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Vertex-disjoint triangles in $K_{1,t}$-free graphs with minimum degree at least t

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ABSTRACT

A graph is said to be $K_{1,t}$-free if it does not contain an induced subgraph isomorphic to $K_{1,t}$. Let $h(t,k)$ be the smallest integer m such that every $K_{1,t}$-free graph of order greater than m and with minimum degree at least t contains k vertex-disjoint triangles. In this paper, we obtain a lower bound of $h(t,k)$ by a constructive method. According to the lower bound, we totally disprove the conjecture raised by Hong Wang [H. Wang, Vertex-disjoint triangles in claw-free graphs with minimum degree at least three, Combinatorica 18 (1998) 441–447]. We also obtain an upper bound of $h(t,k)$ which is related to Ramsey numbers $R(3,t)$. In particular, we prove that $h(4,k)=9(k-1)$ and $h(5,k)=14(k-1)$.

1. Introduction

In this paper, all graphs are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of G. If $uv\in E(G)$, then u is said to be the *neighbor* of v. We use $N(v)$ to denote the set of neighbors of a vertex v. The *degree* $d(v)=|N(v)|$. For a subset U of $V(G)$, $G[U]$ denotes the subgraph of G induced by U. The *join* $G=G_1\vee G_2$ of graph G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph $G_1\bigcup G_2$ together with all the edges jointing V_1 and V_2. For any positive integers k and l, the *Ramsey number* $R(k,l)$ is the smallest integer n such that every graph on n vertices contains either a clique of k vertices or an independent set of l vertices. A (k,l)*-Ramsey graph* is a graph on $R(k,l)-1$ vertices that contains neither a clique of k vertices nor an independent set of l vertices. By the definition of $R(k,l)$, (k,l)-Ramsey graph does exist for all $k\geq 2$ and $l\geq 2$. The graph C_k is a cycle with k vertices and we call C_3 a triangle. We use mQ to represent m vertex-disjoint copies of graph Q. Other notations can be found in [1].

$K_{1,t}$ is the star of order $t+1$. A graph is said to be $K_{1,t}$-free if it does not contain an induced subgraph isomorphic to $K_{1,t}$ $(t\geq 2)$. Let $h(t,k)$ be the smallest integer m such that every $K_{1,t}$-free graph of order greater than m and with minimum degree at least t contains k vertex-disjoint triangles. Wang [5] proved that $h(3,k)=6(k-1)$ for any $k\geq 2$, and he put forward the following conjecture.

Conjecture 1 ([5]). *For each integer $t\geq 4$, there exists an integer k_t depending on t only such that $h(t,k)=2t(k-1)$ for all integers $k\geq k_t$.*

In Section 2, we get a proper lower bound of $h(t,k)$ by a constructive method that $h(4,k)\geq 9(k-1)$ and $h(t,k)\geq (4t-9)(k-1)$ for any $t\geq 5$. Since $4t-9>2t$ for any $t\geq 5$, we totally disprove Conjecture 1. In Section 3, we give an upper bound of $h(t,k)$, which is related to $R(3,t)$. In particular, we prove that $h(4,k)=9(k-1)$ and $h(5,k)=14(k-1)$. In Section 4, we give some remarks on $h(t,k)$ and list some interesting open problems. The paper ends with one conjecture.

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2. A lower bound of $h(t, k)$

Let $G_{n,m}$ be the graph whose vertices are $0, 1, \ldots, n-1$ where two vertices i and j are adjacent if and only if $(i-j) \in \{\pm m, \pm(m+1), \ldots, \pm(2m-1)\}$.

Lemma 1 ([4])**.** *If $n \geq 6m-2$ then $G_{n,m}$ is a triangle-free regular graph whose degree is equal to $2m$. Furthermore, if $n \leq 8m-3$, then the independent number of $G_{n,m}$ is equal to $2m$.*

Similarly, we define $H_{n,m}$ to be the graph whose vertices are $0, 1, \ldots, n-1$ where two vertices i and j are adjacent if and only if $(i-j) \in \{\pm m, \pm(m+1), \ldots, \pm(2m-1), \pm\lfloor \frac{n}{2} \rfloor\}$.

