph in those classes is either bipartite or co-diamond-free or an odd hole. For co-diamond-free graphs the problem is polynomially solvable since it is a subclass of $3K_2$-free graphs. Solvability of the problem for cycles follows from the fact that these are graphs of bounded clique-width [10]. The same is true for bipartite P_5-free and chair-free graphs since these are subclasses of $S_{1,2,3}$-free bipartite graphs, whose clique-width is at most five [21]. □

Remark 2. Both (bull, P_5)-free and (bull, chair)-free graphs extend cographs but neither of these classes is of bounded clique-width. Indeed, all three forbidden graphs contain the complement to a triangle and hence both classes include all co-bipartite graphs.

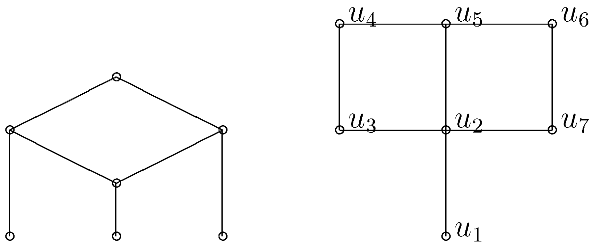

Fig. 6. Graphs Sun_3 and $\Phi(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$.

3.2. Neighborhood reduction

Let G be a graph with vertices $v_1, v_2, \dots, v_n$, and let G_j denote the induced subgraph of G obtained by deleting the closed neighborhood of vertex v_j.

Lemma 4. *Polynomial solvability of the weighted ID-set problem for graphs G_1, $G_2, \dots, G_n$ implies polynomial solvability of the problem for G.*

Proof. Given a vertex v in a graph G, any maximal under inclusion IS in G contains at least one vertex in $N[v]$. For a vertex $u \in N[v]$, denote by S_u an ID-set of minimum weight in the subgraph $G - N[u]$. Then a set of form $S_u \cup \{u\}$ with minimum weight gives a solution to the problem in the graph G. □

We now apply the preceding lemma to prove polynomial solvability of the problem in the class of $(S_{2,2,2}, Sun_3)$-free bipartite graphs (see Fig. 6 for definition of Sun_3). Note that the problem is NP-hard in the class of Sun_3-free bipartite graphs due to Theorem 1.

We show polynomial time solvability of the problem in the class of $(S_{2,2,2}, Sun_3)$-free bipartite graphs by reducing it to the class of bipartite permutation graphs, i.e. the intersection of the class of bipartite graphs with that of permutation graphs. To derive the result we use the characterization of bipartite permutation graphs in terms of forbidden induced subgraphs: these are precisely the bipartite graphs containing no $S_{2,2,2}$, Sun_3, Φ or C_n with $n > 5$ as induced subgraphs (the forbidden induced subgraph characterization of permutation graphs can be found in [13]).

Theorem 5. *If G is a connected $(S_{2,2,2}, Sun_3)$-free bipartite graph, then for any vertex v of G, subgraph $G - N[v]$ is a bipartite permutation graph.*

Proof. According to the induced subgraph characterization of bipartite permutation graphs, we have to show only that $G - N[v]$ contains no Φ and no cycle C_n with $n > 5$. Assume the contrary and let first $G - N[v]$ contain a C_n with vertices $u_1, u_2, \dots, u_n$ ($n > 5$). We consider a shortest path ($u_j = x_0, x_1, x_2, \dots, x_k = v$) of length $k \geqslant 2$ connecting the cycle to vertex v. It is easy to see that x_1 is not adjacent to u_{j+2} else G contains a Sun_3

induced by vertices $u_{j-1}, u_j, u_{j+1}, u_{j+2}, u_{j+3}, x_1, x_2$. Symmetrically, x_1 is not adjacent to u_{j-2}. But then an induced $S_{2,2,2}$ arises.

Now let $G - N[v]$ contain a Φ induced by vertices $u_1, \ldots, u_7$ as shown in Fig. 6. Again, consider a shortest path $(u_j = x_0, x_1, x_2, \ldots, x_k = v)$ of length $k \geqslant 2$ connecting the Φ to vertex v. Up to the symmetry, we have to analyze 5 cases: $j = 1, 2, 3, 4, 5$. In the analysis, we denote $U = \{x_1, x_2, u_1, \ldots, u_7\}$.

