Journal of Symbolic Computation 45 (2010) 1341–1358

Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

The first rational Chebyshev knots[☆]

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ARTICLE INFO

Article history:
Received 25 October 2009
Accepted 30 April 2010
Available online 22 June 2010

Keywords:
Polynomial curves
Rational knots
Two-bridge knots
Chebyshev curves
Real root isolation
Computer algebra
Algorithms

ABSTRACT

A Chebyshev knot $\mathcal{C}(a, b, c, \varphi)$ is a knot which has a parametrization of the form $x(t) = T_a(t);\ y(t) = T_b(t);\ z(t) = T_c(t + \varphi)$, where a, b, c are integers, $T_n(t)$ is the Chebyshev polynomial of degree n and $\varphi \in \mathbf{R}$. We show that any rational knot is a Chebyshev knot with $a = 3$ and also with $a = 4$. For every a, b, c integers ($a = 3, 4$ and a, b coprime), we describe an algorithm that gives all Chebyshev knots $\mathcal{C}(a, b, c, \varphi)$. We deduce the list of minimal Chebyshev representations of rational knots with 10 or fewer crossings.

1. Introduction

It is known that every knot may be obtained from a polynomial embedding $\mathbf{R} \to \mathbf{R}^3 \subset \mathbf{S}^3$, where $\mathbf{S}^3$ is the one-point compactification of $\mathbf{R}^3$ (Vassiliev, 1990; Durfee and O'Shea, 2006; Shastri, 1992; Madeti and Mishra, 2009). The degrees of the polynomials may be quite large, and the plane projections of these knots quite complicated. In this paper we compute explicit three-parameter representations for the first rational knots in the very simple Chebyshev form $x(t) = T_a(t);\ y(t) = T_b(t);\ z(t) = T_c(t + \varphi)$, where $a = 3$ or $a = 4$, b, c are integers and φ is a rational number. For the first time, we produce a data base with very simple polynomial diagrams.

Chebyshev knots are introduced in Koseleff and Pecker (in press-a). A Chebyshev knot $\mathcal{C}(a, b, c, \varphi)$ is a knot which has a parametrization of the form $x(t) = T_a(t);\ y(t) = T_b(t);\ z(t) = T_c(t + \varphi)$,

☆ Expanded version of a conference presented at Mega 2009 (Barcelona, June 2009).

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0747-7171/$ – see front matter © 2010 Published by Elsevier Ltd
doi:10.1016/j.jsc.2010.06.014

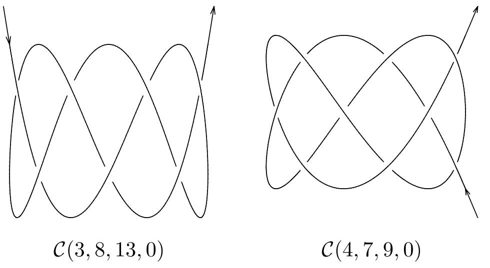


Fig. 1. Two Chebyshev knots: $\overline{7}_7$ and 7_5 (Conway–Rolfsen numbering).

where a, b are coprime integers, c is an integer, $T_n(t)$ is the Chebyshev polynomial of degree n and $\varphi \in \mathbf{R}$. Chebyshev knots are polynomial analogues of Lissajous knots, which admit parametrizations of the form $x = \cos(at);\ y = \cos(bt + \varphi);\ z = \cos(ct + \psi)$, where a, b, c are pairwise coprime integers. These knots, first defined in Bogle et al. (1994), have been studied by many authors: (Jones and Przytycki, 1998; Lamm, 1997; Hoste and Zirbel, 2007).

Apparently, Chebyshev knots are a very particular class of knots (Fig. 1). It is proved in Koseleff and Pecker (in press-a), that, surprisingly, every knot is a Chebyshev knot. The proof uses theorems on braids by Hoste and Zirbel (2007) and Lamm (1999), a density argument (Kronecker theorem) and is non-effective.

In this paper we will give an effective method to find Chebyshev representations of an important and well-understood family of knots: the rational knots.

A rational knot (or a two-bridge knot) is a knot which is isotopic to a compact space curve such that the x-coordinate has only two maxima and two minima. When $a = 3$ or $a = 4$ the Chebyshev knot $K = \mathcal{C}(a, b, c, \varphi)$ is a rational knot K. Now, this knot needs to be identified. Its projection onto the (x, y)-plane is the Chebyshev curve $\mathcal{C}(a, b) : T_b(x) = T_a(y)$. We obtain a regular diagram of K once we know the (under/over) nature of the crossings. When $a = 3$ or $a = 4$ this diagram is in classical Conway normal form (see Conway, 1970; Murasugi, 1996). Using the remarkable Schubert theorem, this gives us an easy way to identify these knots using their classical Schubert fractions (Murasugi, 1996; Schubert, 1956).

Here, we develop a method that enumerates all the knots $\mathcal{C}(a, b, c, \varphi)$, $\varphi \in \mathbf{R}$ where $a = 3$ or $a = 4$, a and b coprime. We also give an algorithm that determines a minimal Chebyshev parametrization $\mathcal{C}(a, b, c, \varphi)$, with $a = 3$ and also with $a = 4$, for any rational knot.

In this paper we develop several algorithms:

(1) Let K be a rational knot. For $a = 3$ and $a = 4$ we determine the minimal integer b such that the Chebyshev curve $\mathcal{C}(a, b) : x = T_a(t),\ y = T_b(t)$ is a plane projection of K. These two algorithms are based on continued fraction expansions.
(2) Let $\mathcal{Z}_{a,b,c}$ be the set of φ such that $\mathcal{C}(a, b, c, \varphi)$ is singular. $\mathcal{Z}_{a,b,c}$ is finite. The knot type of $\mathcal{C}(a, b, c, \varphi)$ is constant over any interval of $\mathbf{R} - \mathcal{Z}_{a,b,c}$. Using resultant computations and multi-precision interval arithmetic, we determine a rational number in each component of $\mathbf{R} - \mathcal{Z}_{a,b,c}$.
(3) We determine the Schubert fraction of the knot $\mathcal{C}(a, b, c, r)$ by computing the (under/over) nature of the crossings. This amounts to evaluate the signs of polynomials at the real solutions of a zero-dimensional system.
(4) We use a sieve method to obtain an exhaustive list of parametrizations for the first rational knots with 10 or fewer crossings.

In Section 2, we will recall some basic results on rational knots and their diagrams. We will show how Chebyshev curves $\mathcal{C}(a, b, c, \varphi)$ define rational knots when $a = 3$ or $a = 4$. We also show that, conversely, rational knots admit Chebyshev parametrizations for $a = 3$ and for $a = 4$. In Section 3, we determine the knot type of $\mathcal{C}(a, b, c, r)$ when $a = 3$ or $a = 4$. In Section 4, we present our sieve method to determine the minimal parametrizations for a given rational knot. At the end, we give an exhaustive and certified list of Chebyshev parametrizations for the rational knots with 10 or fewer crossings. We give minimal parametrizations for the lexicographic degree and also for the total

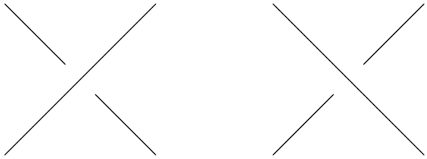


Fig. 2. The right twist and the left twist.

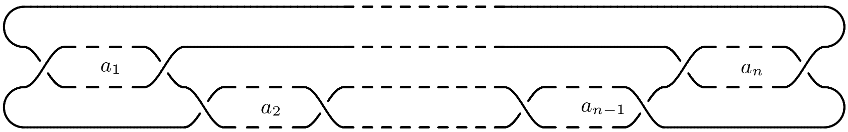


Fig. 3. Conway normal form, *n* odd.