Lemma 2. *$H_{8m-2,m}$ is a triangle-free regular graph whose degree is equal to $2m+1$ and its independent number is equal to $2m+1$.*

Proof. Suppose, to the contrary, that $H_{8m-2,m}$ contains a triangle, say $t_0t_1t_2t_0$ where $0 \leq t_0 < t_1 < t_2 \leq 8m-3$. Then $t_j - t_i \in \{m, m+1, \ldots, 2m-1, 4m-1\}$ for $0 \leq i < j \leq 2$. So $t_2 - t_0 = (t_2 - t_1) + (t_1 - t_0) \geq m + m = 2m$ which implies that $t_2 - t_0 = 4m-1$. Since $t_i \neq t_j$ for $0 \leq i < j \leq 2$, $t_1 - t_0 \leq 2m-1$ and $t_2 - t_1 \leq 2m-1$. This implies that $t_2 - t_0 = (t_2 - t_1) + (t_1 - t_0) \leq 4m-2$, a contradiction. So $H_{8m-2,m}$ is a triangle-free graph.

Let $S = \{\pm m, \pm(m+1), \ldots, \pm(2m-1), 4m-1\}$. Then for any $i, j \in S$, $(i-j) \notin S$. Since $|S| = 2m+1$, $\alpha(H_{8m-2,m}) \geq 2m+1$.

Consider $2m+1$ numbers $0 \leq t_0 < t_1 < \cdots < t_{2m} \leq n-1$ and suppose that $(t_j - t_i) \notin \{\pm m, \pm(m+1), \ldots, \pm(2m-1)\}$ for any i and j. Put $s_i = t_{i+1} - t_i$ $(i = 1, 2, \ldots, 2m-1)$, $s_0 = n + t_0 - t_{2m}$. It is clear that $s_i \leq m-1$ or $s_i \geq 2m$ for any $i = 0, 1, \ldots, 2m-1$. Let r be equal to the number of members s_i which satisfy $s_i \geq 2m$. If $r \geq 3$, then $n \geq r \cdot 2m + (2m+1-r) \cdot 1 = r(2m-1) + 2m + 1 \geq 8m-2$, that contradicts the assumption of the lemma. If $r \leq 2$ then there exists i such that $s_{i+j} < m$ for every $j = 0, 1, \ldots, m-1$ (we mean that $s_{2m+1} = s_0, s_{2m+2} = s_1, \ldots$). Denote $p_0 = 0$, $p_j = s_i + s_{i+1} + \cdots + s_{i+j-1}$ $(j = 1, 2, \ldots, m)$. Hence $p_j \equiv (t_{i+j} - t_i) (\mathrm{mod}\ n)$. Since every $s_{i+j} \geq 1$, $p_m \geq m$. Let $j = \min\{l : p_l \geq m\}$. So $p_j \geq m$, $p_{j-1} \leq m-1$, $p_j = p_{j-1} + s_{i+j} \leq (m-1) + (m-1) \leq 2m-1$. Therefore, $(t_i - t_{i+j}) \in \{\pm m, \pm(m+1), \ldots, \pm(2m-1)\}$, which leads to a contradiction. □

Theorem 3. *For each integer $k \geq 2$, $h(4, k) \geq 9(k-1)$.*

Proof. Let W be a wheel of order 9. Label W's center by v_0 and its neighbors by $v_1, v_2, \ldots, v_8$. Let H be a graph obtained from W by adding two edges v_1v_5 and v_2v_6. It is obvious that H does not contain two vertex-disjoint triangles. Set $P(H) = \{v_3, v_4, v_7, v_8\}$. Let Π_k be the set of graphs of order $9(k-1)$ such that a graph G belongs to Π_k if and only if it is obtained from $k-1$ vertex-disjoint copies $H_1, \ldots, H_{k-1}$ of H by adding $2(k-1)$ new edges on $\bigcup_{i=1}^{k-1} P(H_i)$ so that these new edges form a perfect matching. It is easy to check that every graph H belonging to $\prod_k$ is the $K_{1,4}$-free graph which contains at most $k-1$ vertex-disjoint triangles and $\delta(G) \geq 4$. So $h(4, k) \geq 9(k-1)$. □

