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ORIGINAL ARTICLE

Complete Ranking of Intuitionistic Fuzzy Numbers

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Abstract Fuzzy number was introduced by Dubois and Prade [10] to handle imprecise numerical quantities. Later it was generalized to intuitionistic fuzzy number by Burillo et al. [5]. Ranking intuitionistic fuzzy numbers plays an important role in decision making and information systems. All over the world many researchers have proposed different score functions for ranking intuitionistic fuzzy numbers but unfortunately every method produces some anti-intuitive results in certain places. A complete ranking on the entire class of fuzzy numbers have been achieved by W. Wang, Z. Wang [22] using upper dense sequence defined in $(0, 1]$. But a complete ranking on the set of all intuitionistic fuzzy number remains an open problem till today. Complete ranking on the class of intuitionistic fuzzy interval number was done by Geetha et al. [13]. In this paper, total ordering on the entire class of intuitionistic fuzzy number (IFN) using upper lower dense sequence is proposed and compared with existing techniques using illustrative examples. This new total ordering on intuitionistic fuzzy numbers (IFNs) generalizes the total ordering defined in W. Wang, Z. Wang [22] for fuzzy numbers (FNs).

Keywords Upper lower dense sequence · Total order relation · Intuitionistic fuzzy number · Intuitionistic fuzzy interval number · Trapezoidal intuitionistic fuzzy num-

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ber · Lexicography.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [27]. Atanassov generalizes this idea to intuitionistic fuzzy sets (IFSs) [1], and later there has been much progress in the study of IFSs. As a special case of fuzzy sets, fuzzy number was introduced by Dubois and Prade [10] to handle imprecise numerical quantities. The ranking of intuitionistic fuzzy numbers plays an important role in real life problems involving uncertainties, imprecise and incomplete information. Since intuitionistic fuzzy sets are characterized by membership, non-membership functions, it is a powerful alternative tool to characterize uncertainty, imprecision and vagueness in many fields such as decision making, logic programming, machine learning, information systems etc. The concept of real valued intutionistic fuzzy numbers was introduced by Xu [24] and it has been generalized to interval valued intuitionistic fuzzy number by Xu and Chen [25]. Total ordering on the class of real valued IFNs is given by Xu [26]. Later in 2009, Wang et al. [23] gave a complete ranking procedure on the class of intuitionistic fuzzy interval numbers (IFINs). Ranking fuzzy numbers and intuitionistic fuzzy numbers have started long back, but till date there is no common method is available to rank any two given IFNs. The difficulty of defining total ordering for all intuitionistic fuzzy numbers is that there is no effective tool to identify an arbitrarily given intuitionistic fuzzy number by only finitely many real-valued parameters. In this work by establishing a new decomposition theorem for IFSs, any IFN can be identified by countable number of parameters.

W. Wang and Z. Wang [22] introduced the concept of upper dense sequence to identify any two given fuzzy numbers by countable number of real valued parameters. In the same way in this paper, a new concept of upper lower dense sequence in $[0, 1] \times [0, 1]$ is introduced. Actually there are many upper lower dense sequences are available in $[0, 1] \times [0, 1]$. A new decomposition theorem for intuitionistic fuzzy sets is established by the use of any one of the upper lower dense sequences defined in $[0, 1] \times [0, 1]$. Then using a chosen upper lower dense sequence as one of the necessary reference systems, infinitely many total orderings on the set of all IFNs can be well defined. This paper is divided into seven sections. After introduction, some basic definitions are given in Section 2. The concept of upper lower dense sequence in $[0, 1] \times [0, 1]$ is introduced and a new decomposition theorem for intuitionistic fuzzy sets is established in Section 3. Total ordering on the class of intuitionistic fuzzy number using upper lower dense sequence is achieved in Section 4. Comparison between our proposed method with other existing ones are explained in Section 5 with illustrative examples. Application of our proposed method in solving multi-criteria decision making problem is represented in Section 6. Finally comes the conclusions in Section 7.

2. Preliminaries

Here we give a brief review of some preliminaries.

Definition 2.1 [24] *Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A in X is defined by* $A = (\mu_A, \nu_A)$*, where* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *with the conditions* $0 \le \mu_A(x) + \nu_A(x) \le 1, \forall x \in X$. *The numbers* $\mu_A(x), \nu_A(x) \in [0,1]$ *denote the degree of membership and non-membership of x to lie in A respectively. For each intuitionistic fuzzy subset A in X,* $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ *is called hesitancy degree of x to lie in A.*

Definition 2.2 [20] *Let X be a nonempty universal set and A be a intuitionistic fuzzy set of X with membership function* μ_A *and with a nonmembership function* ν_A. *The* (α, β)*-cut and strong* (α, β)*-cut of A are denoted by* ${}^{(\alpha,\beta)}A$ *and* ${}^{(\alpha,\beta)+}A$ *respectively, that is,* ${}^{(\alpha,\beta)}A = \{x \mid \mu_A(x) \ge \alpha, \text{ and } \nu_A(x) \le \beta, \ x \in X\}$*, equivalently* ${}^{(\alpha,\beta)}A = {}^{\alpha}A \cap {}^{\beta}A$ *and* ${}^{(\alpha,\beta)+}A = \{x \mid \mu_A(x) > \alpha, \text{ and } \nu_A(x) < \beta, \ x \in X\}$*, equivalently* ${}^{(\alpha,\beta)+}A = {}^{\alpha+}A \cap {}^{\beta+}A$ *for all* $\alpha, \beta \in [0,1]$. *The Level set of A is defined by* $L_A = \{(\alpha, \beta) \mid \mu_A(x) = \alpha, \text{ and } \nu_A(x) = \beta, \ x \in X\}$ *for all* $\alpha, \beta \in [0,1]$.

One way of representing a fuzzy set is by special fuzzy sets on α*-cuts and another way of representing a fuzzy set is by special fuzzy sets on strong* α*-cuts. As a generalization of fuzzy sets, any intuitionistic fuzzy set can also be represented by the use of special intuitionistic fuzzy sets on* (α, β) *cuts and special intuitionistic fuzzy sets on strong* (α,β) *cuts. The special intuitionistic fuzzy sets* ${}_{(\alpha,\beta)}A = {}_{\alpha}A \cap {}_{\beta}A$*, and* ${}_{(\alpha,\beta)+}A = {}_{\alpha+}A \cap {}_{\beta+}A$ *are defined as follows,*

$$ {}_{(\alpha,\beta)}A(x) = \begin{cases} (\alpha, \beta), \ x \in^{\alpha} \mu_A \text{ and } x \in {}^{\beta}\nu_A, \\ (0, 1), \ \text{otherwise}, \end{cases} $$

$$ {}_{(\alpha,\beta)+}A(x) = \begin{cases} (\alpha, \beta), \ x \in^{\alpha+} \mu_A \text{ and } x \in {}^{\beta+}\nu_A, \\ (0, 1), \ \text{otherwise}. \end{cases} $$

Let $I \subset [0,1] \times [0,1]$ *be the set which is used to give values for* α, β *in* (α,β)*-cut, and it is defined as* $I = \{(\alpha,\beta) \in [0,1] \times [0,1] \mid \alpha + \beta \le 1\}$.

Definition 2.3 [24] *An intuitionistic fuzzy number* $A = (\mu_A, \nu_A)$ *in the set of real numbers* $\Re$*, is defined as*

$$ \mu_A(x) = \begin{cases} f_A(x), & \text{if } a \le x \le b_1, \\ 1, & \text{if } b_1 \le x \le b_2, \\ g_A(x), & \text{if } b_2 \le x \le c, \\ 0, & \text{otherwise}, \end{cases} $$

and

$$ \nu_A(x) = \begin{cases} h_A(x), & \text{if } e \le x \le f_1, \\ 0, & \text{if } f_1 \le x \le f_2, \\ k_A(x), & \text{if } f_2 \le x \le g, \\ 1, & \text{otherwise}, \end{cases} $$

where $0 \le \mu_A(x) + \nu_A(x) \le 1$ *and* $a, b_1, b_2, c, e, f_1, f_2, g \in \Re$ *such that* $e \le a, f_1 \le b_1 \le b_2 \le f_2, c \le g$*, and four functions* $f_A, g_A, h_A, k_A : \Re \to [0,1]$ *are called the legs of membership function* μ_A *and nonmembership function* ν_A*. The functions* f_A *and* k_A *are nondecreasing continuous functions and the functions* h_A *and* g_A *are nonincreasing continuous functions.*

An intuitionistic fuzzy number $\{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ with $(e, f_1, f_2, g) \le (a, b_1,$

$b_2, c)^c$ is shown in Fig.1.