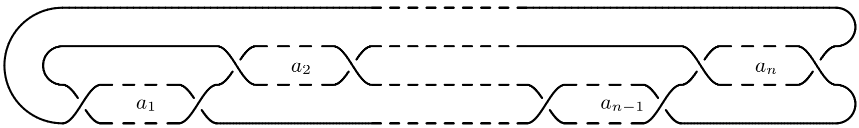


Fig. 4. Conway normal form, *n* even.

degree. The projections of these curves have few crossing points. On the other hand, their degrees may be quite big although they are minimal. This fully justifies the use of certified algorithms and exact computations.

2. Rational knots and their diagrams

A diagram of a knot K is given by a plane projection of K and the knowledge of the (under/over) nature of the crossings. Once we have chosen the (x, y) coordinates, there are two cases of crossings: the right twist and the left twist (see Murasugi, 1996, p. 178). We exclude the case of vertical tangents (Fig. 2).

A rational knot (or link) admits a diagram in Conway normal form (Conway, 1970; Murasugi, 1996). This form, denoted by $C(a_1, a_2, \ldots, a_n)$ where a_i are integers, is explained by the following pictures (Figs. 3 and 4). The number of twists is denoted by the integer $|a_i|$, and the sign of a_i is defined as follows: if i is odd, then the right twist is positive, if i is even, then the right twist is negative. On Figs. 3 and 4, the a_i are positive (the first a_1 twists are right twists).

Schubert discovered the spectacular classification of rational links (Schubert, 1956). He introduced the Schubert fraction of a rational link:

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots + \cfrac{1}{a_m}}}} = [a_1, \ldots, a_m], \quad \alpha > 0. \tag{1}$$

Theorem 1 (Schubert, 1956)**.** *Two rational links of fractions* $\frac{\alpha}{\beta}$ *and* $\frac{\alpha'}{\beta'}$ *are equivalent if and only if* $\alpha = \alpha'$ *and* $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$.

Schubert also proved that the integer α is the classical determinant of the link, it is odd for a knot, and even for a multi-component link. If $K = S(\frac{\alpha}{\beta})$, its mirror image is $\overline{K} = S(\frac{\alpha}{-\beta})$. We will write $\frac{\alpha}{\beta} \approx \frac{\alpha}{\beta'}$ when $S(\frac{\alpha}{\beta})$ and $S(\frac{\alpha}{\beta'})$ are equivalent.

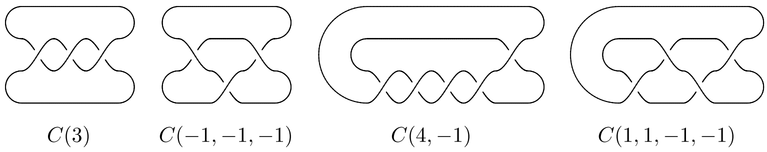


Fig. 5. Trefoil diagrams.

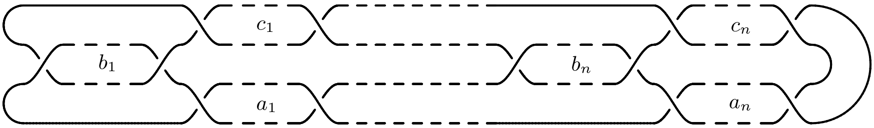


Fig. 6. A knot isotopic to $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \ldots, b_n, a_n + c_n)$.

Example. Fig. 5 shows standard trefoil diagrams.

We shall also need to study knots with a diagram illustrated by Fig. 6.

In this case, the a_i and the c_i are positive if they are left twists, the b_i are positive if they are right twists (on our figure a_i, b_i, c_i are positive). Such a knot is equivalent to a knot with Conway normal form $C(b_1, a_1 + c_1, b_2, a_2 + c_2, \ldots, b_n, a_n + c_n)$ (Murasugi, 1996, p. 183-184).

Chebyshev curves

Chebyshev curves were defined in Fischer (2001). The classical Chebyshev polynomials T_n are defined by $T_n(t) = \cos n\theta$, where $\cos\theta = t$. These polynomials satisfy the linear recurrence $T_0 = 1$, $T_1 = t$, $T_{n+1} = 2t\,T_n - T_{n-1}$, $n \geq 1$, from which we deduce that T_n is a polynomial of degree n and leading coefficient 2^{n-1}.

Proposition 2 (*Fischer, 2001; Koseleff and Pecker, in press-a*)**.** *Let a and b be relatively prime positive integers. The affine Chebyshev curve $\mathcal{C}(a, b)$ defined by*

$$\mathcal{C}(a, b) : T_b(x) - T_a(y) = 0 \tag{2}$$

admits the parametrization $x = T_a(t),\ y = T_b(t)$. This curve has $\frac{1}{2}(a-1)(b-1)$ singular points which are crossing points. The pairs (t, s) giving a crossing point are

$$t = \cos\left(\frac{k}{a} + \frac{h}{b}\right)\pi, \qquad s = \cos\left(\frac{k}{a} - \frac{h}{b}\right)\pi, \tag{3}$$

where k, h are positive integers such that $\dfrac{k}{a} + \dfrac{h}{b} < 1$.

Note that the crossing points of the Chebyshev curve $\mathcal{C}(a, b) : x = T_a(t),\ y = T_b(t)$ lie on the $(b-1)$ vertical lines $T'_b(x) = 0$ and on the $(a-1)$ horizontal lines $T'_a(y) = 0$.

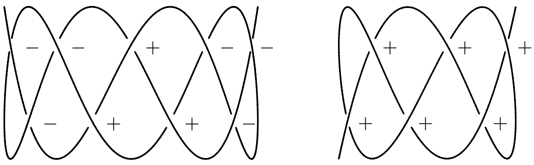
It is remarkable that the (x, y)-diagram of the Chebyshev knot $\mathcal{C}(3, b, c, \varphi)$ is already in Conway normal form (see Fig. 7). For example we obtain the torus knot $7_1 = C(-1, -1, -1, 1, 1, 1, -1, -1, -1)$, and the Fibonacci knot $6_3 = C(1, 1, 1, 1, 1, 1)$.

We get the fraction $\dfrac{7}{-6} \approx 7$ for 7_1 and the fraction $\dfrac{13}{8}$ for the knot 6_3. Consequently, the Conway form of such a knot is given by a $C(\pm 1, \pm 1, \ldots, \pm 1)$.

Fig. 8 shows the examples $\overline{4}_1$: $C(1, 0, 1, 2)$ and 9_{20}: $C(1, 2, 1, 2, 1, 2)$.

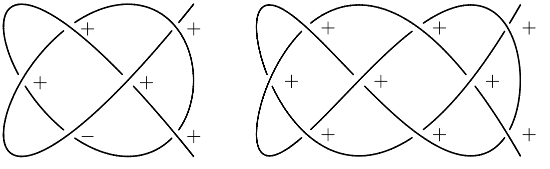
In the case of Chebyshev knots $\mathcal{C}(4, b, c, \varphi)$ we obtain diagrams like in Fig. 6. We thus deduce that the Conway notation for a knot $\mathcal{C}(4, b, c, \varphi)$ is $C(a_1, b_1, \ldots, a_n, b_n)$ where $a_i = \pm 1, b_i = 0, \pm 2$.

In conclusion, we see that, if $a = 3$ or $a = 4$, the knot $\mathcal{C}(a, b, c, \varphi)$ is a rational knot. The nature of the crossings gives the Schubert fraction, which identifies the knot.



7_1 6_3

Fig. 7. Chebyshev diagrams, $a = 3$.



4_1 9_{20}

Fig. 8. Chebyshev diagrams, $a = 4$.

Chebyshev diagrams of rational knots

Conversely, we will show that every rational knot admits a Chebyshev diagram with $a = 3$ and also with $a = 4$. First, we show that any rational number $\frac{\alpha}{\beta}$ may be expressed as a continued fraction corresponding to a Chebyshev diagram $\mathcal{C}(3, b)$.