Theorem 4. *For each integers $t \geq 5$ and $k \geq 2$,*

$$h(t, k) \geq \begin{cases} (4t-6)(k-1), & \text{if } t \text{ is odd;} \\ (4t-9)(k-1), & \text{if } t \text{ is even.} \end{cases}$$

Proof. Let $G = (k-1)(K_1 \vee G_{8m-3,m})$. Then $|V(G)| = (8m-2)(k-1)$ and $\delta(G) = 2m+1$. By Lemma 1, G is a $K_{1,2m+1}$-free graph which contains at most $k-1$ vertex-disjoint triangles. So $h(2m+1, k) \geq (8m-2)(k-1)$. Let $t = 2m+1$. Then $h(t, k) \geq (4t-6)(k-1)$. Similarly, we put $H = (k-1)(K_1 \vee H_{8m-2,m})$. Then $|V(G)| = (8m-1)(k-1)$ and $\delta(G) = 2m+2$. By Lemma 2, H is a $K_{1,2m+2}$-free graph which contains at most $k-1$ vertex-disjoint triangles. So we also have $h(2m+2, k) \geq (8m-1)(k-1)$. Let $t = 2m+2$. Then $h(t, k) \geq (4t-9)(k-1)$. □

By Theorems 3 and 4, we totally disprove Conjecture 1.

3. An upper bound of $h(t, k)$

In this section, we continue to consider $K_{1,t}$-free graphs and give an upper bound of $h(t, k)$. First, we introduce a useful lemma, which is known as *Ramsey's Theorem*.

Lemma 5 ([1] (Ramsey's Theorem))**.** *For any two integers $k \geq 2$ and $l \geq 2$, $R(k, l) \leq R(k, l-1) + R(k-1, l)$. Furthermore, if $R(k, l-1)$ and $R(k-1, l)$ are both even, then the strict inequality holds.*

In [2] (also see page 7 in [3]), Burr et al. proved that $R(k, t) \geq R(k-1, t) + 2t - 3$ for $k, t \geq 3$. It follows that $R(3, t) \geq R(2, t-1) + 2t - 3 = 3t - 3$ for $t \geq 3$. So we have the following lemma.

Lemma 6. *For each integer $t \geq 4$, $\max\left\{\left\lfloor \frac{3(t-1)}{2} \right\rfloor, 2t-2, \frac{5}{2}t-4, 3t-6\right\} \leq R(3, t-1) + t - 4$.*

Theorem 7. *For each integer $t \geq 4$, $h(t,k) \leq g(t)(k-1)$ where*

$$g(t) = \begin{cases} R(3,t-1)+t-1, & \text{if } R(3,t-1) \text{ and } t \text{ are both even;} \\ R(3,t-1)+t, & \text{otherwise.} \end{cases}$$

Proof. If $k \leq 1$, the theorem is obvious. So we assume that $k \geq 2$. Suppose that the theorem is false. Let s be the greatest integer such that G contains s vertex-disjoint triangles, say $T_1, \ldots, T_s$. Then $s < k$. Define $T = \{T_1, \ldots, T_s\}$, $S = \bigcup_{i=1}^{s} V(T_i)$ and $H = G - S$. Since G is $K_{1,t}$-free and $\delta(G) \geq t$, we have that

(1) $\Delta(H) \leq t-1$, and

(2) every vertex must be contained in a triangle.

By the maximality of s, we have that

(3) any triangle must have at least one vertex in S.