Case $j = 1$: If x_1 is adjacent to u_3, then G contains a Sun_3 induced either by $U - \{u_5, u_6\}$ (if $(x_1, u_7) \notin E$) or by $U - \{u_1, u_5\}$ (if $(x_1, u_7) \in E$). By symmetry, x_1 is not adjacent to u_7. But then $U - \{x_2, u_5\}$ induces a $S_{2,2,2}$.

Case $j = 2$: If x_1 is adjacent to u_4, then G contains a Sun_3 induced either by $U - \{u_3, u_7\}$ (if $(x_1, u_6) \notin E$) or by $U - \{u_1, u_2\}$ (if $(x_1, u_6) \in E$). By symmetry, x_1 is not adjacent to u_6. But then $U - \{u_1, u_5\}$ induces a $S_{2,2,2}$.

Case $j = 3$: If x_1 is not adjacent to u_1, then G contains a Sun_3 induced either by $U - \{x_2, u_7\}$ (if $(x_1, u_5) \notin E$) or by $U - \{u_4, u_7\}$ (if $(x_1, u_5) \in E$). Therefore, x_1 is adjacent to u_1 and consequently to u_7 else $U - \{u_5, u_6\}$ induces a Sun_3. But now a Sun_3 is induced by $U - \{u_1, u_5\}$.

Case $j = 4$: If x_1 is adjacent to u_6, then G contains a Sun_3 induced by $U - \{u_1, u_2\}$. If x_1 is not adjacent to u_6, then G contains a Sun_3 induced either by $U - \{x_2, u_7\}$ (if $(x_1, u_2) \notin E$) or by $U - \{u_3, u_7\}$ (if $(x_1, u_2) \in E$).

Case $j = 5$: If x_1 has no neighbors in $\{u_3, u_7\}$, then a $S_{2,2,2}$ arises induced by vertices $U - \{u_1, u_2\}$. If x_1 is adjacent to, say, u_3, then x_1 is not adjacent to u_7 else G contains a Sun_3 induced by $U - \{u_1, u_5\}$. But now G contains a Sun_3 induced either by $U - \{u_4, u_7\}$ (if $(x_1, u_1) \notin E$) or by $U - \{u_3, u_6\}$ (if $(x_1, u_1) \in E$). □

Combining Lemma 4 and Theorem 5 with a polynomial time solution to the problem in bipartite permutation graphs (see, e.g., [5]), we derive the conclusion.

Corollary 1. *Weighted independent domination is polynomially solvable in the class of* $(S_{2,2,2}, Sun_3)$*-free bipartite graphs.*

Remark 3. Note that the clique-width of $(S_{2,2,2}, Sun_3)$-free bipartite graphs is unbounded since they contain all bipartite permutation graphs.

3.3. Decreasing graphs

Now let us consider the class of claw-free graphs. By Theorem 2, the ID-set problem is NP-hard in this class. In contrast, the maximum IS problem has a polynomial time solution for claw-free graphs [24,26]. The idea to solve the latter problem is based on finding augmenting graphs. This suggests an approach that can hopefully lead to efficient algorithms for the ID-set problem in subclasses of claw-free graphs. The idea is as follows.

Let G be a graph and S an ID-set in G. We call the vertices in S *white* and the remaining vertices of G *black*.

Assume G contains an induced bipartite subgraph $H = (W, B, E)$ with set of white vertices W and set of black vertices B satisfying the following conditions: $|B| < |W|$, and $S' = (S - W) \cup B$ is an ID-set in G. Since the size of S' is strictly smaller than that of S, we call the subgraph H *decreasing* for S, and say that S admits the decreasing graph H.

Conversely, assume the cardinality of S is not minimum and let S' denote a smaller ID-set in G. Then obviously the subgraph of G induced by set $(S - S') \cup (S' - S)$ is decreasing for S. We thus have proved the following theorem.

Theorem 6. *An ID-set S in a graph G is minimum if and only if S admits no decreasing graph.*

For any induced bipartite subgraph $H = (W, B, E)$ with set of white vertices W and set of black vertices B, the value of $|W| - |B|$ will be called the *decrement* of H. We call $H = (W, B, E)$ *even* if $|W| = |B|$, or *odd* otherwise. Clearly, every decreasing graph is odd else its decrement is zero.