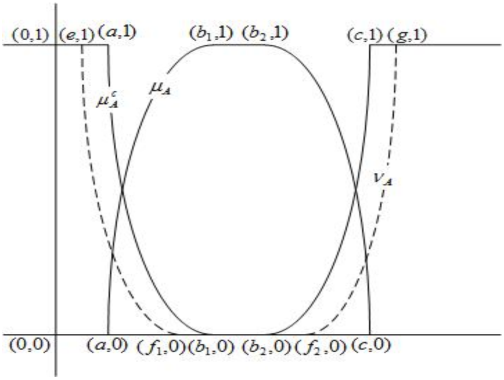

Fig. 1 Intuitionistic fuzzy number

Definition 2.4 *A trapezoidal intuitionistic fuzzy number A with parameters* $e \leq a, f_1 \leq b_1 \leq b_2 \leq f_2, c \leq g$, *is denoted as* $A = \{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ *in the set of real numbers* $\Re$ *is an intuitionistic fuzzy number whose membership function and non-membership function are given as*

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & \text{if } a_1 \leq x \leq a_2, \\ 1, & \text{if } a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3}, & \text{if } a_3 \leq x \leq a_4, \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_A(x) = \begin{cases} \dfrac{x - c_2}{c_1 - c_2}, & \text{if } c_1 \leq x \leq c_2, \\ 0, & \text{if } c_2 \leq x \leq c_3, \\ \dfrac{x - c_3}{c_4 - c_3}, & \text{if } c_3 \leq x \leq c_4, \\ 1, & \text{otherwise,} \end{cases}$$

If $a_2 = a_3$ (and $c_2 = c_3$) in a trapezoidal intuitionistic fuzzy number A, we have the triangular intuitionistic fuzzy numbers as special case of the trapezoidal intuitionistic fuzzy numbers. A trapezoidal intuitionistic fuzzy number $A = \{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ with $f_1 \leq b_1$, $f_2 \geq b_2$, $e \leq a$, and $g \geq c$ is shown in Fig. 2.

We note that the condition $(e, f_1, f_2, g) \leq (a, b_1, b_2, c)^c$ of the trapezoidal intuitionistic fuzzy number $A = \{(a, b_1, b_2, c), (e, f_1, f_2, g)\}$ whose membership and non-membership fuzzy numbers of A are (a, b_1, b_2, c) and (e, f_1, f_2, g) implies $f_1 \leq b_1$, $f_2 \geq b_2$, $e \leq a$, and $g \geq c$ on the legs of trapezoidal intuitionistic fuzzy number.

Definition 2.5 [25] *Let* $D[0, 1]$ *be the set of all closed subintervals of the interval* $[0, 1]$. *An interval valued intuitionistic fuzzy set on a set* $X \neq \phi$ *is an expression given*

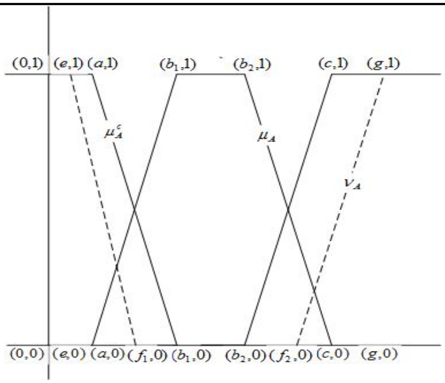

Fig. 2 Trapezoidal intuitionistic fuzzy number

by $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$*, where* $\mu_A : X \to D[0,1], \nu_A : X \to D[0,1]$ *with the condition* $0 < \sup_x \mu_A(x) + \sup_x \nu_A(x) \leq 1$.

The intervals $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of belongingness and non-belongingness of the element x to the set A. Thus for each $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals whose lower and upper end points are, respectively, denoted by $\mu_{A_L}(x)$, $\mu_{A_U}(x)$ and $\nu_{A_L}(x)$, $\mu_{A_U}(x)$.

We denote $A = \{\langle x, [\mu_{A_L}(x), \mu_{A_U}(x)], [\nu_{A_L}(x), \nu_{A_U}(x)]\rangle \mid x \in X\}$ where $0 < \mu_A(x) + \nu_A(x) \leq 1$.

For each element $x \in X$, we can compute the unknown degree (hesitance degree) of belongingness $\pi_A(x)$ to A as $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = [1 - \mu_{A_U}(x) - \nu_{A_U}(x), 1 - \mu_{A_L}(x) - \nu_{A_L}(x)]$. We denote the set of all IVIFSs in X by IVIFS(X). An intuitionistic fuzzy interval number (IFIN) is denoted by $A = ([a,b],[c,d])$ for convenience.

Definition 2.6 [22] *Orderings: Let X be a non-empty set. Any subset of the cartesian product X × X is called a relation, denoted by R, on X. We write aRb iff* $(a,b) \in R$*. Relation R is called a partial ordering on X if it is reflexive, antisymmetric, and transitive. A Partial ordering R on X is called Total ordering if either aRb or bRa for any* $a, b \in X$*. Two total orderings* R_1 *and* R_2 *are different iff there exist* $a, b \in X$ *with* $a \neq b$ *such that* aR_1b *but* bR_2a*. For any given total ordered infinite set, there are infinitely many different ways to redefine a new total ordering on it. Relation R is called an equivalence relation if it is reflexive, symmetric and transitive.*

3. A Complete Ranking of Intuitionistic Fuzzy Numbers

The difficulty of defining total ordering on the set of IFNs is that there is no effective tool to identify any given IFN by finitely many real valued parameters. In this section, the total ordering on the class of intuitionistic fuzzy numbers is achieved by establishing a new decomposition theorem for IFSs. In this section first we will see total ordering defined on several common proper subsets of IFNs, before defining a complete ordering on the set of IFNs. We may write any IFIN A as $A = ([a,b],[c,d])$. Clearly, we know that the set of IFIN is a proper subset of IFNs.

A complete ranking of an IFIN using membership, nonmembership, vague and precise score functions has been done by Geetha et al. [13]. In this section, the ranking methodology of Geetha et al. [13] is reviewed briefly.

Definition 3.1 [17] *For any IFIN* $A = ([a,b],[c,d])$*, the membership score function is defined as* $L(A) = \dfrac{a+b-c-d+ac+bd}{2}$.

Definition 3.2 [13] *For any IFIN* $A = ([a,b],[c,d])$*, the non-membership score function is defined as* $LG(A) = \dfrac{-a-b+c+d+ac+bd}{2}$.

Definition 3.3 [13] *For any IFIN* $A = ([a,b],[c,d])$*, the vague score function is defined as* $P(A) = \dfrac{a-b-c+d+ac+bd}{2}$.

Definition 3.4 [13] *For any IFIN* $A = ([a,b],[c,d])$*, the imprecise score function is defined as* $IP(A) = \dfrac{-a+b-c+d-ac+bd}{2}$.

Let $A = ([a_1,b_1],[c_1,d_1]), B = ([a_2,b_2],[c_2,d_2])$ be any two IFINs. The complete ranking $\leq$ on the class of IFIN may be defined to one of the following criterion:

(1) $L(A) < L(B)$, or

(2) $L(A) = L(B)$ but $-LG(A) < -LG(B)$, or

(3) $L(A) = L(B)$ and $LG(A) = LG(B)$ but $P(A) < P(B)$, or

(4) $L(A) = L(B), LG(A) = LG(B)$ and $P(A) = P(B)$ but $-IP(A) \leq -IP(B)$.

The above way of defining a total ordering is often referred to as lexicographic in literature [19].

Note 3.1 Let $A = ([a_1,b_1],[1-b_1,1-a_1])$ and $B = ([a_2,b_2],[1-b_2,1-a_2])$ be any two IFIN. From Definitions 3.1 and 3.2 it is very clear that, $L(A) = \dfrac{3(a_1+b_1)}{2} - (1+a_1b_1)$, $LG(A) = (1-a_1b_1) - \dfrac{(a_1+b_1)}{2}$, and $L(B) = \dfrac{3(a_2+b_2)}{2} - (1+a_2b_2)$, $LG(B) = (1-a_2b_2) - \dfrac{(a_2+b_2)}{2}$. Suppose $L(A) = L(B) \cdots (I_1)$ and $LG(A) = LG(B) \cdots (I_2)$, then $3 \times (I_1) + (I_2)$ gives $a_1b_1 = a_2b_2 \cdots (I_3)$ and (I_1)-(I_2) gives $a_1 + b_1 = a_2 + b_2 \cdots (I_4)$. Hence from (I_3) and (I_4) we get $a_1 = a_2$ and $b_1 = b_2$.

That is if $A = ([a_1,b_1],[1-b_1,1-a_1])$ and $B = ([a_2,b_2],[1-b_2,1-a_2])$, then (1) and (2) (L and LG) are enough to cover any two arbitrary IFINs using new decomposition theorem.