Thus, we can divide $V(H)$ into three disjoint subsets V_1, V_2 and V_3 by the following steps. Let $x \in V(H)$ and C_x the set of triangles incident with x. First, if there is a triangle $C \in C_x$, say $C = xyzx$, and a $T_m \in T$ such that $x, z \in V(H)$ and $y \in V(T_m)$, then we put x into V_1 and say that x is dominated by T_m at y. Otherwise, any triangle containing x must has two vertices contained in S. Then, if there exist a $C \in C_x$, say $C = xyzx$, and a $T_m \in T$ such that $y, z \in V(T_m)$, then we put x into V_2 and say that x is dominated by T_m at y and z. Finally, we left the case that for any triangle $C \in C_x$, the two vertices in C different from x must contain in different triangles in T. Thus we choose a triangle $C \in C_x$, say $C = xyzx$, and two triangles $T_m, T_n \in T$ such that $y \in V(T_m)$ and $z \in V(T_n)$ where $1 \leq m < n \leq s$. Now we put x into V_3 and say that x is dominated by both T_m at y and T_n at z. Moreover, this partition of $V(H)$ should also satisfies

- $|V_1|$ is maximum, and subject to the condition,
- $|V_2|$ is maximum.

Setting this way, we will have $V_i \bigcap V_j = \emptyset$ for any $1 \leq i < j \leq 3$ and moreover, if two vertices in $V_2 \bigcup V_3$ have a common neighbor in S, they are not adjacent (by the choice of V_i's). In the following, we call a vertex xi-vertex if $x \in V_i$ $(1 \leq i \leq 3)$ and always assume that for $x \in V_1 \bigcup V_2$, if there are two or more triangles which can dominate x, we only choose one; and for $x \in V_3$, if it is dominated by at least two pairs of triangles, then we choose only one pair of triangles in T to dominate x.

Let $T_m = xyzx$ be a triangle in the set T. For any $v \in T_m$, we define $S_i(T_m, v)$ to be the set of i-vertices dominated by T_m at v and then $S_i(T_m) = \bigcup_{v \in T_m} S_i(T_m, v)$, where $1 \leq i \leq 3$. Then

(4) $\max\{|S_3(T_m,x)|, |S_3(T_m,y)|, |S_3(T_m,z)|\} \leq t-2$.

Since if $|S_3(T_m,x)| \geq t-1$, that is, x is adjacent to $t-1$ 3-vertices $x_1, \ldots, x_{t-1}$ dominated by T_m, then $G[\{x, x_1, \ldots, x_{t-1}, z\}] \simeq K_{1,t}$, a contradiction. So $|S_3(T_m,x)| \leq t-2$. Similarly, we have $|S_3(T_m,y)| \leq t-2$ and $|S_3(T_m,z)| \leq t-2$. Hence (4) holds.

(5) $|S_2(T_m)| \leq \left\lfloor \frac{3(t-1)}{2} \right\rfloor$.

Since if $|S_2(T_m)| > \left\lfloor \frac{3(t-1)}{2} \right\rfloor$, there must exist a vertex, say x, such that T_m dominates at least t 2-vertices at x. Then these t 2-vertices along with x forms a $K_{1,t}$, a contradiction.

(6) $\max\{|S_2(T_m,x) \bigcup S_3(T_m,x)|, |S_2(T_m,y) \bigcup S_3(T_m,y)|, |S_2(T_m,z) \bigcup S_3(T_m,z)|\} \leq t-1$.

Since if $|S_2(T_m,x) \bigcup S_3(T_m,x)| \geq t$, $G\left[\{x\} \bigcup S_2(T_m,x) \bigcup S_3(T_m,x)\right] \supseteq K_{1,t}$, a contradiction.

Let $v \in V_1 \bigcup V_2$. If v is dominated by some $T_m \in T$, then we define $a(v, T_m) = 1$. Otherwise, we define $a(v, T_m) = 0$. Let $v \in V_3$ and v is dominated by two triangles $T_i = xyzx$ and $T_j = abca$ at x and a, respectively. If $\max\{|S_1(T_i,y)|, |S_1(T_i,z)|\} \geq 1$ and $S_1(T_j,b) = S_1(T_j,c) = 0$, we define $a(v,T_j) = 1$ and $a(v,T_m) = 0$ for all $m \neq j$. Otherwise, we define $a(v,T_i) = a(v,T_j) = \frac{1}{2}$ and $a(v,T_m) = 0$ for any $m \in \{1, 2, \ldots, s\} \setminus \{i, j\}$.