Algorithm 1 (*Koseleff and Pecker, in press-b*)**.** Let $\frac{\alpha}{\beta}$ be a rational number. There exists a sequence $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_i = \pm 1$, such that $\frac{\alpha}{\beta} = [\varepsilon_1, \ldots, \varepsilon_n]$.

Proof. We have $-[a_1, \ldots, a_m] = [-a_1, \ldots, -a_m]$ so we only need to prove the existence for $\alpha, \beta > 0$, by induction on the height $h(\frac{\alpha}{\beta}) = \max(\alpha, \beta)$.

- If $h = 1$ then $\frac{\alpha}{\beta} = 1 = [1]$ and the result is true.
- If $\alpha > \beta$, we have $\frac{\alpha}{\beta} = [1, \frac{\beta}{\alpha - \beta}]$. Since $h(\frac{\beta}{\alpha - \beta}) < h(\frac{\alpha}{\beta})$, we get our continued fraction by induction.
- If $\beta > \alpha$ we have $\frac{\alpha}{\beta} = [1, -1, -\frac{\beta - \alpha}{\alpha}]$. And we also get the continued fraction.

This completes the construction of our continued fraction expansion $[\pm 1, \ldots, \pm 1]$. $\square$

It is proved in Koseleff and Pecker (in press-b) that the continued fraction expansion $\frac{\alpha}{\beta} = [\varepsilon_1, \ldots, \varepsilon_n]$, $\varepsilon_i = \pm 1$, is unique and of minimal length if there is no two consecutive sign changes, and $\varepsilon_{n-1}\varepsilon_n > 0$.

Now, let us show that any rational number $\frac{\alpha}{\beta}$, α odd and β even, may be expressed as a continued fraction corresponding to a Chebyshev diagram $\mathcal{C}(4, b)$.

Algorithm 2 (*Koseleff and Pecker, 2009*)**.** Let $\frac{\alpha}{\beta}$ be a rational number, β even. There exists a sequence $\varepsilon_1, \ldots, \varepsilon_{2n}, \varepsilon_i = \pm 1$, such that $\frac{\alpha}{\beta} = [\varepsilon_1, 2\varepsilon_2, \ldots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$.

Proof. Let us prove the existence for $\alpha, \beta > 0$, by induction on the height.

- If $h(\frac{\alpha}{\beta}) = 2$, then $\alpha = 1$ and $\beta = 2$ and we have $r = [1, -2]$.

- If $0 < 2\beta < \alpha$ then we write $\frac{\alpha}{\beta} = [1, 2, -1, 2, \frac{\alpha - 2\beta}{\beta}]$. We have $h(\frac{\alpha-2\beta}{\beta}) < h(\frac{\alpha}{\beta})$ and we conclude by induction.
- If $\beta < \alpha < 2\beta$ then we write $\frac{\alpha}{\beta} = [1, 2, \frac{\alpha-\beta}{3\beta - 2\alpha}]$. We have $|3\beta - 2\alpha| \le \alpha$ and $|\alpha - \beta| < \alpha$ and we conclude by induction.
- If $0 < \alpha < \beta$ we write $\frac{\alpha}{\beta} = [1, -2, \frac{\alpha-\beta}{2\alpha-\beta}]$. From $|2\alpha - \beta| \le \beta$ we have $h(\frac{\alpha-\beta}{2\alpha-\beta}) < h(\frac{\alpha}{\beta})$ and we conclude by induction.

The existence of a continued fraction $[1, \pm 2, \ldots, \pm 1, \pm 2]$ is proved. □

It is proved in Koseleff and Pecker (2009) that the continued fraction expansion $[\varepsilon_1, 2\varepsilon_2, \ldots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$, $\varepsilon_i = \pm 1$, is unique if there is no three consecutive sign changes.

We therefore deduce

Corollary 3. *Every rational knot has a Chebyshev diagram* $\mathcal{C}(3, b)$, $b \not\equiv 0 \pmod 3$. *Every rational knot has a Chebyshev diagram* $\mathcal{C}(4, b)$, $b \equiv 1 \pmod 2$.

Proof. Let us consider a knot $K = S(\frac{\alpha}{\beta})$.

Using Algorithm 1, we can write $\frac{\alpha}{\beta} = [\varepsilon_1, \ldots, \varepsilon_n]$, $\varepsilon_i = \pm 1$. From $[\varepsilon, \frac{a}{b}] = \frac{\varepsilon a + b}{a}$, one sees by induction that $n \not\equiv 2 \pmod 3$ since α is odd. K is isotopic to $C(\varepsilon_1, \ldots, \varepsilon_n)$ which corresponds to a Chebyshev diagram $\mathcal{C}(3, n+1) : x = T_3(t), y = T_{n+1}(t)$.

If β is even, we can write, using Algorithm 2, $\frac{\alpha}{\beta} = [\varepsilon_1, 2\varepsilon_2, \ldots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$, $\varepsilon_i = \pm 1$. If β is odd, we consider $\frac{\alpha}{\beta - \alpha}$. The knot K is isotopic to $C(\varepsilon_1, 2\varepsilon_2, \ldots, \varepsilon_{2n-1}, 2\varepsilon_{2n})$ which corresponds to a Chebyshev diagram $\mathcal{C}(4, 2n+1) : x = T_3(t), y = T_{2n+1}(t)$. □

Corollary 4. *Every rational knot is a Chebyshev knot* $\mathcal{C}(3, b, c, \varphi)$. *Every rational knot is a Chebyshev knot* $\mathcal{C}(4, b, c, \varphi)$.

Proof. Using Kronecker theorem, it is proved in Koseleff and Pecker (in press-a) that there exist φ and c such that $\mathcal{C}(3, b, c, \varphi) = C(\varepsilon_1, \ldots, \varepsilon_n)$. The case $a = 4$ is similar. □

Unfortunately, Corollary 4 does not provide c nor φ and not even a bound for c. We will show now that there is only a finite number of distinct knots $\mathcal{C}(a, b, c, \varphi)$. This allows us to develop a sieve method to determine a minimal Chebyshev parametrization for every rational knot.

3. Description of Chebyshev diagrams

It is convenient to consider the polynomials in $S = s + t$ and $T = st$:

$$P_n(S, T) = \frac{T_n(t) - T_n(s)}{(t - s)}, \qquad Q_n(S, T, \varphi) = \frac{T_n(t + \varphi) - T_n(s + \varphi)}{(t - s)}. \tag{4}$$

The study of the family $\mathcal{C}(a, b, c, \varphi)$ is connected with the description of the algebraic variety

$$\mathcal{V}_{a,b,c} = \{(S, T, \varphi),\ P_a(S, T) = 0,\ P_b(S, T) = 0, P_c(S, T, \varphi) = 0\}. \tag{5}$$

We denote by $\mathcal{Z}_{a,b,c}$ the set of *critical values* φ, such that $\mathcal{C}(a, b, c, \varphi)$ is a singular curve.

Proposition 5. *Let a, b and c be positive integers, a and b being relatively prime.* $\mathcal{V}_{a,b,c}$ *is 0-dimensional and we have* $\left|\mathcal{Z}_{a,b,c}\right| \le \frac{1}{2}(a-1) \times (b-1) \times (c-1)$.

Proof. $\mathcal{C}(a, b, c, \varphi)$ is a singular space curve iff there exist $S = s + t,\ T = st \in \mathbf{R}$ such that

$$P_a(S, T) = 0, \qquad P_b(S, T) = 0, \qquad Q_c(S, T, \varphi) = 0. \tag{6}$$

From Proposition 2, $\mathcal{C}(a, b)$ has exactly $\frac{1}{2}(a - 1)(b - 1)$ crossing points corresponding to $\{(S, T),\ P_a(S, T) = 0,\ P_b(S, T) = 0\}$. For each of these elements, the set $\{\varphi \in \mathbf{R},\ Q_c(S, T, \varphi) = 0\}$ has at most $c - 1$ elements because the leading monomial of $Q_c(S, T, \varphi)$ is $c(2\varphi)^{c-1}$. Therefore, $\mathcal{Z}_{a,b,c}$ has at most $\frac{1}{2}(a - 1) \times (b - 1) \times (c - 1)$ elements. □

In the following lemma, we see that the nature of the crossing is given by the sign of a symmetrical polynomial.