We now turn our attention to some properties of claw-free decreasing graphs. By definition, any decreasing graph is bipartite. Obviously any connected component in a claw-free bipartite graph is either a cycle or a path. We can say more when restricting ourselves to decreasing graphs that are minimal under inclusion.

Lemma 5. *Let S be an ID-set and $H = (W, B, E)$, a disconnected claw-free decreasing graph for S which is minimal under inclusion. Denote by $H_1 = (W_1, B_1, E_1)$ any proper collection of connected components of H with positive decrement and let $H_2 = (W_2, B_2, E_2)$ be the rest of H. Then there is a black vertex outside H that has exactly one neighbor in W_1 and exactly one neighbor in B_2.*

Proof. Exchanging white vertices of H_1 with its black vertices produces an independent set S' of smaller size than S. Obviously, S' is not dominating else H_1 would be a decreasing graph for S contradicting minimality of H. Hence, there must be a black vertex v non-adjacent to any vertex in $S' \supseteq B_1 \cup W_2$. Since S is a dominating set, vertex v has a neighbor in W_1. Furthermore, v has a neighbor in B_2, otherwise $(S - W) \cup B$ is not a dominating set. Any other neighbor of v in W_1 or B_2 would lead to an induced claw with center v. □

Lemma 6. *In the class of claw-free graphs every minimal decreasing graph $H = (W, B, E)$ is cycle-free.*

Proof. Assume H contains a connected component H_2 being a cycle. Then $H_1 = H - H_2$ has a positive decrement and hence meets the condition of Lemma 5. Consequently, there is a vertex that has exactly one neighbor in H_2, and clearly this neighbor is the center of a claw. □

The general properties of claw-free decreasing graphs described in the two proceeding lemmas are not sufficient to solve the problem. However, they can be helpful in solving the problem in certain subclasses of claw-free graphs. An example of such a class follows below.

Lemma 7. *In the class of* $(P_6, claw)$*-free graphs any minimal decreasing graph is connected.*

Proof. Let G be a $(P_6$,claw)-free graph, S a maximal independent set in G, and $H=(W,B,E)$ a minimal decreasing graph for S. At least one connected component of H is a path P_k with positive decrement, i.e. $k>2$. Assume by contradiction that H is disconnected and let $H_2=H-P_k$. By Lemma 5, there exists a vertex v that has a neighbor u in P_k and a neighbor w in H_2. Obviously, u is an endpoint of the path P_k else it is the center of a claw. But now vertices of P_k together with v, w and a neighbor of w in H_2 induce a path with at least 6 vertices, a contradiction. □

We now summarize all the above arguments in the following theorem.

Theorem 7. *Given a* $(P_6, claw)$*-free graph* G *with* n *vertices, one can find an ID-set of minimum cardinality in* G *in time* $O(n^3)$.

Proof. Let S be an ID-set in G. If S admits a decreasing graph, then minimal of such a graph is either a P_3 or P_5 due to the lemma above. Determining whether S admits a decreasing P_3 or P_5 can be trivially implemented in time $O(n^2)$. Since the number of decreasing steps is at most n, the total time to solve the problem is $O(n^3)$. □

Remark 4. Both a P_6 and a claw contain the complement to a triangle as an induced subgraph. Therefore, the class of $(P_6$,claw)-free graphs includes all co-bipartite graphs, which means that the clique-width of $(P_6$,claw)- free graphs is unbounded.

The result for $(P_6$,claw)-free graphs is sharp in the following sense. In the class of $(P_k$,claw)-free graphs with $k>6$, there are minimal decreasing graphs with arbitrary many vertices. To show this, consider a graph G with $3m$ vertices $a_1,\dots,a_m,b_1,\dots,b_m$, $c_1,\dots,c_m$. Assume vertices $a_1,\dots,a_m$ form a clique in G, every vertex of form c_j is of degree 1, and every vertex of form b_j has exactly two neighbors in G, namely a_j and c_j. It is easy to see that G is a $(P_k$,claw)-free graph for any $k>6$. Consider an independent set $S_1=\{a_1,c_1,\dots,c_m\}$ in G. It is maximal but not minimum, since set $S_2=\{b_1,\dots,b_m\}$ is a maximal independent set of smaller size. It is not hard to verify that the subgraph of G induced by vertices $S_1\cup S_2$ is a minimal decreasing graph for set S_1, which is not connected.