For each $T_m \in T$, we define its dominatingcapacity $ca(T_m) = \sum_{x \in V(H)} a(x, T_m)$. Since any vertex in $V(H)$ is dominated by some $T_i \in T$, $\sum_{i=1}^{s} ca(T_i) = \sum_{i=1}^{s} \sum_{x \in V(H)} a(x,T_i) = \sum_{x \in V(H)} \sum_{i=1}^{s} a(x,T_i) \geq |V(H)| \geq g(t)(k-1)+1-3s \geq (g(t)-3)s+1$. This implies that there is a triangle T_α, say $T_\alpha = xyzx$, such that $ca(T_\alpha) > g(t)-3$ for some $1 \leq \alpha \leq s$.

Case 1. T_α dominates no 3-vertices.

Suppose T_α dominates no 1-vertices, then by (5) and Lemma 6, we have $ca(T_\alpha) \leq \left\lfloor \frac{3(t-1)}{2} \right\rfloor \leq R(3,t-1)+t-4 \leq g(t)-3$. So without loss of generality, we can assume x_1 is a 1-vertex dominated by T_α at x. Then by the definition of V_1, there exists another vertex $x_2 \in V_1$ such that $xx_2, x_1x_2 \in E(G)$. By the maximality of s, if v is a 2-vertex dominated by T_α, we must have $vx \in E(G)$. Suppose $S_1(T_\alpha,y) \bigcup S_1(T_\alpha,z) \subseteq N(x)$, then $ca(T_\alpha) \leq \Delta(G)-2 \leq R(3,t)-3 \leq g(t)-3$ by Lemma 5, a contradiction. So without loss of generality, we can assume that $S_1(T_\alpha,y) \setminus N(x) \neq \emptyset$. This implies that there exists a 1-vertex dominated by T_α at y, say y_1, such that $yy_1 \in E(G)$ and $xy_1 \notin E(G)$. At the same time, there also exist another vertex $y_2 \in V_1$ such that $yy_2, y_1y_2 \in E(G)$. By the maximality of s, we must have $y_2 \in \{x_1, x_2\}$. Without loss of generality, we assume $y_2 = x_1$ which implies $x_1y_1, x_1y \in E(G)$. By the maximality of s, we have $S_2(T_\alpha, z) = \emptyset$. Since $vy_1 \notin E(G)$ for any $v \in S_2(T_\alpha,y)$, $|S_2(T_\alpha)| = |S_2(T_\alpha,y)| \leq t-2$. Suppose $S_1(T_\alpha,z) \neq \emptyset$ or $S_1(T_\alpha,x) \setminus \{x_1,x_2\} \neq \emptyset$, then for any 1-vertex v dominated by T_α, we must have $vx_1 \in E(G)$. For otherwise, we can replace T_α with two new vertex-disjoint triangles which are also vertex-disjoint to any triangle in $T \setminus \{T_\alpha\}$, a contradiction. Since $d_H(x_1) \leq t-1$ by (1), $S_1(T_\alpha) \leq t-1+1 = t$. So $ca(T_\alpha) = |S_1(T_\alpha)| + |S_2(T_\alpha)| \leq t+t-2 = 2t-2 \leq g(t)-3$ by Lemma 6, a contradiction. So $S_1(T_\alpha, z) = S_1(T_\alpha, x) \setminus \{x_1, x_2\} = \emptyset$. By the maximality of s, $S_1(T_\alpha, y) \setminus \{x_1, x_2\}$ along with z forms an independent set in G, so $|S_1(T_\alpha, y) \setminus \{x_1, x_2\}| \le t-2$ which implies $S_1(T_\alpha) \le t-2+2 = t$. So we also have $ca(T_\alpha) = |S_1(T_\alpha)| + |S_2(T_\alpha)| \le t + t - 2 = 2t-2 \le g(t) - 3$ by Lemma 6, a contradiction.

Case 2. T_α dominates a 3-vertex at x and $S_1(T_\alpha, y) = S_1(T_\alpha, z) = \emptyset$.