Lemma 6. *Let a, b and c be integers, a and b being relatively prime. Consider the diagram of the curve $\mathcal{C}(a, b, c, \varphi)$. Let $s \neq t$ be parameters such that $T_a(t) = T_a(s)$ and $T_b(t) = T_b(s)$ and let $S = s + t$, $T = st$,*

$$D(S, T, \varphi) = Q_c(S, T, \varphi)P_{b-a}(S, T). \tag{7}$$

Then $D(S, T, \varphi) > 0$ if and only if the crossing is a right twist.

Proof. Let (s, t) be the parameters of a double point of $\mathcal{C}(a, b)$. The crossing is a right twist if and only if

$$\Big(z(t) - z(s)\Big)\left(\frac{y'(t)}{x'(t)} - \frac{y'(s)}{x'(s)}\right) > 0. \tag{8}$$

Using Proposition 2, we get $s = \cos\sigma$ and $t = \cos\tau$. A simple computation gives $\sin\tau\ x'(t) = -\sin\sigma\ x'(s)$, $\sin\tau\ y'(t) = \sin\sigma\ y'(s)$ and $\dfrac{y'(t)}{x'(t)} = -\dfrac{y'(s)}{x'(s)}$. The slopes of the corresponding tangents are opposite. We therefore deduce that (using $x \sim y$ for $\operatorname{sign}(x) = \operatorname{sign}(y)$)

$$x'(t)y'(t) \sim -x'(s)y'(s) \sim \frac{y'(t)}{x'(t)} - \frac{y'(s)}{x'(s)} \sim x'(t)y'(t) - x'(s)y'(s). \tag{9}$$

But

$$x'(t)y'(t) = ab\frac{\sin a\tau\ \sin b\tau}{\sin^2\tau} \sim 2\sin a\tau\ \sin b\tau = T_{a-b}(t) - T_{a+b}(t). \tag{10}$$

Consequently $x'(t)y'(t) - x'(s)y'(s) \sim (T_{a-b}(t) - T_{a+b}(t)) - (T_{b-a}(s) - T_{a+b}(s))$. On the other hand, using the identities $T_{b+a} + T_{b-a} = 2T_aT_b$, $T_a(t) = T_a(s)$ and $T_b(t) = T_b(s)$, we conclude that $x'(t)y'(t) - x'(s)y'(s) \sim T_{b-a}(t) - T_{b-a}(s)$, which gives the announced result. □

We thus deduce

Proposition 7. *The number of distinct knots $\mathcal{C}(a, b, c, \varphi)$, $\varphi \in \mathbf{R}$ is at most $\frac{1}{2}(a-1) \times (b-1) \times (c-1)$.*

Proof. The curves $\mathcal{C}(a, b, c, \varphi)$, $\varphi \in \mathbf{R}$ have the same projection $\mathcal{C}(a, b)$ on the (x, y)-plane. The nature of the crossings corresponding to parameters (s, t) are given by the sign of $Q_c(S, T, \varphi)$ where $S = s + t, T = st$. Let $\varphi_1 < \varphi_2 < \cdots < \varphi_N$ be the critical values. When $\varphi \in (\varphi_i, \varphi_{i+1})$, the signs of $D(S, T, \varphi)$ are constant and so is the diagram of $\mathcal{C}(a, b, c, \varphi)$. There are at most $N + 1$ different knots. When φ is big enough, $T_c(t + \varphi)$ is increasing in $[-1, 1]$ and the knot is trivial. When φ is small enough, the knot is also trivial. □

Remark 8. We see that $\mathcal{C}(a, b, c, -\varphi)$ is isotopic to $\mathcal{C}(a, b, c, \varphi)$ up to mirroring. $\mathcal{Z}_{a,b,c}$ is then symmetrical about the origin. Note that $\varphi = 0$ is a critical value when c and ab are not coprime. Then, there are at most $\frac{1}{2}(a - 1)(b - 1)\lfloor\frac{c-1}{2}\rfloor$ different knots, up to mirroring.

Lemma 9. *There exists $R_{a,b,c} \in \mathbf{Q}[\varphi]$ with degree $\deg R_{a,b,c} \leq \frac{1}{2}(a - 1)(b - 1)(c - 1)$ such that $\mathcal{Z}_{a,b,c} = Z(R_{a,b,c})$.*

Proof. Choose a generator R of the principal ideal $\langle P_a, P_b, Q_c\rangle \bigcap \mathbf{Q}[\varphi]$. □

Computations of the diagrams

a, b and c being fixed, there are mainly two algorithms for the determination of all possible knots $\mathcal{C}(a, b, c, \varphi)$.

(1) Determine test values, that is to say rational numbers $r_0, \dots, r_N$, such that $\mathcal{C}(a, b, c, \varphi)$ is one of the $\mathcal{C}(a, b, c, r_i)$.
(2) Given a rational number r, determine the diagram of $K = \mathcal{C}(a, b, c, r)$. When $a = 3$ or $a = 4$, determine as well the rational knot K by computing its Schubert form.

As T_n satisfies the linear recurrence of order 2: $T_{n+1} + T_{n-1} = 2tT_n$, we deduce that Q_n (Formula (4)) satisfies the linear recurrence of order 4:

$$Q_0 = 0, \qquad Q_1 = 1, \qquad Q_2 = 2S + 4\varphi, \qquad Q_3 = -4T + 12\varphi S + 4S^2 + 12\varphi^2 - 3.$$
$$Q_{n+4} = 2(S + 2\varphi)(Q_{n+3} + Q_{n+1}) - 2\left(2\varphi^2 + 2T + 2\varphi S + 1\right) Q_{n+2} - Q_n. \tag{11}$$

For $P_n(S, T) = Q_n(S, T, 0)$ we find

$$P_0 = 0, \qquad P_1 = 1, \qquad P_2 = 2S, \qquad P_3 = -4T + 4S^2 - 3.$$
$$P_{n+4} = 2S(P_{n+3} + P_{n+1}) - (4T + 2) P_{n+2} - P_n. \tag{12}$$

As $T_m(T_n) = T_{mn}$ we deduce that $Q_{nm} = Q_n(T_m)Q_m$ so $Q_m | Q_{mn}$.