3.1. Upper Lower Dense Sequence

In this section, the concept of upper lower dense sequence on I is introduced in order to establish new decomposition theorem for IFSs. This upper lower dense sequence on I gives values for α, β in the (α,β)-cut of IFNs. The already defined upper dense sequence [22] on $(0,1]$ is sufficient to give value for α in the α-cut of an FN. But for IFNs, two sequences are needed to give value for α & β in the (α,β)-cut where

$\alpha \in (0,1]$ & $\beta \in [0,1)$. The upper dense sequence defined in [22] is sufficient to give value for α in the (α,β)-cut of IFN but for giving values to β it is needed for us to define a new lower dense sequence in the interval $[0,1)$, since $\beta \in [0,1)$. In this section, the new lower dense sequence in the interval $[0,1)$ is introduced and some properties of this new lower dense sequence is studied.

Let us recall the concept of upper dense sequence before defining a lower dense sequence in $[0,1)$.

Definition 3.5 [22] *Let D be a set of real numbers in* $(0,1]$. *A set D is dense in* $(0,1]$ *if for every point* $[0,1)$ *and any* $\epsilon > 0$, *there exists* $\delta \in D$ *such that* $|x-\delta| < \epsilon$.

Definition 3.6 [22] *Let D be a set of real numbers in* $(0,1]$. *A set D is upper dense in* $(0,1]$ *if, for every point* $x \in (0,1]$ *and* $\epsilon > 0$, *there exists* $\delta \in D$ *such that* $\delta \in [x, x+\epsilon)$. *A set D is lower dense in* $(0,1]$ *if, for every point* $x \in (0,1]$ *and any* $\epsilon > 0$, *there exists* $\delta \in D$ *such that* $\delta \in (x-\epsilon, x]$.

Theorem 3.1 [22] *If D is dense in* $(0,1]$ *and* $1 \in D$, *then it is upper dense in* $(0,1]$.

Theorem 3.2 [22] *If D is dense in* $(0,1]$, *then it is lower dense in* $(0,1]$.

Definition 3.7 *Let D be a set of real numbers in* $[0,1)$. *A set* $D \subset [0,1)$ *is said to be upper dense in* $[0,1)$ *if, for every point* $x \in [0,1)$ *and* $\epsilon > 0$, *there exists* $\delta \in D$ *such that* $\delta \in [x, x+\epsilon)$. *A set* $D \subset [0,1)$ *is said to be lower dense in* $[0,1)$ *if, for every point* $x \in [0,1)$ *and any* $\epsilon > 0$, *there exists* $\delta \in D$ *such that* $\delta \in (x-\epsilon, x]$.

Definition 3.8 *Let D be a set of real numbers in* $[0,1)$. *A set D is dense in* $[0,1)$ *if, for every point* $[0,1)$ *and any* $\epsilon > 0$, *there exists* $\delta \in D$ *such that* $|x-\delta| < \epsilon$.

Theorem 3.3 *If D is dense in* $[0,1)$ *and* $0 \in D$, *then it is lower dense in* $[0,1)$.

Proof The proof of this theorem is similar to the proof of Theorem 1 in [22].

Theorem 3.4 *If D is dense in* $[0,1)$, *then it is upper dense in* $[0,1)$.

Proof The proof of this theorem is similar to the proof of Theorem 2 in [22].

From Theorems 3.1 and 3.2 it is noted that, any upper dense sequence in $(0,1]$ is nothing but a dense sequence with real number 1 and is also a lower dense sequence. Similarly, from Theorems 3.3 and 3.4 it is noted that, any lower dense sequence in $[0,1)$ is nothing but a dense sequence with real number 0 and is also an upper dense sequence.

Definition 3.9 *Let* $D_\alpha = \{\alpha_i \mid i = 1,2,\cdots\}$ *be the upper dense sequence in* $(0,1]$ *and let* $D_\beta = \{\beta_i \mid i = 1,2,\cdots\}$ *be the lower dense sequence in* $[0,1)$. *Then the upper lower dense sequence is of the form:* $D_{(\alpha,\beta)} = \{(\alpha_i,\beta_i) \mid \alpha_i + \beta_i \leq 1,\ where\ i = 1,2,\cdots\} \subset I$.

Examples for upper dense sequence in $(0,1]$ and lower dense sequence in $[0,1)$ are given as follows.

Example 3.1 Let $D_\alpha = \{d_{\alpha_i} \mid i = 1,2,\cdots\}$ be the set of all rational numbers in $(0,1]$, where $d_{\alpha_1} = 1, d_{\alpha_2} = \frac{1}{2}, d_{\alpha_3} = \frac{1}{3}, d_{\alpha_4} = \frac{2}{3}, d_{\alpha_5} = \frac{1}{4}, d_{\alpha_6} = \frac{3}{4}, d_{\alpha_7} = \frac{1}{5}, d_{\alpha_8} = \frac{2}{5}, d_{\alpha_9} = \frac{3}{5}, d_{\alpha_2} = \frac{4}{5}, \cdots$.

Clearly, D_α is an upper dense sequence in $(0,1]$. If we allow a number to have multiple occurrences in the sequence, the general members in upper dense sequence $S_\alpha = \{s_{\alpha_i} \mid i = 1,2,\cdots\}$ can be expressed by $s_{\alpha_i} = (\frac{i}{k} - \frac{k-1}{2}), i = 1,2,\cdots$, where $k = \left\lceil \sqrt{2i + \frac{1}{4}} - \frac{1}{2} \right\rceil$.

That is, $s_{\alpha_1} = 1$, $s_{\alpha_2} = \frac{1}{2}$, $s_{\alpha_3} = 1$, $s_{\alpha_4} = \frac{1}{3}$, $s_{\alpha_5} = \frac{2}{3}$, $s_{\alpha_6} = 1$, $_{\alpha_7} = \frac{1}{4}$, $s_{\alpha_8} = \frac{2}{4}$, $s_{\alpha_9} = \frac{3}{4}$. In sequence S_α, for instance, s_{α_3} is the same real number as s_{α_1}.

Example 3.2 Let $D_\beta = \{d_{\beta_i} \mid i = 1,2,\cdots\}$ be the set of all rational numbers in $[0,1)$, where $d_{\beta_1} = 0, d_{\beta_2} = \frac{4}{9}, d_{\beta_3} = \frac{3}{5}, d_{\beta_4} = \frac{2}{7}, d_{\beta_5} = \frac{2}{3}, d_{\beta_6} = \frac{2}{9}, d_{\beta_7} = \frac{5}{7}, d_{\beta_8} = \frac{1}{2}, d_{\beta_9} = \frac{1}{3}, \cdots$, and $d_{\beta_i} = (\frac{9(k^2+k-2i}{20k})$, $i = 1,2,\cdots$ and $k = \left\lceil \sqrt{2i + \frac{1}{4}} - \frac{1}{2} \right\rceil$. Therefore $D_\beta = \left\{0, \frac{4}{9}, \frac{3}{5}, \frac{2}{7}, \frac{2}{3}, \frac{2}{9}, \frac{5}{7}, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$. Now sequence D_β is lower dense sequence in $[0,1)$.

Example 3.3 In Examples 3.1 and 3.2, let $D_{(\alpha,\beta)} = \left\{(d_{\alpha_i}, d_{\beta_i}) \mid d_{\alpha_i} \in D_\alpha \& d_{\beta_i} \in D_\beta\right\} = \{(1,0), (1/2,4/9), (1/3,3/5), (2/3,2/7), (1/4,2/3), (3/4,2/9), (1/5,5/7), (2/5,1/2), \cdots\}$. Clearly, $D_{(\alpha,\beta)}$ is a upper lower dense sequence with $d_{\alpha_i} + d_{\beta_i} \leq 1$.

3.2. Decomposition Theorem for Intuitionistic Fuzzy Number Using Upper Lower Dense Sequence

In this subsection, the total ordering on the class of intuitionistics fuzzy numbers is achieved by establishing a new decomposition theorem for IFSs. Before establishing a new decomposition theorem, we recall existing decomposition theorems for intuitionistic fuzzy sets. The following decomposition theorems will show the representation of an arbitrary IFS in terms of the special IFSs $_{(\alpha,\beta)}A$.

Theorem 3.5 (first decomposition theorem of IFS [20]) *Let X be a non-empty set. For an intuitionistic fuzzy subset A in X,* $A = \bigcup_{\alpha,\, \beta \in [0,1]}\ {}_{(\alpha,\, \beta)}A$.

Theorem 3.6 (second decomposition theorem of IFS [20]) *Let X be a non-empty set. For an intuitionistic fuzzy subset A in X,* $A = \bigcup_{\alpha,\, \beta \in [0,1]}\ {}_{(\alpha,\, \beta)+}A$.

Theorem 3.7 (third decomposition theorem of IFS [20]) *Let X be a non-empty set. For an intuitionistic fuzzy subset A in X,* $A = \bigcup_{\alpha,\, \beta\, \in\, L_A}\ {}_{(\alpha,\beta)}A$, *where* L_A *is a Level set of A.*

Regarding IFN as special intuitionistic fuzzy subsets of $\Re$, these decomposition theorems are also available for IFNs. Unfortunately, all the above three decomposition theorems are failed to identify an arbitrary IFN by countable number of real-valued parameters. Therefore, establishing a new decomposition theorem, which identifies any IFN by only countably many real valued parameters is essential.