Suppose $S_1(T_\alpha, x) \neq \emptyset$. Select $u \in S_3(T_\alpha, x)$. Set $L = G[N_H(x) \bigcup \{x, y, z\}/\{u\}]$. Then L is a $K_{1,t-1}$-free graph, for otherwise, there must exist an independent set $M \subseteq V(L)$ of size $t-1$, then $G[M \bigcup \{x, u\}] \simeq K_{1,t}$, a contradiction. By the maximality of s, $L \not\supseteq 2C_3$. It follows that $d_H(x) = d_L(x) + 1 - 2 \le R(3, t-1) - 2$. Since $S_1(T_\alpha, x) \neq \emptyset$, $a(w, T_\alpha) \le \frac{1}{2}$ for any $w \in S_3(T_\alpha, y) \bigcup S_3(T_\alpha, z)$ and there is no vertex $v \in V_2$ such that $vy, vz \in E(G)$. That is, $S_2(T_\alpha, y) \bigcup S_2(T_\alpha, z) \subseteq N(x)$. By (4), we have $|S_3(T_\alpha, y)| \le t-2$ and $|S_3(T_\alpha, z)| \le t-2$. Then $ca(T_\alpha) \le d_H(x) + \frac{1}{2}(t-2) + \frac{1}{2}(t-2) \le R(3, t-1) + t - 4 \le g(t) - 3$, a contradiction. So $S_1(T_\alpha, x) = \emptyset$.

Suppose $|S_2(T_\alpha)| = 0$, then by (4), we have $\max\{|S_3(T_\alpha, x)|, |S_3(T_\alpha, y)|, |S_3(T_\alpha, z)|\} \le t-2$. This implies $ca(T_\alpha) \le 3(t-2) \le g(t) - 3$ by Lemma 6, a contradiction. So $|S_2(T_\alpha)| = m > 0$. Without loss of generality, we can select a vertex $w \in S_2(T_\alpha)$ such that $wy, wz \in E(G)$. Now, we claim that $a(v, T_\alpha) = \frac{1}{2}$ for any $v \in S_3(T_\alpha, x)$. By the definition of V_3, for such a v, there exists another triangle, say $T_\gamma = dpqd$, such that v is dominated by T_γ at d. Then by the maximality of s, we must have $S_1(T_\gamma, p) = S_1(T_\gamma, q) = \emptyset$. By the definition of the function $a(\cdot, \cdot)$, we have $a(v, T_\alpha) = \frac{1}{2}$ since $S_1(T_\alpha, y) = S_1(T_\alpha, z) = \emptyset$. Let $|S_2(T_\alpha, x)| = a_x$, $|S_2(T_\alpha, y)| = a_y$ and $|S_2(T_\alpha, z)| = a_z$, then $a_x + a_y + a_z = 2m$. By (6), we also have $|S_3(T_\alpha, x)| \le t-1-a_x$, $|S_3(T_\alpha, y)| \le t-1-a_y$ and $|S_3(T_\alpha, z)| \le t-1-a_z$. Suppose $m \ge 2$, then $a_y + a_z \ge 3$ and $ca(T_\alpha) \le m + \frac{1}{2}(t-1-a_x) + (t-1-a_y) + (t-1-a_z) = \frac{5}{2}(t-1) - \frac{1}{2}(a_y + a_z) \le \frac{5}{2}t - 4 \le g(t) - 3$, a contradiction. So we have $m = 1$ and then $a_x = 0$, $a_y = a_z = 1$, which implies $ca(T_\alpha) \le 1 + \frac{1}{2}(t-2) + (t-2) + (t-2) = \frac{5}{2}(t-1) - 4 \le g(t) - 3$, a contradiction.

Case 3. T_α dominates a 3-vertex at x but $S_1(T_\alpha, y) \neq \emptyset$ (the case when $S_1(T_\alpha, z) \neq \emptyset$ is similar).

Select $u \in S_3(T_\alpha, x)$, then by the definition of V_3, there exists another triangle, say $T_\beta = abca$, such that $u \in S_3(T_\beta, a)$. That is, $xa, xu, au \in E(G)$. Choose $y_1 \in S_1(T_\alpha, y)$. Then $a(w, T_\alpha) \le \frac{1}{2}$ for any $w \in S_3(T_\alpha, x) \bigcup S_3(T_\alpha, z)$. Suppose $\max\{|S_1(T_\beta, b)|, |S_1(T_\beta, c)|\} \ge 1$. Without loss of generality, we assume $|S_1(T_\beta, b)| \ge 1$ and $b_1 \in S_1(T_\beta, b)$. Then by the definition of V_1, there exist a vertex $y_2 \in N(y)$ and $b_2 \in N(b)$ such that $y_1y_2 \in E(G)$, $b_1b_2 \in E(G)$.