- **Computation of** $R_{a,b,c}$: `PhiProjection`
  A straightforward way is to compute a Gröbner basis of $\langle P_a(S, T), P_b(S, T), Q_c(S, T, \varphi)\rangle$ for a so called *elimination order* (see Cox et al. (1998)). Triangular decompositions provide a suitable alternative, or, more basically, iterative resultants in *generic* situations.
  One can also use any method that first rewrites the system $\{P_a(S, T), P_b(S, T)\}$ as a rational parametrization (as in Rouillier (1999) or Giusti et al. (2001)). We obtain $\{S = S(u), T = T(u), M(u) = 0\}$ where $S(u)$ and $T(u)$ are polynomials in u whose minimal polynomial is M. We compute $Q_c(S(u), T(u), \varphi) \pmod{M(u)}$ using the recurrence formula (11). We then obtain $R_{a,b,c}$ as the resultant between $M(u)$ and $Q_c(S(u), T(u), \varphi)$. We will not discuss here this general method, we shall only describe the cases $a = 3$ and $a = 4$.
- **Computation of the test values**: `PhiSampling`
  Such a function can easily be implemented using any solver that is able to *isolate* real roots of univariate polynomials (say providing non-overlapping intervals with rational bounds around all the real roots). This solver must be able to discriminate multiple roots from clusters of roots, real roots from complex roots with a small imaginary part (which excludes many numerical methods and most of implementations using hardware floats). One can use methods based on Sturm sequences or the Descartes rule of signs (see Basu et al. (2003) for an overview), but also many strategies using interval analysis. Due to the high degree of the polynomials, our choice is to use algorithms based on the Descartes rule of signs using multi-precision interval arithmetic as in Rouillier and Zimmermann (2003).
- **Computation of the diagram** $\mathcal{C}(a, b, c, r)$: `SignSolve`.
  We have to determine the signs of $D(S, T, r)$ where r is a rational number and (S, T) belong to $\{P_a(S, T) = 0, P_b(S, T) = 0\}$. The determination of the sign of a polynomial over a zero-dimensional system is difficult to certify when using numerical method. There are few exact/certified existing methods/implementations for this problem. The strategy is naturally linked to the implementation of the function `PhiProjection` since the zero-dimensional system to be considered by `SignSolve` is a subsystem of the one which is to be considered by `PhiProjection`. One can use the *generalized Hermite method* for zero-dimensional systems as in Pedersen et al. (1993), which makes use of Gröbner bases. If we have polynomial parametrization (that we obtained for `PhiProjection`), we apply any algorithm that computes the sign of an univariate polynomial at a real algebraic number. This last step can be done by extending methods based on Sturm theorem or based on the Descartes rule of signs. Due to our implementation of `PhiProjection` and to the degrees of the polynomials, we base our implementation on the Descartes rule of signs.

The case $a = 3$

We first study a simple example.

The family of knots $\mathcal{C}(3, 5, 7, \varphi) : \Big(T_3(t),\ T_5(t),\ T_7(t+\varphi)\Big)$**.**
Double points of $\mathcal{C}(3, 5, 7, \varphi)$ satisfy $P_3(S, T) = 0,\ P_5(S, T) = 0,\ Q_7(S, T, \varphi)$. From $P_3(S, T) = 4(T - S^2 + \frac{3}{4}) = 0$, we deduce the polynomial parametrization

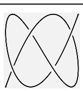
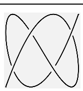
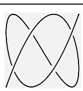
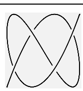
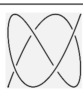
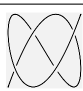
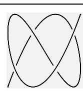
$$S = S, \qquad T = S^2 - \tfrac{3}{4}, \qquad P_5(S, S^2 - \tfrac{3}{4}) = 0.$$

The curve $\mathcal{C}(3, 5, 7, \varphi)$ is singular iff $16S^4 - 12S^2 + 1$ and $Q_7(S, S^2 - \frac{3}{4}, \varphi)$ have a common root, that is to say, iff φ is a root of $R_{3,5,7} = \mathrm{Res}_S\Big(P_5(S, S^2 - \frac{3}{4}), Q_7(S, S^2 - \frac{3}{4}, \varphi)\Big)$.

$$\begin{array}{rcl} P_5(S, S^2 - \frac{3}{4}) & = & -16S^4 + 12S^2 - 1, \\ Q_7(S, S^2 - \frac{3}{4}, \varphi) & = & 64\,S^6 - 16\,\left(84\,\varphi^2 + 5\right) S^4 - 112\,\varphi\,\left(20\,\varphi^2 + 1\right) S^3 \\ & & +24\,\left(42\,\varphi^2 + 1\right) S^2 + 28\,\varphi\,\left(48\,\varphi^4 + 80\,\varphi^2 + 3\right) S \\ & & +448\,\varphi^6 + 1120\,\varphi^4 + 84\,\varphi^2 - 1. \end{array} \qquad (13)$$

$R_{3,5,7}$ has degree $24 = \frac{1}{2}(3-1)(5-1)(7-1)$ and 12 real roots $\varphi_1, \ldots, \varphi_{12}$. We choose 13 rational values $r_0 < \varphi_1 < r_1 < \cdots < \varphi_{12} < r_{12}$.

Let us now determine the nature of $\mathcal{C}(3, 5, 7, r)$. We have to evaluate $D(S, T, \varphi) = Q_7(S, S^2 - \frac{3}{4}, r) \times P_2(S, S^2 - \frac{3}{4})$ when $P_5(S, S^2 - \frac{3}{4}) = 0$. Let $S_1 < S_2 < S_3 < S_4$ be the 4 real roots of $P_5(S, S^2 - \frac{3}{4})$. They correspond to parameters $(s_1, t_1), \ldots, (s_4, t_4)$ such that $s_i + t_i = S_i,\ s_i t_i = T_i = S_i^2 - \frac{3}{4}$. We have $T_3(s_2) < T_3(s_1) < T_3(s_4) < T_3(s_5)$ and the knot $\mathcal{C}(3, 5, 7, r)$ is given by the continued fraction expansion $\alpha/\beta = [D_2, -D_1, D_4, -D_3]$ where $D_i = \mathrm{sign}\,(Q_7(S_i, S_i^2 - \frac{3}{4}, r) \times P_2(S_i, S_i^2 - \frac{3}{4}))$. We obtain the following diagrams:

$r_7 = 0$	$r_8 = \frac{1}{15}$	$r_9 = \frac{1}{5}$	$r_{10} = \frac{1}{4}$	$r_{11} = \frac{1}{2}$	$r_{12} = \frac{2}{3}$	$r_{13} = 1$
						
$\frac{\alpha}{\beta} = -\frac{5}{3}$	$\frac{\alpha}{\beta} = \frac{1}{3}$	$\frac{\alpha}{\beta} = 1$	$\frac{\alpha}{\beta} = 1$	$\frac{\alpha}{\beta} = -1$	$\frac{\alpha}{\beta} = -1$	$\frac{\alpha}{\beta} = -1$

The only nontrivial knot is the figure-eight knot $4_1 = S(\frac{5}{2})$. It is obtained for $r = 0$.

The general case.
We will proceed as follows. The crossing points of $\mathcal{C}(3, b)$ correspond to the $(b-1)$ parameters $\{(S, T),\ P_3(S, T) = 0,\ P_b(S, T) = 0\}$. Let (S, T) such that $P_3(S, T) = 0,\ P_b(S, T) = 0$. We obtain $T = S^2 - \frac{3}{4}$ and $P_b(S, S^2 - \frac{3}{4}) = 0$, that gives a polynomial parametrization of (S, T). The polynomial $P_b(S, S^2 - \frac{3}{4})$ has $(b-1)$ roots in $\mathbf{C}$ and thus $\deg(P_b(S, S^2 - \frac{3}{4})) = b - 1$. The set $\mathcal{Z}_{3,b,c}$ of critical values φ is exactly the set of the roots of the polynomial $R_{3,b,c}$ of degree $(b-1)(c-1)$ (Fig. 9):

$$R_{3,b,c} = \mathrm{Res}_S\Big(P_b(S, S^2 - \tfrac{3}{4}), Q_c(S, S^2 - \tfrac{3}{4}, \varphi)\Big). \qquad (14)$$

Let $A(S)$ be a crossing point corresponding to parameter S (and $T = S^2 - \frac{3}{4}$). Its abscissa is $T_3(t) = T_3(s) = -T_3(S) = S(3 - 4S^2)$.

We define the order relation $A(S) <_3 A(S')$ if $T_3(S) > T_3(S')$. The Conway notation of $\mathcal{C}(3, b, c, r)$ is $C(D(A_1), -D(A_2), \ldots, (-1)^{b-1} D(A_{b-1}))$ where $A_1 <_3 A_2 <_3 \cdots <_3 A_{b-1}$ and $D(A)$ is $D(S, T, r)$ defined in Formula (7) (Fig. 10).