Theorem 3.8 (fourth decomposition theorem of IFS) *Let A be an intuitionistic fuzzy subset of X, and let S be a given upper lower dense sequence in I. Then* $A = \bigcup_{(\alpha,\, \beta) \in S}\ {}_{(\alpha,\, \beta)}A$, *where* ${}_{(\alpha,\, \beta)}A = (\alpha,\, \beta)^{(\alpha,\, \beta)}A$.

Proof Let A be an intuitionistic fuzzy subset of X, and S be a given upper lower dense sequence in I.

Claim: $A = \bigcup_{(\alpha,\beta)\in S} \ {}_{(\alpha,\beta)}A$.

Since $S \subseteq I$, we have $\bigcup_{(\alpha,\beta)\in S} \ {}_{(\alpha,\beta)}A \subseteq \bigcup_{(\alpha,\beta)\in I} \ {}_{(\alpha,\beta)}A = A$, i.e., $\sup_{(\alpha,\beta)\in[0,1]\times[0,1]} \mu_{{}_{(\alpha,\beta)}A}(x) \geq \sup_{(\alpha,\beta)\in S} \mu_{{}_{(\alpha,\beta)}A}(x)$ and $\inf_{(\alpha,\beta)\in[0,1]\times[0,1]} \nu_{{}_{(\alpha,\beta)}A}(x) \leq \inf_{(\alpha,\beta)\in S} \nu_{{}_{(\alpha,\beta)}A}(x)$ for every $x \in X$ which implies that,

$$A \supseteq \bigcup_{(\alpha,\beta)\in S} \ {}_{(\alpha,\beta)}A. \tag{3.1}$$

Now we have to show that $\mu_A(x) \leq \sup_{(\alpha,\beta)\in S} \mu_{{}_{(\alpha,\beta)}A}(x)$ and $\nu_A(x) \geq \inf_{(\alpha,\beta)\in S} \nu_{{}_{(\alpha,\beta)}A}(x)$ for every $x \in X$.

Let x be an arbitrary element in X and let $\mu_A(x) = a$ and $\nu_A(x) = b$. By second decomposition theorem, we have for each $x \in X$,

$$\begin{aligned}(\mu_A(x),\ \nu_A(x)) &= (\mathrm{Sup}_{(\alpha,\beta)\in I}\ \mu_{{}_{(\alpha,\beta)+}A}(x), \mathrm{Inf}_{(\alpha,\beta)\in I}\ \nu_{{}_{(\alpha,\beta)+}A}(x))\\ &= (\mathrm{Sup}_{(\alpha,\beta)\in[0,a)\times[0,1]\subseteq I}\ \mu_{{}_{(\alpha,\beta)+}A}(x), \mathrm{Inf}_{(\alpha,\beta)\in[0,1]\times(b,1]\subseteq I}\ \nu_{{}_{(\alpha,\beta)+}A}(x)).\end{aligned}$$

For each $\alpha \in [0, a)$ and $\beta \in (b, 1]$, since S is upper lower dense in I, we may find $(K_1,\ K_2) \in S$ such that $K_1 \geq \alpha$, and $K_2 \leq \beta$, which implies that

$$(\mu_{{}_{(\alpha,\beta)+}A}(x), \nu_{{}_{(\alpha,\beta)+}A}(x)) = (\alpha,\ \beta) = (\mu_{{}_{(\alpha,\beta)}A}(x), \nu_{{}_{(\alpha,\beta)}A}(x)).$$

Since $\mu_{{}_{(\alpha,\beta)}A}(x) \leq \mu_{{}_{(K_1,K_2)+}A}(x)$, and $\nu_{{}_{(\alpha,\beta)}A}(x) \geq \nu_{{}_{(K_1,K_2)+}A}(x)$, we get

$$\begin{aligned}(\mu_{{}_{(\alpha,\beta)}A}(x), \nu_{{}_{(\alpha,\beta)}A}(x)) &\leq (\mu_{{}_{(K_1,K_2)+}A}(x), \nu_{{}_{(K_1,K_2)+}A}(x))\\ &= (K_1,\ K_2) = (\mu_{{}_{(K_1,K_2)}A}(x), \nu_{{}_{(K_1,K_2)}A}(x)),\end{aligned}$$

which implies that

$$\mu_{{}_{(\alpha,\beta)}A}(x) \leq \mu_{{}_{(K_1,K_2)}A}(x) \leq \mathrm{Sup}_{(K_1,K_2)\in S}\ \mu_{{}_{(K_1,K_2)}A}(x)$$

and

$$\nu_{{}_{(\alpha,\beta)}A}(x) \geq \nu_{{}_{(K_1,K_2)}A}(x) \geq \mathrm{Inf}_{(K_1,K_2)\in S}\ \nu_{{}_{(K_1,K_2)}A}(x)).$$

Since $\mathrm{Sup}_{(K_1,K_2)\in S}\ \mu_{{}_{(K_1,K_2)}A}(x)$ is an upper bound for $\mu_{{}_{(\alpha,\beta)}A}(x)$ and $\mathrm{Inf}_{(K_1,K_2)\in S}\ \nu_{{}_{(K_1,K_2)}A}(x)$ is an lower bound for $\nu_{{}_{(\alpha,\beta)}A}(x)$, we have

$$\mathrm{Sup}_{(\alpha,\beta)\in I}\ \mu_{{}_{(\alpha,\beta)}A}(x) \leq \mathrm{Sup}_{(K_1,K_2)\in S}\ \mu_{{}_{(K_1,K_2)}A}(x)$$

and

$$\inf_{(\alpha,\beta)\in I}\ \nu_{{}_{(\alpha,\beta)}A}(x) \geq \mathrm{Inf}_{(K_1,K_2)\in S}\ \nu_{{}_{(K_1,K_2)}A}(x).$$

Therefore

$$\mu_A(x) \leq \sup_{(\alpha,\beta)\in S}\ \mu_{{}_{(\alpha,\beta)}A}(x) \text{ and } \nu_A(x) \geq \inf_{(\alpha,\beta)\in S}\ \nu_{{}_{(\alpha,\beta)}A}(x)\ \forall x \in X. \tag{3.2}$$

The proof is now concluded from (3.1) and (3.2).

4. Total Ordering on the Set of All Intuitionistic Fuzzy Numbers

The fourth decomposition theorem established in Section 3 identifies an arbitrary intuitionistic fuzzy number by countably many real-valued parameters. It provides us with a powerful tool for defining total order in the class of IFN by using extended lexicography.

Note 4.1 Let $S = \{(\alpha_i,\beta_i) \mid i = 1, 2, \cdots\} \in [0, 1]$ be an upper lower dense sequence for any given intuitionistic fuzzy number A. Since we know that the (α,β)-cut of an IFN A at each α_i,β_i , $i = 1, 2, \cdots$, is a closed interval which is obtained by taking intersection between the α-cut of a membership function of A and the β-cut of a non-membership function of A and it is denoted by $[a_i, b_i]$. Let $C_{2i-1} = \frac{3(a_i + b_i)}{2} - (1 + a_ib_i)$, $C_{2i} = \frac{-(a_i + b_i)}{2} + (1 - a_ib_i)$ where $i = 1, 2, \cdots$. By fourth decomposition theorem these countably many parameters $\{C_j \mid j = 1, 2, \cdots\}$ identify the intuitionistic fuzzy number. Using these parameters, we define a relation on the set of all intuitionistic fuzzy number as follows.

Definition 4.1 *Let A and B be any two IFNs. For any given upper lower dense sequence* $S = \{(\alpha_i, \beta_i) \mid i = 1, 2, \cdots\}$ *in I, we use* $C_j(A)$ *and* $C_j(B)$ *to denote above mentioned C′s for A and B respectively. We say that* $A = B$ *iff their* (α, β)*-cuts at each* α_i, β_i *are equal to each other, that is,* ${}^{(\alpha_i,\beta_i)}A = {}^{(\alpha_i,\beta_i)}B$ *for all* $i = 1, 2, \cdots$ *and we say that* $A \prec B$ *iff there exists a positive integer j such that* $C_j(A) < C_j(B)$ *and* $C_i(A) = C_i(B)$ *for all positive integers* $i < j$*. We say that* $A \preceq B$ *iff* $A \prec B$ *or* $A = B$*.*

Theorem 4.1 *Relation* $\preceq$ *is a total ordering on the set of all intuitionistic fuzzy number.*

Proof We claim that $\preceq$ is total ordering on the set of all intuitionistic fuzzy numbers. To prove $\preceq$ is total ordering we need to show the following (a) $\preceq$ is a partial ordering on the set of intuitionistic fuzzy numbers, (b) Any two elements in the set of intuitionistic fuzzy numbers are comparable.

(a) To show $\preceq$ is a partial ordering on the set of intuitionistic fuzzy numbers: We need to prove

(i) $\preceq$ is reflexive: Which is obvious.

(ii) $\preceq$ is antisymmetric:

If $A \preceq B$ and $B \preceq A$, then $A = B$.