Set $U = \{y_1, y_2\} \bigcup \{b_1, b_2\}$. Then by the maximality of s, we have

(a) $|U| \le 3$

(b) There is no vertex $v \in V(H) \setminus U$ such that $vx, vz \in E(G)$ or $vy, vz \in E(G)$. In particular, $S_2(T_\alpha, x) \bigcap S_2(T_\alpha, z) = \emptyset$ and $S_2(T_\alpha, y) \bigcap S_2(T_\alpha, z) = \emptyset$.

(c) $\{z\} \bigcup N_H(x) \setminus U$, $\{z\} \bigcup N_H(y) \setminus \{b_1, b_2\}$, $\{x\} \bigcup N_H(z) \setminus U$ are three independent sets. In particular, $\max\{|N_H(x) \setminus U|, |N_H(y) \setminus \{b_1, b_2\}|, |N_H(z) \setminus U|\} \le t-2$.

Next, we claim that $|S_1(T_\alpha)| \le t-1$. First, we consider the case when $|U| = 2$. Suppose $S_1(T_\alpha, x) \neq \emptyset$ (the case when $S_1(T_\alpha, z) \neq \emptyset$ is similar). If $S_1(T_\alpha, y) \setminus U \neq \emptyset$ or $S_1(T_\alpha, z) \neq \emptyset$, then every vertex in $S_1(T_\alpha) \setminus U$ must be adjacent to the same vertex in U. Since $\Delta(H) \le t-1$ by (1) and $b_1 \notin S_1(T_\alpha)$, $|S_1(T_\alpha)| \le t-1$. If $S_1(T_\alpha, y) \setminus U = S_1(T_\alpha, z) = \emptyset$, then by (c) we have $|N_H(x) \setminus U| \le t-2$ since G is a $K_{1,t}$-free graph. Note again that $b_1 \notin S_1(T_\alpha)$, we also have $|S_1(T_\alpha)| \le t-2+1 = t-1$. So we assume $S_1(T_\alpha, x) = S_1(T_\alpha, z) = \emptyset$. Then by (c), we have $|N_H(y) \setminus \{b_1, b_2\}| \le t-2$ which implies $|S_1(T_\alpha)| \le t-2+1 = t-1$ since $b_1 \notin S_1(T_\alpha)$. Second, we consider the case when $|U| = 3$. Since $|U| = 3$, $|\{y_1, y_2\} \bigcap \{b_1, b_2\}| = 1$. Let $v \in \{y_1, y_2\} \bigcap \{b_1, b_2\}$. Then by the maximality of s, every vertex in $S_1(T_\alpha) \setminus U$ must be adjacent to v. Since $d_H(v) \le t-1$ by (1) and $b_1 \notin S_1(T_\alpha)$, $|S_1(T_\alpha)| \le t-1$.