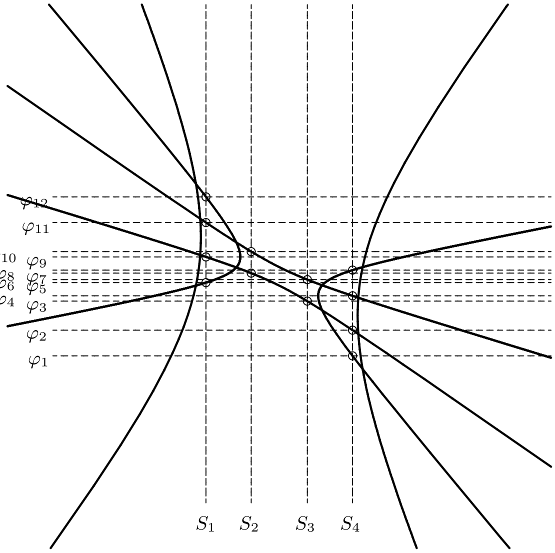


Fig. 9. The curve $Q_7(S, S^2 - \frac{3}{4}, \varphi) = 0$ and the lines $P_5(S, S^2 - \frac{3}{4}) = 0$.

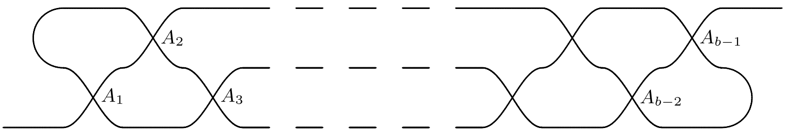


Fig. 10. $\mathcal{C}(3, b)$, b even.

The case $a = 4$

We get $P_4(S, T) = 8\,S\left(S^2 - 2\,T - 1\right)$. We thus obtain two parametrizations corresponding to two families of double points for $\mathcal{C}(4, b)$:

$$\mathcal{A} : \{S = 0,\ P_b(0, T) = 0\}, \qquad \mathcal{B} : \left\{T = \tfrac{1}{2}\left(S^2 - 1\right), P_b\left(S, \tfrac{1}{2}(S^2 - 1)\right) = 0\right\}. \tag{15}$$

These two families correspond to diff When $S = 0$, we have $T_b(t) = -T_b(t) = 0$. The elements of $\mathcal{A}$ lie on $y = 0$. Let $n = \frac{1}{2}(b - 1)$. We have $|\mathcal{A}| = n$ from which we deduce that $\deg_T P_b(0, T) = n$.

Elements of $\mathcal{B}$ are symmetric with respect to $y = 0$. They lie on the two lines $y = \pm\dfrac{1}{\sqrt{2}}$. From $|\mathcal{B}| = 2n$, we deduce that $\deg_S P_b(S, \frac{1}{2}(S^2 - 1)) = 2n$. As the leading coefficient of $Q_c(S, T, \varphi)$ is $c(2\varphi)^{c-1}$ we deduce that

$$R_1(\varphi) = \mathrm{Res}_T(P_b(0, T), Q_c(0, T, \varphi)) \tag{16}$$

has degree $n(c - 1)$ and

$$R_2(\varphi) = \mathrm{Res}_S\left(P_b\left(S, \tfrac{1}{2}\left(S^2 - 1\right)\right), Q_c\left(S, \tfrac{1}{2}\left(S^2 - 1\right), \varphi\right)\right) \tag{17}$$

has degree $2n(c - 1)$. $\mathcal{Z}_{4,b,c}$ is exactly the set of the real roots of $R_{4,b,c} = R_1 \times R_2$.

The abscissa of $A(T) \in \mathcal{A}$ is given by $T_4(t) = T_4(s) = 1 + 8\,T + 8\,T^2$. The abscissa of $B(S) \in \mathcal{B}$ is given by $T_4(t) = T_4(s) = -1 + 4\,S^2 - 2\,S^4$. We have to separately sort the crossing points of $\mathcal{A}$ and $\mathcal{B}$ by increasing abscissae: $A_1, \ldots, A_n$ and $B_1, B'_1, \ldots, B_n, B'_n$ (Fig. 11). Note that B_i and B'_i have the same

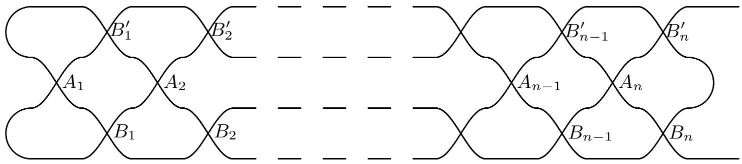


Fig. 11. $\mathcal{C}(4, 2n + 1)$.

abscissa. The Conway notation for the knot we obtain is then

$$C\Big(D(A_1), -(D(B_1) + D(B'_1)), \dots, D(A_n), -(D(B_n) + D(B'_n))\Big), \tag{18}$$

where $D(A)$ (resp. $D(B)$) is $D(S, T, r)$ defined in Formula (7).

4. Results

In this section we present some results we have obtained using certified implementations of the three black boxes on which our algorithms are based. They are easily implementable in any high level language.

There are numerous choices for the implementations, but our requirements are strict: we must certify all the results since our goal is to obtain a classification; bearing in mind that the systems of polynomial equations have thousands of roots.

For the experiments, we used the MAPLE environment. `PhiProjection` is based on resultants computations (Section 3). We use the MAPLE function `Isolate` for `PhiSampling` (without constraints) and for `SignSolve` (with constraints). In the univariate case, this function is based on the algorithm described in Rouillier and Zimmermann (2003). Other computations have been straightforwardly implemented according to the descriptions proposed in Section 3.

Computing minimal diagrams

The next table gives the number $\mathcal{K}_N$ of rational knots with crossing number N, up to mirror symmetry (see Ernst and Sumners, 1987 for a formula).

N	3	4	5	6	7	8	9	10
$\mathcal{K}_N$	1	1	2	3	7	12	24	45

The minimal b for a Chebyshev diagram $\mathcal{C}(3, b)$ of $K = S(\frac{\alpha}{\beta})$, $\frac{\alpha}{\beta} > 1$, is obtained with $b = n + 1$ where n is the length of the continued fraction of $\frac{\alpha}{\beta}$ or $\frac{\alpha}{\alpha - \beta}$ (see Koseleff and Pecker , in press-b). This allows us, using Algorithm 1, to know the minimal b for which $\mathcal{C}(3, b)$ is a projection of a given rational knot K. We obtain all rational knots with N crossings (here $N \le 10$) with Chebyshev diagrams $\mathcal{C}(3, b)$ where $b \le \frac{3}{2}N - 1 \le 14$. Note that this last result is proved in Koseleff and Pecker (in press-b) for any rational knot with crossing number N.

In a similar manner, let $K = S(\frac{\alpha}{\beta})$, $\frac{\alpha}{\beta} > 1$, β even. The minimal integer $b = 2n+1$ for which there exists a continued fraction expansion $r = [\varepsilon_1, 2\varepsilon_2, \dots, \varepsilon_{2n-1}, 2\varepsilon_{2n}]$, $\varepsilon_i = \pm 1$, such that $K = S(r)$, is the smallest length of the continued fraction expansion $[\pm 1, \pm 2, \dots, \pm 1, \pm 2]$ of either $\frac{\alpha}{\beta}$, $\frac{\alpha}{2\alpha - \beta}$, $\frac{\alpha}{\beta'}$ or $\frac{\alpha}{2\alpha - \beta'}$ where $0 < \beta' < \alpha$, β' even and $\beta\beta' = \pm 1 \pmod{\alpha}$.