Suppose $A \neq B$, then from the hypothesis $A \prec B$ and $B \prec A$. From Definition 4.1, we can find j_1 such that $C_{j_1}(A) < C_{j_1}(B)$ and $C_j(A) = C_j(B)$ for all positive integers $j < j_1$. Similarly, we are able to find j_2 such that $C_{j_2}(A) < C_{j_2}(B)$ and $C_j(A) = C_j(B)$ for all positive integers $j < j_2$. Then $j_1 \& j_2$ must be the same, let it to be j_0. But $C_{j_0}(A) < C_{j_0}(B)$, and $C_{j_0}(B) < C_{j_0}(A)$ this contradicts our hypothesis. Therefore our assumption $A \neq B$ is wrong. Hence $A = B$.

(iii) Now we prove $\preceq$ is transitive: If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.

Let A, B, C be three IFNs. Let us assume $A \preceq B$ and $B \preceq C$. Therefore from $A \preceq B$, we can find a positive integer k_1 such that $C_{k_1}(A) < C_{k_1}(B)$ and $C_k(A) = C_k(B)$ for all positive integer $k < k_1$. Similarly from $B \preceq C$, we can find a positive integer k_2 such that $C_{k_2}(B) < C_{k_2}(C)$ and $C_k(B) = C_k(C)$ for all positive integer $k < k_2$. Now taking $j_0 = \min(k_1, k_2)$, we have $C_{k_0}(A) < C_{k_0}(C)$ and $C_k(A) = C_k(C)$ for all positive integer $k < k_0$, i.e., $A < C$. Hence $\preceq$ is transitive.

Therefore from (i), (ii), and (iii), we proved the relation $\preceq$ is partial ordering on the set of all IFNs.

(b) Any two elements in the set of intuitionistic fuzzy numbers are comparable. For any two IFNs A and B, they are either $A = B$, or $A \neq B$. In the latter case, there are some integers j such that $C_j(A) \neq C_j(B)$. Let $J = \{j \mid C_j(A) \neq C_j(B)\}$. Then J is lower bounded by 0 and therefore, according to the well ordering property, J has unique smallest element, denoted by j_0. Thus we have $C_j(A) = C_j(B)$ for all positive integers $j < j_0$, and either $C_{j_0}(A) < C_{j_0}(B)$ or $C_{j_0}(A) > C_{j_0}(B)$, that is, either $A \prec B$ or $B \prec A$ in this case. So, for these two IFNs, either $A \preceq B$ or $B \preceq A$. This means that partial ordering $\preceq$ is a total ordering on the set of all intuitionistic fuzzy numbers. Hence the proof.

Similar to the case of total orderings on the real line $(-\infty, +\infty)$, total ordering on the class of IFNs shown in Section 3, infinitely many different total orderings on the set of IFNs can be defined. Even using a given upper lower dense sequence in I, there are still infinitely many different ways to determine a total ordering on the set of IFNs. A notable fact is that each of them is consistent with the natural ordering on the set of all real numbers. This can be regarded as a fundamental requirement for any practice of ordering method on the set of all IFNs.

Numerical Examples:

The following examples show that, how the total orderings work for ranking of IFNs. Many researchers have proposed different ranking methods on IFNs, but none of them has covered the entire class of IFNs, and also almost all the methods have the disadvantage that at some point of time they ranked two different numbers as the same. But the proposed total ordering can order any given IFNs.

Example 4.1 Let $M = \langle(0.4, 0.5, 0.7, 0.9), (0.3, 0.4, 0.8, 0.9)\rangle$ and $N = \langle(0.4, 0.5, 0.7, 0.8), (0.2, 0.3, 0.7, 0.85)\rangle$ be two IFNs. The total ordering $\preceq$ defined by using upper lower dense sequence $D_{(\alpha,\beta)}$ given in Example 3.3 and the way shown in Definition 4.1, Note 4.1 are now adapted. Since ${}^{(\alpha_1,\beta_1)}M =^{\alpha} M \cap {}^{\beta}M$. For $i = 1, (\alpha_1, \beta_1) = (1, 0)$, We have ${}^{(1,0)}M =^{1} M \cap {}^{0}M = [0.5, 0.7]$ and $C_1(M) = C_1(N) = 0.45$, $C_2(M) = C_2(N) = -0.05$. For $i = 2, (\alpha_2, \beta_2) = (1/2, 4/9)$, we have $C_3(M) = 0.515, C_3(N) = 0.4625$, i.e., $C_3(M) > C_3(N)$. Hence $M > N$.

Example 4.2 Let $A = \langle(0.1, 0.3, 0.4, 0.45), (0, 0.2, 0.45, 0.5)\rangle$ and $B = \langle(0.2, 0.3, 0.3, 0.4), (0, 0.1, 0.4, 0.5)\rangle$ be two IFNs. For $i = 1, (\alpha_1, \beta_1) = (1, 0)$, we have $C_1(A) = -0.07, C_1(B) = -0.19$, i.e., $C_1(A) > C_1(B)$. Hence $A > B$.

5. Significance of the Proposed Method

Ranking intuitionistic fuzzy numbers plays an important role in decision making and information systems. All over the world many researchers have proposed different score functions for ranking intuitionistic fuzzy numbers but unfortunately every method produces some anti-intuitive results in certain places. In this section, significance of our proposed method over some existing methods are explained with an examples. Table 1 shows how the illogicalities of two different IFNs are ranked equally by existing methods are rectified by our proposed method.

For example, let $A_1 = \langle(0, 0.25, 0.3), 1, 0\rangle$ & $A_2 = \langle(0.1, 0.2, 0.4), 1, 0\rangle$ be two different TrIFNs. Deng-Feng Li's method ranks A_1 and A_2 equally which is illogical, and it is rectified by our proposed method. From Table 1 we observed that the proposed ranking method overcome the shortcomings of existing ranking methods ([4, 7-9, 15, 21]).

Comparision of Proposed Method with Xu [26] and Wang et al. [23]

In this subsection, our proposed method is compared with Xu [26] and Wang et al. [23] ranking methods. Xu [26] gave total order on the collection of real valued intuitionistic fuzzy numbers but his method can not be applied to IFINs and TrIFNs (TIFNs) which are generalizations of real valued IFNs. Wang et al. [23] gave total order on the class of IFINs which supports our proposed method and it is shown

Table 1: Significance of proposed method.