Let $|S_1(T_\alpha, x) \setminus U| = b_x$, $|S_1(T_\alpha, y) \setminus U| = b_y$, $|S_1(T_\alpha, z) \setminus U| = b_z$ and $|S_2(T_\alpha)| = m$. Note that $b_1 \notin S_1(T_\alpha)$ and $|U| \le 3$, we have $|U \bigcap S_1(T_\alpha)| \le 2$. This implies $|S_1(T_\alpha)| - 2 \le b_x + b_y + b_z \le |S_1(T_\alpha)| - 1$. By (b) and (c), we have $|S_3(T_\alpha, x)| \le t-2-m-b_x$ and $|S_3(T_\alpha, z)| \le t-2-b_z$. Suppose $b_x + b_y + b_z = |S_1(T_\alpha)| - 1$, then by (c), we also have $|S_3(T_\alpha, y)| \le t-2-m-b_y$. This implies $ca(T_\alpha) \le |S_1(T_\alpha)| + m + \frac{t-2-m-b_x}{2} + (t-2-m-b_y) + \frac{t-2-b_z}{2} \le \frac{|S_1(T_\alpha)|-1}{2} + 1 + 2(t-2) \le \frac{5}{2}t - 4 \le g(t) - 3$ by Lemma 6, a contradiction. So $b_x + b_y + b_z = |S_1(T_\alpha)| - 2$ which implies $|U| = 3$. By (c), we have $|N_H(y) \setminus U| \le |N_H(y) \setminus \{b_1, b_2\}| - 1 \le t-3$ and then $|S_3(T_\alpha, y)| \le t-3-m-b_y$. So $ca(T_\alpha) \le |S_1(T_\alpha)| + m + \frac{t-2-m-b_x}{2} + (t-3-m-b_y) + \frac{t-2-b_z}{2} \le \frac{|A_2|-2}{2} + 2 + 2t - 5 \le \frac{5}{2}t - 4 \le g(t) - 3$ by Lemma 6, a contradiction.

So for any 3-vertex $v \in S_3(T_\alpha, x)$ where $T_\alpha = xyzx$, $\max\{|S_1(T_\alpha, y)|, |S_1(T_\alpha, z)|\} \ge 1$ and there must exist a triangle $T_\beta = abca$ such that $v \in S_3(T_\beta, a)$ and $|S_1(T_\beta, b)| = |S_1(T_\beta, c)| = 0$. Then for any $v \in S_3(T_\alpha)$, $a(v, T_\alpha) = 0$. By the similar proof as in Case 1, we also have $ca(T_\alpha) \le g(t) - 3$, a contradiction.

Hence, for each integer $t \ge 4$,

$$h(t, k) \le \begin{cases} (R(3, t-1) + t - 1)(k-1), & \text{if } R(3, t-1) \text{ and } t \text{ are both even;} \\ (R(3, t-1) + t)(k-1), & \text{for otherwise.} \end{cases}$$

We complete the proof of the theorem. □

By Theorems 3, 4 and 7, we have the following result.

Corollary 8. $h(4, k) = 9(k-1)$ *and* $h(5, k) = 14(k-1)$.

4. Conclusions

In Section 2, we constructively obtain a lower bound of $h(t, k)$. We firstly construct a $K_{1,t}$-free graph (in fact, a graph with independent number no more than $t - 1$) with minimum degree at least t but containing at most one vertex-disjoint triangle, then we make $k - 1$ copies of it. The resulting graph just implies the lower bound of $h(t, k)$. In view of this, consider a $(3, t)$-Ramsey graph R (that is a triangle-free graph with its independent number no more than $t - 1$). The join graph $K_1 \vee R$ must be a $K_{1,t}$-free graph on $R(3, t)$ vertices but containing at most one vertex-disjoint triangle. But we do not know whether $\delta(K_1 \vee \mathcal{R}) \geq t$ or not. In particular, we have the following question.

Question 1. *Does there exist a $(3, t)$-Ramsey graph R such that $\delta(R) \geq t - 1$?*

If such a graph R do exist, then $(k - 1)(K_1 \vee R)$ is a $K_{1,t}$-free graph on $R(3, t)(k - 1)$ vertices but containing at most $k - 1$ vertex-disjoint triangles. This implies $h(t, k) \geq R(3, t)(k - 1)$. This lower bound seems more beautiful and reasonable, but whether it is proper is still unknown. Note that $R(3, 3) = 6$, $R(3, 4) = 9$ and $R(3, 5) = 14$ (see [1] on page 106). Wang [5] proved $h(3, k) = 6(k - 1) = R(3, 3)(k - 1)$. In Section 3, we prove $h(4, k) = R(3, 4)(k - 1)$ and $h(5, k) = R(3, 5)(k - 1)$. These results imply that the answer of the above question is "yes" for $3 \leq t \leq 5$. Thus we pose the following conjecture to end this paper.

Conjecture 2. *For each integer $t \geq 3$, $h(t, k) = R(3, t)(k - 1)$.*

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