For some knots, there may be a shorter continued fraction expansion, allowing 0 instead of some ± 2. The list of these knots (up to crossing number 10) is 8_{12}, 9_{13}, 9_{15}, 9_{26}, 10_8, 10_{12}, 10_{13}, 10_{25}, 10_{29}, 10_{38}, 10_{42}. For example $8_{12} = S(\frac{29}{12})$ and we have $\frac{29}{12} = [1, 0, 1, 2, 1, 0, 1, 2]$ while it is not possible to get a shorter continued fraction corresponding to 8_{12}. We have enumerated all possible continued fraction expansions corresponding to diagrams $\mathcal{C}(4, b)$ in order to determine the minimal b corresponding to a rational knot K. We have obtained all rational knots with 10 crossings or fewer with Chebyshev diagrams $\mathcal{C}(4, b)$ where $b \le 13$.

Let K be a rational knot. Once we know what we can expect as a diagram $\mathcal{C}(a, b)$ for K, we look for it as a Chebyshev knot $\mathcal{C}(a, b, c, \varphi)$. We see from the previous table that every rational knot with crossing number $N \le 10$ is a Chebyshev knot $\mathcal{C}(3, b, c, r)$ with $b \le 14$ and a Chebyshev knot $\mathcal{C}(4, b, c, r)$ with $b \le 13$. In comparison, the number of crossing points for Lissajous diagrams is far greater (see Boocher et al., 2009).

Computing the minimal parametrization

Once we know the minimal b for each knot, we seek for the minimal c such that $\mathcal{C}(a, b, c, \varphi)$ parametrizes K or $\overline{K}$. We have limited ourselves to the bounds $b \le 21$, $c \le 300$ and $(b-1)(c-1) \le 13 \times 299 = 3887$. The degrees of the polynomials $R_{a,b,c}(\varphi)$ giving the critical values are bounded by 3887 when $a = 3$ and 5382 when $a = 4$.

A remarkable fact is that these polynomials have a large number of real roots (in average 58% when $a = 3$ and 57% when $a = 4$). The proportion of nontrivial knots $\mathcal{C}(a, b, c, r)$ is approx. 25% when $a = 3$ and 39% when $a = 4$.

We conclude our paper with the list of the minimal Chebyshev parametrizations for the first 95 rational knots with 10 or fewer crossings. We give the Conway–Rolfsen numbering, their Schubert fraction (up to mirror symmetry) and their presentation as Chebyshev knots. Most of them have a parametrization with the minimal b and $c < 300$. All of them have a parametrization with a minimal $(b-1)(c-1)$. For the extremal values of $(b-1)(c-1)$, the computation of the resultants and their roots may take at this stage many hours. We have recently proposed an alternative technique to get the minimal parametrizations with higher degrees.

The family of knots $\mathcal{C}(3, 14, 292, \varphi)$

$P_{14}(S, S^2 - \frac{3}{4})$ is a product of 4 factors of degrees [6, 3, 3, 1]. We have $292 = 4 \times 73$. We know that $Q_{73} | Q_{146} | Q_{292}$. The polynomial $Q_{73}(S, S^2 - \frac{3}{4}, \varphi)$ is irreducible and has degree 72. The polynomial Q_{146} is a product of polynomials with degrees [72, 72, 1]. At the end, Q_{292} is a product of 5 factors with degrees [144, 72, 72, 2, 1]. We compute $R_{3,14,292}$ as the product of 20 resultants between factors of P_{14} and Q_{292}. The polynomial $R_{3,14,292}(\varphi)$ has degree 3783 and exactly 2185 distinct real roots. We compute the 1093 Schubert fractions $\mathcal{C}(3, 14, 292, r_i)$ where $r_i > 0$. We obtain 275 nontrivial knots and eventually 34 distinct knots. One of these has crossing number greater that 10, it is the knot $\overline{12}_{518} = S(\frac{157}{34})$. We obtain the knot $10_{20} = S(\frac{35}{11})$ with $\mathcal{C}(3, 14, 292, 1/94)$ and the knot $10_{29} = S(\frac{63}{17})$ with $\mathcal{C}(3, 14, 292, 1/93)$. These two parametrizations are of minimal lexicographic degrees.

The family of knots $\mathcal{C}(4, 13, 267, \varphi)$

The polynomials $P_{13}(S, T)$, $P_{13}(0, T)$ (of degree 6) and $P_{13}(S, \frac{1}{2}(S^2 - 1))$ (of degree 12) are irreducible. We have $267 = 3 \times 89$. We know that $Q_{89} | Q_{267}$. The polynomial Q_{267} is the product of Q_3, Q_{89} and a polynomial of degree 176 in φ. The polynomial $Q_{267}(S, T, \varphi)$ is a product of polynomials of degrees [176, 88, 2].

We thus obtain $R_1 = \mathrm{Res}_T\Big(P_{13}(0, T), Q_{267}(0, T)\Big)$ as a product of polynomials of degrees [1056, 528, 12] and $R_2 = \mathrm{Res}_S\Big(P_{13}(S, \frac{1}{2}(S^2 - 1)), Q_{267}(S, \frac{1}{2}(S^2 - 1))\Big)$ as a product of polynomials of degrees [2112, 1056, 24].

The polynomial $R_{4,13,267} = R_1 \times R_2$ has degree 4788. It has 2882 distinct real roots. We compute the 1442 Schubert fractions $\mathcal{C}(4, 13, 267, r_i)$ where $r_i > 0$. We obtain 710 nontrivial knots. 72 of these are distinct knots whose crossing numbers take all values between 3 and 16.

5. Conclusion

We have shown that every rational knot is a Chebyshev knot with $a = 3$ and also with $a = 4$. For every a, b, c integers ($a = 3, 4$ and a, b coprime), we have described an algorithm that gives all Chebyshev knots $\mathcal{C}(a, b, c, \varphi)$.

We have given parametrizations of every rational knot as $\mathcal{C}(3, b, c, \varphi)$ and $\mathcal{C}(4, b, c, \varphi)$ where (b, c) were minimal for the lexicographic order ($c \leq 300$). For 11 knots we know the minimal b and that $c > 300$.

Our experiments fully justify the use of certified algorithms and exact computations since numerical methods would certainly have failed in finding for example the knot $10_{20} = S(\frac{35}{11})$ with $\mathcal{C}(3, 14, 292, 1/94)$ and the knot $10_{29} = S(\frac{63}{17})$ with $\mathcal{C}(3, 14, 292, 1/93)$. A further objective is now to consolidate and speed up our algorithms in order to increase their capabilities.

As the zero-dimensional systems we study have a triangular structure, we currently try to directly get an exhaustive list of Chebyshev knots $\mathcal{C}(a, b, c, \varphi)$ without computing additional resultants.

If a, b and c are relatively coprime, we expect that the real variety $\mathcal{V}_{a,b,c}$ has only single points and try to directly get all possible signs. In that case, our three black-boxes could be implemented using exclusively univariate functions that compute recursively the roots of the systems to be solved without any additional rewriting.

Note added in proof

We have recently proposed in Koseleff et al. (2010) an alternative method based on an explicit factorization of $R_{a,b,c}(\varphi)$ in terms of second-degree polynomials in $\mathbf{Q}(\cos\frac{\pi}{a}, \cos\frac{\pi}{b}, \cos\frac{\pi}{c})[\varphi]$.

This allows us to separately compute the roots of these second-degree polynomials, using multi-precision interval arithmetic, even if $a > 4$. The problem remains to ensure that we obtain the roots of $R_{a,b,c}$ with their multiplicities. With this method, we find the eleven missing knots.

For example, $R_{4,13,856}$ has degree 15 390 and 9246 real roots (0 has multiplicity 18). We get 2050 nontrivial knots, 83 of them are distinct, and 63 have less than 10 crossings. We obtain 10_{33} as $\mathcal{C}(4, 13, 856, 1/328)$. This is the minimal parametrization for $a = 4$. Such a result was unreachable with resultant calculations.