Other existing methods	Shortcomings of exisiting methods	Numerical example (shortcomings)	Proposed method
Deng-Feng Li [7] $R(A,\lambda) = \frac{a_1+4a_2+a_3}{2(3+a_3-a_1)}$	$A_1 = \langle(a_1, a_2, a_3), 1, 0\rangle$, $A_2 = \langle(a_1-\epsilon, a_2+\frac{\epsilon}{2}, a_3-\epsilon), 1, 0\rangle$ $R(A_1) = R(A_2) \Rightarrow A_1 = A_2$	$A_1 = \langle(0, 0.25, 0.3), 1, 0\rangle$, $A_2 = \langle(0.1, 0.2, 0.4), 1, 0\rangle$ $R(A_1) = R(A_2) = 0.19697 \Rightarrow A_1 = A_2$	$C_1(A_1) = -0.44$, $C_1(A_2) = -0.3125$ $C_1(A_2) > C_1(A_1) \Rightarrow A_2 > A_1$
Deng-Feng Li, et al. [8] $V_\lambda(A) = \lambda V_\mu(A) + (1-\lambda)V_\nu(A)$ where $V_\mu(A) = \frac{(a_1+4a_2+a_3){w_A}^2}{6}$, $V_\nu(A) = \frac{(a_1+4a_2+a_3)(1-u_A)^2}{6}$ $A_\lambda(A) = \lambda A_\mu(A) + (1-\lambda)A_\nu(A)$ where $A_\mu(A) = \frac{(a_3-a_1){w_A}^2}{6}$, $A_\nu(A) = \frac{(a_3-a_1)(1-u_A)^2}{6}$	$A_1 = \langle(a_1, a_2, a_3), 1, 0\rangle$ $A_2 = \langle(a_1-\epsilon, a_2+\frac{\epsilon}{2}, a_3-\epsilon), 1, 0\rangle$ $V_\lambda(A_1) = V_\lambda(A_2)$ and $A_\lambda(A_1) = A_\lambda(A_2) \Rightarrow A_1 = A_2$	$A_1 = \langle(0, 0.25, 0.3), 1, 0\rangle$, $A_2 = \langle(0.1, 0.2, 0.4), 1, 0\rangle$ $V_\lambda(A_1) = V_\lambda(A_2) = 0.2167$ and $A_\lambda(A_1) = A_\lambda(A_2) = 0.05 \Rightarrow A_1 = A_2$	$C_1(A_1) = -0.44, C_1(A_2) = -0.3125$ $C_1(A_2) > C_1(A_1) \Rightarrow A_2 > A_1$
Hassan Mishmasi Nehi [15] $C^k_\mu(A) = \frac{a_2+a_3}{2} + \frac{a_1-a_2+a_4-a_3}{2(k+2)}, k \in \Re^+$ $C^k_\nu(A) = \frac{b_1+b_4}{2} + \frac{b_2-b_1+b_3-b_4}{2(k+2)}, k \in \Re^+$	$A = \langle(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)\rangle$ $B = \langle(a_1+\epsilon, a_2+\epsilon, a_3-\epsilon, a_4-\epsilon)$, $(b_1+\epsilon, b_2+\epsilon, b_3-\epsilon, b_4-\epsilon)\rangle$, $C^k_\mu(A) = C^k_\mu(B),\ and\ C^k_\nu(A) = C^k_\nu(B)$ $\Rightarrow A = B$	$A = \langle(0.2, 0.3, 0.5, 0.6), (0, 0.1, 0.6, 0.7)\rangle$ $B = \langle(0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.5, 0.6)\rangle$ $C^k_\mu(A) = C^k_\mu(B) = 0.4, C^k_\nu(A) = C^k_\nu(B) = 0.35$ $\Rightarrow A = B$	$C_1(A) = 0.05, C_1(B) = 0.04$ $C_1(A) > C_1(B) \Rightarrow A > B$
P.K. De [9] $V(\overline{A}) = V_\mu(\overline{A}) + \lambda(V_\nu(\overline{A}) - V_\mu(\overline{A}))$ $A(\overline{A_1}) = A_\nu(\overline{A}) + \lambda(A_\nu(\overline{A}) - A_\mu(\overline{A}))$ where $V_\mu(\overline{A}) = \frac{(a_1+2a_2+2a_3+a_4)(w_{\overline{A}})}{6}$ $V_\nu(\overline{A}) = \frac{(a_1+2a_2+2a_3+a_4)(1-u_{\overline{A}})}{6}$ $A_\mu(\overline{A}) = \frac{(a_4-a_1-2a_2+2a_3)(w_{\overline{A}})}{6}$ $A_\nu(\overline{A}) = \frac{(a_4-a_1-2a_2+2a_3)(1-u_{\overline{A}})}{6}$	$\overline{A_1} = \langle(a_1, a_2, a_3, a_4), 1, 0\rangle$ $\overline{A_2} = \langle(a_1-\epsilon, a_2+\frac{\epsilon}{2}, a_3+\frac{\epsilon}{2}, a_4-\epsilon), 1, 0\rangle$ $V(\overline{A_1}) = V(\overline{A_2})$ and $A(\overline{A_1}) = A(\overline{A_2})$ $\Rightarrow \overline{A_1} = \overline{A_2}$	$\overline{A_1} = \langle(0.20, 0.35, 0.40, 0.55), 1, 0\rangle$, $\overline{A_2} = \langle(0.10, 0.40, 0.45, 0.45), 1, 0\rangle$ $V(\overline{A_1}) = V(\overline{A_2}) = 0.2333$ and $A(\overline{A_1}) = A(\overline{A_2}) = 0.0833 \Rightarrow \overline{A_1} = \overline{A_2}$	$C_1(\overline{A_1}) = -0.015, C_1(\overline{A_2}) = 0.095$ $C_1(\overline{A_2}) > C_1(\overline{A_1}) \Rightarrow \overline{A_2} > \overline{A_1}$
Amit Kumar [4] $M^{\beta,k}_\mu(A) = \frac{\beta(a_2+a_3)}{2} + \frac{(1-\beta)(a_1+a_4)}{2} + \frac{(1-\beta)(a_2-a_1+a_3-a_4)(k+1)}{2(k+2)}$, $M^{\beta,k}_\nu(A) = \frac{\beta(a'_2+a'_3)}{2} + \frac{(1-\beta)(a'_1+a'_4)}{2} + \frac{(1-\beta)(a'_2-a'_1+a'_3-a'_4)(k+1)}{2(k+2)}$ $, k \in \Re^+ \& \beta \in [0,1]$	$A = \langle(a_1, a_2, a_3, a_4), (a'_1, a'_2, a'_3, a'_4)\rangle$ $B = \langle(a_1-\epsilon, a_2-\epsilon, a_3+\epsilon, a_4+\epsilon)$, $(a'_1-\epsilon, a'_2-\epsilon, a'_3+\epsilon, a'_4+\epsilon)\rangle$ $M^{\beta,k}_\nu(A) = M^{\beta,k}_\nu(B),\ and\ M^{\beta,k}_\nu(A) = M^{\beta,k}_\nu(B)$ $\Rightarrow A = B$	$A = \langle(0.2, 0.3, 0.5, 0.6), (0, 0.1, 0.6, 0.7)\rangle$ $B = \langle(0.3, 0.4, 0.4, 0.5), (0.1, 0.2, 0.5, 0.6)\rangle$ $M^{\beta,k}_\nu(A) = M^{\beta,k}_\nu(B) = 0.4$, $M^{\beta,k}_\nu(A) = M^{\beta,k}_\nu(B) = 0.35$ $\Rightarrow A = B$	$C_1(A) = 0.05, C_1(B) = 0.04$ $C_1(A) > C_1(B) \Rightarrow A > B$
S.P. Wan and J.Y. Dong [21] $m_\mu(A) = \frac{1}{2}[\underline{m}_\mu(A) + \overline{m}_\mu(A)]$ $m_\nu(A) = \frac{1}{2}[\underline{m}_\nu(A) + \overline{m}_\nu(A)]$ $m_\mu(A) = \frac{1}{6}(a+2b+2c+d)w_A$ $m_\nu(A) = \frac{1}{6}(a+2b+2c+d)(1-u_A)$	$A_1 = \langle(a_1, a_2, a_3, a_4), 1, 0\rangle$ $A_2 = \langle(a_1-\epsilon, a_2-\epsilon, a_3+\epsilon, a_4+\epsilon), 1, 0\rangle$ $m_\mu(A_1) = m_\nu(A_1) = m_\mu(A_2) = m_\nu(A_2)$ $\Rightarrow A_1 = A_2$	$A_1 = \langle(0.40, 0.45, 0.50.55), 1, 0\rangle$, $A_2 = \langle(0.30, 0.35, 0.60, 0.65), 1, 0\rangle$ $m_\mu(A_1) = m_\nu(A_1) = m_\mu(A_2) = m_\nu(A_2) = 0.475$ $\Rightarrow A_1 = A_2$	$C_1(A_1) = 0.2, C_1(A_2) = 0.215$ $C_1(A_2) > C_1(A_1) \Rightarrow A_2 > A_1$

in Table 2. His method also can be applied to IFINs and real valued IFNs alone, if we are given with TrIFNs (TIFNs) (generalization of IFINs and real valued IFNs), then our proposed method is the right choice for ranking IFNs. Hence by using our method we are able to rank all types of IFNs such as real valued IFNs, IFINs, TrIFNs (TIFNs) effectively. For example, let $A_1 = ([0.1, 0.4], [0.18, 0.42])$, $A_2 = ([0.15, 0.35], [0.26, 0.34])$ be two IFINs. Then using Wang et al.'s [23] score functions we get, $S(A_1) = S(A_2) = -0.1, H(A_1) = H(A_2) = 1.1$, $T(A_1) = T(A_2) = 0.06$ but $G(A_1) = 0.54$ and $G(A_2) = 0.28$ which implies that A_2 is better than A_1. If we apply our proposed method to A_1 and A_2, we get $C_1(A_1) = -0.202$, $C_1(A_2) = -0.1884 \Rightarrow$ A_2 is better than A_1 which is supported by Wang et al.'s [23] approach.

Table 2: Comparison of proposed method with Xu [26] and Wang et al. [23].