References

[1] V.E. Alekseev, On the number of maximal independent sets in graphs from hereditary classes, in: A.A. Markov (Ed.), Combinatorial–algebraic Methods in Discrete Optimization, University of Nizhny Novgorod, Nizhny, Novgorod, 1991, pp. 5–8 (in Russian).

[2] E. Balas, Ch.S. Yu, On graphs with polynomially solvable maximum-weight clique problem, Networks 19 (1989) 247–253.

[3] P. Bertolazzi, C. De Simone, A. Galuccio, A nice class for the vertex packing problem, Discrete Appl. Math. 76 (1997) 3–19.

[4] T.A. Beyer, A. Proskurowski, S.T. Hedetniemi, S. Mitchell, Independent domination in trees, Congr. Numer. 19 (1977) 321–328.
[5] A. Brandstädt, D. Kratsch, On domination problems on permutation and other graphs, Theoret. Comput. Sci. 54 (1987) 181–198.
[6] A. Brandstädt, V.V. Lozin, On the linear structure and clique-width of bipartite permutation graphs, Ars Combin., to appear.
[7] M.-S. Chang, Efficient algorithms for the domination problems on interval and circle-arc graphs, SIAM J. Comput. 27 (1998) 1671–1694.
[8] B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraphs grammars, J. Comput. System Sci. 46 (1993) 218–270.
[9] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, Theory Comput. Systems 33 (2000) 125–150.
[10] B. Courcelle, S. Olariu, Upper bounds to the clique-width of a graphs, Discrete Appl. Math. 101 (2000) 77–114.
[11] P. Damaschke, H. Müller, D. Kratsch, Domination in convex and chordal bipartite graphs, Inform. Process. Lett. 36 (1990) 231–236.
[12] C. De Simone, On the vertex packing problem, Graphs Combin. 9 (1993) 19–30.
[13] P. Duchet, Classical perfect graphs, Ann. Discrete Math. 21 (1984) 67–96.
[14] M. Farber, Independent domination in chordal graphs, Oper. Res. Lett. 1 (1982) 134–138.
[15] J.-L. Fouquet, A decomposition for a class of $(P_{t5}, \overline{P_5})$-free graphs, Discrete Math. 121 (1993) 75–83.
[16] J.-L. Fouquet, V. Giakoumakis, On semi-P_4-sparse graphs, Discrete Math. 165–166 (1997) 277–300.
[17] V. Giakoumakis, I. Rusu, Weighted parameters in $(P_5, \overline{P_5})$-free graphs, Discrete Appl. Math. 80 (2–3) (1997) 255–261.
[18] M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, Internat. J. Found. Comput. Sci. 11 (2000) 423–443.
[19] C.T. Hoang, B.A. Reed, Some classes of perfectly orderable graphs, J. Graph Theory 13 (1989) 445–463.
[20] D. Kratsch, L. Stewart, Domination of co-comparability graphs, SIAM J. Discrete Math. 6 (3) (1993) 400–417.
[21] V.V. Lozin, Bipartite graphs without a skew star, Discrete Math. 257 (2002) 83–100.
[22] J.A. Makowsky, U. Rotics, On the clique-width of graphs with few P_4's, Internat. J. Found. Comput. Sci. 10 (3) (1999) 329–348.
[23] R.M. McConnell, J.P. Spinrad, Modular decomposition and transitive orientation, Discrete Math. 201 (1999) 189–241.
[24] G.J. Minty, On maximal independent sets of vertices in claw-free graphs, J. Combin. Theory, Ser. B 28 (1980) 284–304.
[25] E. Prisner, Graphs with few cliques, Graph Theory, Combinatorics, and Algorithms, vol. 1, 2, Kalamazoo, MI, 1992, Wiley-Interscience, Wiley, New York, 1995, pp. 945–956.
[26] N. Sbihi, Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile, Discrete Math. 29 (1980) 53–76.
[27] S. Tsukiyama, M. Ide, H. Ariyoshi, I. Shirakawa, A new algorithm for generating all the maximal independent sets, SIAM J. Comput. 6 (1977) 505–517.