Other existing methods	Examples of existing methods	Numerical example	Proposed method
Xu (2007) [26]			
$S(A) = \mu_A - \nu_A$	$A = (\mu_A, \nu_A)$	$A = (0.3, 0.4)$	$C_1(A) = -0.07$
$H(A) = \mu_A + \nu_A$	$B = (\mu_A + \epsilon, \nu_A + \epsilon)$	$B = (0.4, 0.5)$	$C_1(B) = 0.15$
where $A = (\mu_A, \nu_A)$	$S(A) = S(B) = \mu_A - \nu_A$	$S(A) = S(B) = -0.1$	$C_1(A) < C_1(B) \Rightarrow A < B$
with $\mu_A + \nu_A \leq 1$	but $H(A) = \mu_A + \nu_A$	$H(A) = 0.7 < H(B) = 0.9 \Rightarrow A < B$	
	$< H(B) = \mu_A + \nu_A + 2\epsilon \Rightarrow A < B$		
Wang et al. [23]			
$S(A) = a_1 + a_2 - a_3 - a_4$	$A_1 = ([a_1, a_2], [a_3, a_4])$	$A_1 = ([0.1, 0.4], [0.18, 0.42])$	$C_1(A_1) = -0.202$
$H(A) = a_1 + a_2 + a_3 + a_4$	$A_2 = ([a_1 + \epsilon, a_2 - \epsilon], [a_3 + \epsilon, a_4 - \epsilon])$	$A_2 = ([0.15, 0.35], [0.26, 0.34])$	$C_1(A_2) = -0.1884$
$T(A) = a_2 + a_3 - a_1 - a_4$	$S(A_1) = S(A_2), H(A_1) = H(A_2)$	$S(A_1) = S(A_2) = -0.1, H(A_1) = H(A_2) = 1.1$	$C_1(A_1) < C_1(A_2) \Rightarrow A_1 < A_2$
$G(A) = a_2 + a_4 - a_1 - a_3$	$T(A_1) = T(A_2)$	$T(A_1) = T(A_2) = 0.06$	
where $A = ([a_1, a_2], [a_3, a_4])$	but $G(A_1) = a_2 + a_4 - a_1 - a_3 > G(A_2)$	but $G(A_1) = 0.54 > G(A_2) = 0.28 \Rightarrow A_1 < A_2$	
with $a_2 + a_4 \leq 1$	$= a_2 + a_4 - a_1 - a_3 - 4\epsilon$		
	$\Rightarrow A_1 < A_2$		

6. Trapezoidal Intuitionistic Fuzzy Information System

In this section, we will see the application of our proposed method in information system. Information system (IS) is a decision model that makes decisions rapidly in the selection of best alternative from available alternatives with respect to criteria involved in the evaluations of alternatives. In information system, dominance relation wholly depends on ranking of information. In this section, trapezoidal intuitionistic fuzzy information system (TrIFIS) is defined and decision making from IFIS using a new dominance degree based on the dominance relation on the set of objects is studied.

Definition 6.1 *An information system $S = (U, AT, V, f)$ with $V = \cup_{a \in AT} V_a$, where V_a is a domain of attribute a is called trapezoidal intuitionistic fuzzy information system (IFIS) if V is a set of TrIFN.*

We denote $f(x, a) \in V_a$ by $f(x, a) = \langle (a_1, a_2, a_3, a_4), (a_1', a_2', a_3', a_4') \rangle$, where a_i *and* $a_i^{'} \in [0, 1]$.

Definition 6.2 *A TrIFIS, $S = (U, AT, V, f)$ together with weights $W = \{w_a / a \in AT\}$ is called weighted trapezoidal intuitionistic fuzzy information system (WTrIFIS) and is denoted by $S = (U, AT, V, f, W)$.*

Definition 6.3 *Let $a \in AT$ be a criterion. Let $x, y \in U$. If $C_1(x) > C_1(y)$ or $C_1(x) = C_1(y)$, $C_2(x) > C_2(y)$ with respect to the criterion a, then $x >_a y$ which* *indicates that x is better than (outranks) y with respect to the criterion a. Also* $x =_a y$ *means that x is equally good as y with respect to the criterion a, if* $f(x, a) = f(y, a)$ *with respect to the criterion a.*

Definition 6.4 *Let* $S = (U, AT, V, f, W)$ *be a WTrIFIS and* $A \subseteq AT$. *Let* $B_A(x, y) = \{a \in A \mid x >_a y\}$ *and let* $C_A(x, y) = \{a \in A \mid x =_a y\}$. *The weighted fuzzy dominance relation* $WR_A(x, y) : U \times U \to [0, 1]$ *is defined by*

$$WR_A(x, y) = \sum_{a \in B_A(x,y)} w_a + \frac{\sum\limits_{a \in C_A(x,y)} w_a}{2}.$$

Definition 6.5 *Let* $S = (U, AT, V, f, W)$ *be a TrIFIS and* $A \subseteq AT$. *The entire dominance degree of each object is defined as*

$$WR_A(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} WR_A(x_i, y_j).$$

6.1. Algorithm for Ranking of Objects in TrIFIS

Let $S = (U, AT, V, f, W)$ be a TrIFIS. The objects in U are ranked using the following algorithm.

Algorithm 6.1

1. Using Definition 4.1, Note 4.1 find $C_j's$ accordingly, to decide whether $x_i >_a x_i$ or $x_j >_a x_i$ or $x_i =_a x_j$ for all $a \in A (A \subseteq AT)$ and for all $x_i, x_j \in U$.
2. Enumerate $B_A(x_i, x_j)$ using $B_A(x_i, x_j) = \{a \in A \mid x_i >_a x_j\}$ and $C_A(x_i, x_j)$ using $C_A(x_i, x_j) = \{a \in A \mid x_i =_a x_j\}$.
3. Calculate the weighted fuzzy dominance relation using $WR_A(x, y) : U \times U \to [0, 1]$ defined by
$$WR_A(x_i, x_j) = \sum_{a \in B_A(x_i,x_j)} w_a + \frac{\sum\limits_{a \in C_A(x_i,x_j)} w_a}{2}.$$
4. Calculate the entire dominance degree of each object using
$$WR_A(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} WR_A(x_i, x_j).$$
5. The objects are ranked using entire dominance degree. The larger the value of $WR_A(x_i)$, the better the object is.

6.2. Numerical Illustration

In this subsection, Algorithm 6.1 is illustrated by Example 6.1. In this example, we consider a selection problem of the best supplier for an automobile company from the available alternatives $\{x_i \mid i = 1 \text{ to } 10\}$ of pre evaluated 10 suppliers, based on TrIFIS with attributes $\{a_j \mid j = 1 \text{ to } 5\}$ as product quality, relationship closeness, delivery performance, social responsibility and legal issue.

Example 6.1 A TrIFIS with $U = \{x_1, x_2, \cdots, x_{10}\}$, $AT = \{a_1, a_2, \cdots, a_5\}$ is given in Table 3, and weights for each attribute W_a is given by $W_a = \{0.3, 0.2, 0.15, 0.17, 0.18\}$.

For $i = 1$, $(\alpha, \beta) = (1, 0)$: By Step 1, $C_1(f(x_i, a_j))$ using Definition 4.1 and Note 4.1, for all $a_i \in AT$ and for all $x_i \in U$ is found and tabulated in Table 4. If $C_1(f(x_i, a_j)) = C_1(f(x_j, a_j))$ for any alternatives x_i, x_j, then find C_2 wherever required. The bold letters are used in Table 4 to represent the equality of scores. From Table 4 we observe that in many places $C_j^{\prime}s$ are not distinguishable for different IFNs. Therefore the same procedure which is explained in Step 1 is repeated for $i = 2, (\alpha, \beta) = (1/2, 4/9)$ and it is shown in Table 5.

The weighted fuzzy dominance degree of each object using $WR_A(x, y) = \sum\limits_{a \in B_A(x,y)} w_a + \dfrac{\sum_{a \in C_A(x,y)} w_a}{2}$ is calculated and is tabulated in Table 6. For example, $B_A(x_2, x_5) = \{a_2, a_3, a_4\}$ and $C_A(x_2, x_5) = \emptyset$ and hence $WR_A(x_2, x_5) = 0.2 + 0.15 + 0.17 = 0.52$.

Now the entire dominance degree of each object using $WR_A(x_i) = \dfrac{1}{|U|} \sum_{j=1}^{|U|} WR_A(x_i, y_j)$ is found by Definition 6.4. For example, $WR_A(x_1) = \dfrac{1}{10} \sum_{j=1}^{10} WR_A(x_2, x_j) = 0.267$. So by Step 5, x_9 is selected as the best object from the weighted trapezoidal intuitionistic fuzzy information system is seen from Table 7.

Table 3: TrIFIS to evaluate alternatives with respect to criteria.

	a_1	a_2	a_3	a_4	a_5
x_1	$\langle(0.17, 0.2, 0.2, 0.3), (0.1, 0.1, 0.3, 0.4)\rangle$	$\langle(0.17, 0.19, 0.23, 0.41), (0.12, 0.16, 0.27, 0.52)\rangle$	$\langle(0.2, 0.25, 0.3, 0.4), (0.15, 0.20, 0.30, 0.60)\rangle$	$\langle(0.10, 0.30, 0.40, 0.45), (0, 0.25, 0.50, 0.80)\rangle$	$\langle(0.1, 0.2, 0.4, 0.4), (0, 0, 0.45, 0.50)\rangle$
x_2	$\langle(0.1, 0.1, 0.15, 0.2), (0, 0.10, 0.15, 0.20)\rangle$	$\langle(0.1, 0.2, 0.3, 0.3), (0.10, 0.10, 0.30, 0.45)\rangle$	$\langle(0.10, 0.20, 0.40, 0.50), (0, 0, 0.50, 0.60)\rangle$	$\langle 0.3, 0.3\rangle$	$\langle(0.17, 0.26, 0.29, 0.41), (0.09, 0.11, 0.37, 0.46)\rangle$
x_3	$\langle(0, 10, 0.20, 0.20, 0.20), (0, 0, 0.20, 0.3)\rangle$	$\langle(0.17, 0.20, 0.30, 0.43), (0.10, 0.20, 0.40, 0.50)\rangle$	$\langle(0.31, 0.41, 0.52, 0.71), (0.31, 0.41, 0.52, 0.71)\rangle$	$\langle(0.07, 0.13, 0.21, 0.46), (0.06, 0.11, 0.25, 0.51)\rangle$	$\langle(0.10, 0.20, 0.35, 0.40), (0, 0, 0.40, 0.50)\rangle$
x_4	$\langle(0.11, 0.20, 0.20, 0.34), (0.07, 0.20, 0.70, 0.80)\rangle$	$\langle(0.20, 0.20, 0.20, 0.45), \langle(0.10, 0.20, 0.20, 0.60)\rangle$	$\langle(0.1, 0.2, 0.3, 0.4), (0.07, 0.20, 0.30, 0.47)\rangle$	$\langle(0.10, 0.35, 0.40, 0.40) (0.10, 0.35, 0.40, 0.40)\rangle$	$\langle(0.50, 0.60, 0.70, 0.90), (0.50, 0.60, 0.70, 0.90)\rangle$
x_5	$\langle(0.07, 0.10, 0.17, 0.26), (0, 0.05, 0.21, 0.60)\rangle$	$\langle(0.11, 0.29, 0.31, 0.65), (0.07, 0.13, 0.33, 0.69)\rangle$	$\langle(0.14, 0.19, 0.26, 0.57), (0.10, 0.10, 0.40, 0.60)\rangle$	$\langle(0.01, 0.30, 0.40, 0.45), (0.01, 0.30, 0.42, 0.47)\rangle$	$\langle(0.20, 0.26, 0.29, 0.40), (0.02, 0.11, 0.41, 0.60)\rangle$
x_6	$\langle(0.20, 0.40, 0.40, 0.50), (0.20, 0.40, 0.40, 0.50)\rangle$	$\langle(0.13, 0.20, 0.30, 0.40), (0.10, 0.20, 0.60, 0.70)\rangle$	$\langle(0.10, 0.20, 0.45, 0.60), (0, 0.10, 0.60, 0.70)\rangle$	$\langle(0.20, 0.30, 0.35, 0.50), (0.20, 0.30, 0.35, 0.50)\rangle$	$\langle(0.40, 0.65, 0.75, 0.80), (0.40, 0.65, 0.75, 0.80)\rangle$
x_7	$\langle(0.10, 0.30, 0.50, 0.70), (0.10, 0.30, 0.50, 0.70)\rangle$	$\langle(0.10, 0.25, 0.25, 0.35), (0, 0.25, 0.25, 0.50)\rangle$	$\langle(0.03, 0.13, 0.37, 0.47), (0, 0.13, 0.37, 0.54)\rangle$	$\langle 0.2, 0.2\rangle$	$\langle(0.10, 0.20, 0.30, 0.40), (0, 0.10, 0.40, 0.50)\rangle$
x_8	$\langle(0.10, 0.20, 0.20, 0.30), (0.05, 0.15, 0.30, 0.40)\rangle$	$\langle(0.16, 0.21, 0.34, 0.49), (0.16, 0.20, 0.36, 0.53)\rangle$	$\langle(0.40, 0.50, 0.60, 0.70), (0.30, 0.40, 0.70, 0.80)\rangle$	$\langle(0.31, 0.52, 0.73, 0.84), (0.31, 0.52, 0.73, 0.84)\rangle$	$\langle(0.10, 0.20, 0.40, 0.45), (0.07, 0.10, 0.51, 0.63)\rangle$
x_9	$\langle(0, 0.35, 0.55, 0.60), (0, 0.35, 0.55, 0.60)\rangle$	$\langle(0.61, 0.63, 0.71, 0.83), (0.61, 0.63, 0.71, 0.83)\rangle$	$\langle(0.10, 0.20, 0.30, 0.40), (0, 0, 0.40, 0.50)\rangle$	$\langle(0.10, 0.40, 0.50, 0.60), (0, 0.20, 0.70, 0.70)\rangle$	$\langle(0.10, 0.30, 0.30, 0.70), (0.10, 0.20, 0.50, 0.60)\rangle$
x_{10}	$\langle 0.40, 0.40\rangle$	$\langle(0.20, 0.20, 0.30, 0.40), (0.10, 0.20, 0.30, 0.95)\rangle$	$\langle(0.61, 0.61, 0.63, 0.71), (0.61, 0.61, 0.63, 0.71)\rangle$	$\langle(0.10, 0.20, 0.30, 0.40), (0.10, 0.20, 0.30, 0.40)\rangle$	$\langle(0.13, 0.23, 0.43, 0.73), (0.10, 0.16, 0.51, 0.78)\rangle$

Table 4: C_1 and C_2 for $i = 1, (\alpha, \beta) = (1, 0)$.

	a_1	a_2	a_3	a_4	a_5		a_1	a_2	a_3	a_4	a_5
C_1						C_2					
x_1	**-0.44**	−0.41	−0.25	−0.07	−0.18		**+0.76**				
x_2	−0.64	**-0.31**	−0.18	−0.19	**-0.25**			**+0.69**			**+0.65**
x_3	**-0.44**	**-0.31**	+0.18	−0.52	**-0.31**		**+0.76**	**+0.69**			**-0.69**
x_4	**-0.44**	**-0.44**	−0.313	−0.015	−0.13		**+0.76**	**+0.76**			
x_5	−0.61	−0.44	−0.37	−0.07	**-0.25**						**+0.65**
x_6	**+0.04**	**-0.31**	−0.115	−0.13	+0.61		**+0.44**	**+0.69**			
x_7	−0.05	−0.313	−0.298	−0.44	**-0.31**						**+0.76**
x_8	**-0.44**	−0.246	+0.35	+0.495	−0.18		**+0.76**				
x_9	+0.158	+0.563	−0.31	+0.15	−0.19						
x_{10}	**+0.04**	**-0.31**	+0.476	−0.31	−0.1089		**+0.44**	**+0.69**			

Table 5: C_3 and C_4 for $i = 2, (\alpha, \beta) = (1/2, 4/9)$.

	a_1	a_2	a_3	a_4	a_5		a_1	a_2	a_3	a_4	a_5
C_3						C_4					
x_1	−0.39375	−0.3076	−0.21625	−0.1475	−0.235						
x_2	−0.67125	−0.37	−0.1675	−0.19	−0.22775						
x_3	−0.505	−0.242525	+0.2411	−0.381	−0.26875						
x_4	−0.40435	−0.2775	−0.34	−0.1525	−0.09375						
x_5	−0.568	−0.4375	−0.198	−0.196	−0.217						
x_6	−0.01	−0.28525	−0.06625	−0.09375	+0.543						
x_7	+0.08	−0.34	−0.2836	−0.44	−0.3025						
x_8	−0.4375	−0.1768	+0.3575	+0.474	−0.2012						
x_9	+0.0244	+0.6076	−0.3025	+0.0625	−0.05						
x_{10}	+0.04	−0.245	+0.5113	−0.3025	+0.0356						

Table 6: Weighted fuzzy dominance relation between two alternatives $WR_A(x, y)$.

$WR_A(x, y)$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	0.50	0.67	0.65	0.62	0.67	0	0.70	0.30	0.15	0.17
x_2	0.33	0.50	0.35	0.15	0.52	0	0.50	0	0.15	0.17
x_3	0.35	0.65	0.50	0.35	0.65	0.35	0.70	0	0.15	0.20
x_4	0.38	0.85	0.65	0.50	0.85	0.20	0.55	0.48	0	0.17
x_5	0.33	0.48	0.35	0.15	0.50	0	0.50	0	0.15	0.17
x_6	1	1	0.65	0.80	1	0.50	0.70	0.48	0.33	0.35
x_7	0.30	0.50	0.30	0.45	0.50	0.30	0.50	0.30	0.15	0.30
x_8	0.70	1	1	0.52	1	0.52	0.70	0.50	0.32	0.37
x_9	0.85	0.85	0.85	1	0.85	0.67	0.85	0.68	0.50	0.67
x_{10}	0.83	0.83	0.80	0.83	0.83	0.65	0.70	0.63	0.33	0.50

Table 7: Total dominance degree $R_A(x_i)$.

X_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$R_A(x_i)$	0.443	0.267	0.390	0.463	0.263	0.681	0.360	0.663	0.777	0.693

7. Conclusion

A complete ranking of intuitionistic fuzzy number using upper lower dense sequence is achieved in this paper. The ranking method introduced and discussed in this paper consists with the natural ordering of real numbers. Actually, the complete ranking on IFN defined in this paper generalizes the total ordering on FNs defined by W. Wang and Z. Wang [22]. Therefore, this is a real generalization of the total ordering on the set of all real numbers to the set of IFNs. Up to date different methods are used in decision making and other industrial problems to obtain an optimal solution, but all the available methods are not consistent and also not suitable in certain cases because of its anti-intuitive ranking procedure on IFNs. This method can be the best alternative to all other existing methods.

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