6. Table

Here is the list of the first 95 rational knots. We have given Chebyshev parametrizations for $a = 3$ and $a = 4$. One corresponds to the minimal b and the other to the minimal total degree in b, c. For each parametrization we give the corresponding Schubert fraction (α/β), the number of double points (DP) in the corresponding diagram $\mathcal{C}(a, b)$ so as the degree (deg) of $\mathcal{V}_{a,b,c}$. Note that sometimes we have fewer double points with $a = 4$ than with $a = 3$. For 11 knots (9_2, 9_5, 9_{23}, 10_3, 10_6, 10_{30}, 10_{33}, 10_{36}, 10_{37}, 10_{38}, 10_{39}), Chebyshev parametrizations with b minimal are obtained with $c > 300$ and with the method developed in Koseleff et al. (2010).

When naming knots, we do not distinguish a knot from its mirror image. Nevertheless, this information is easy to find out from the Schubert fractions of our diagrams (in the third column).

Chebyshev parametrizations of the first rational knots

K	minimal b	α/β	DP	deg	min. $(b-1)(c-1)$	α/β	DP	deg
3_1	$\mathcal{C}(3, 4, 5, 0)$	$3/2$	3	12	$\mathcal{C}(3, 4, 5, 0)$	$3/2$	3	12
	$\mathcal{C}(4, 3, 5, 0)$	$-3/2$	3	12	$\mathcal{C}(4, 3, 5, 0)$	$-3/2$	3	12
4_1	$\mathcal{C}(3, 5, 7, 0)$	$5/3$	4	24	$\mathcal{C}(3, 5, 7, 0)$	$5/3$	4	24
	$\mathcal{C}(4, 5, 12, 1/23)$	$5/2$	6	66	$\mathcal{C}(4, 7, 8, 1/3)$	$5/2$	9	63
5_1	$\mathcal{C}(3, 7, 8, 0)$	$-5/4$	6	42	$\mathcal{C}(3, 7, 8, 0)$	$-5/4$	6	42
	$\mathcal{C}(4, 5, 8, 1/23)$	$5/4$	6	42	$\mathcal{C}(4, 5, 8, 1/23)$	$5/4$	6	42

Chebyshev parametrizations of the first rational knots

K	minimal b	α/β	DP	deg	min. $(b-1)(c-1)$	α/β	DP	deg
5_2	$\mathcal{C}(3, 7, 17, 1/50)$	$-7/4$	6	96	$\mathcal{C}(3, 10, 11, 1/16)$	$7/4$	9	90
	$\mathcal{C}(4, 5, 7, 0)$	$7/4$	6	36	$\mathcal{C}(4, 5, 7, 0)$	$7/4$	6	36
6_1	$\mathcal{C}(3, 8, 10, 1/42)$	$-9/5$	7	63	$\mathcal{C}(3, 8, 10, 1/42)$	$-9/5$	7	63
	$\mathcal{C}(4, 7, 16, 1/39)$	$9/4$	9	135	$\mathcal{C}(4, 7, 16, 1/39)$	$9/4$	9	135
6_2	$\mathcal{C}(3, 8, 19, 1/46)$	$-11/7$	7	126	$\mathcal{C}(3, 8, 19, 1/46)$	$-11/7$	7	126
	$\mathcal{C}(4, 5, 11, 0)$	$-11/8$	6	60	$\mathcal{C}(4, 5, 11, 0)$	$-11/8$	6	60
6_3	$\mathcal{C}(3, 7, 11, 0)$	$13/8$	6	60	$\mathcal{C}(3, 7, 11, 0)$	$13/8$	6	60
	$\mathcal{C}(4, 7, 36, 1/42)$	$13/8$	9	315	$\mathcal{C}(4, 9, 14, 1/29)$	$13/8$	12	156
7_1	$\mathcal{C}(3, 10, 11, 0)$	$7/6$	9	90	$\mathcal{C}(3, 10, 11, 0)$	$7/6$	9	90
	$\mathcal{C}(4, 7, 27, 1/68)$	$-7/6$	9	234	$\mathcal{C}(4, 9, 12, 1/18)$	$7/6$	12	132
7_2	$\mathcal{C}(3, 10, 27, 1/50)$	$11/6$	9	234	$\mathcal{C}(3, 10, 27, 1/50)$	$11/6$	9	234
	$\mathcal{C}(4, 7, 9, 1/30)$	$-11/6$	9	72	$\mathcal{C}(4, 7, 9, 1/30)$	$-11/6$	9	72
7_3	$\mathcal{C}(3, 10, 28, 1/47)$	$13/4$	9	243	$\mathcal{C}(3, 10, 28, 1/47)$	$13/4$	9	243
	$\mathcal{C}(4, 7, 27, 1/80)$	$-13/10$	9	234	$\mathcal{C}(4, 9, 15, 1/35)$	$-13/10$	12	168
7_4	$\mathcal{C}(3, 10, 36, 1/306)$	$-15/4$	9	315	$\mathcal{C}(3, 11, 27, 1/238)$	$-15/11$	10	260
	$\mathcal{C}(4, 9, 64, 1/156)$	$15/4$	12	756	$\mathcal{C}(4, 13, 21, 1/24)$	$15/4$	18	360
7_5	$\mathcal{C}(3, 10, 35, 1/60)$	$17/12$	9	306	$\mathcal{C}(3, 13, 14, 1/24)$	$-17/10$	12	156
	$\mathcal{C}(4, 7, 9, 0)$	$-17/10$	9	72	$\mathcal{C}(4, 7, 9, 0)$	$-17/10$	9	72
7_6	$\mathcal{C}(3, 10, 33, 1/46)$	$19/8$	9	288	$\mathcal{C}(3, 14, 15, 1/26)$	$-19/7$	13	182
	$\mathcal{C}(4, 7, 40, 1/51)$	$-19/8$	9	351	$\mathcal{C}(4, 9, 13, 5/44)$	$19/8$	12	144
7_7	$\mathcal{C}(3, 8, 13, 0)$	$21/13$	7	84	$\mathcal{C}(3, 8, 13, 0)$	$21/13$	7	84
	$\mathcal{C}(4, 9, 61, 1/67)$	$21/8$	12	720	$\mathcal{C}(4, 11, 16, 3/10)$	$21/34$	15	225
8_1	$\mathcal{C}(3, 11, 13, 1/60)$	$13/7$	10	120	$\mathcal{C}(3, 11, 13, 1/60)$	$13/7$	10	120
	$\mathcal{C}(4, 9, 118, 1/67)$	$-13/6$	12	1404	$\mathcal{C}(4, 15, 20, 2/15)$	$13/24$	21	399
8_2	$\mathcal{C}(3, 11, 28, 1/80)$	$17/11$	10	270	$\mathcal{C}(3, 11, 28, 1/80)$	$17/11$	10	270
	$\mathcal{C}(4, 9, 35, 1/68)$	$-17/14$	12	408	$\mathcal{C}(4, 13, 17, 1/6)$	$-17/6$	18	288
8_3	$\mathcal{C}(3, 11, 13, 0)$	$17/13$	10	120	$\mathcal{C}(3, 11, 13, 0)$	$17/13$	10	120
	$\mathcal{C}(4, 11, 101, 1/85)$	$17/4$	15	1500	$\mathcal{C}(4, 15, 17, 1/14)$	$17/4$	21	336
8_4	$\mathcal{C}(3, 11, 46, 1/58)$	$19/15$	10	450	$\mathcal{C}(3, 17, 22, 1/144)$	$19/81$	16	336
	$\mathcal{C}(4, 7, 25, 1/75)$	$19/14$	9	216	$\mathcal{C}(4, 11, 12, 3/14)$	$-19/14$	15	165
8_6	$\mathcal{C}(3, 11, 87, 1/70)$	$23/13$	10	860	$\mathcal{C}(3, 17, 22, 1/44)$	$-23/13$	16	336
	$\mathcal{C}(4, 9, 91, 1/66)$	$-23/16$	12	1080	$\mathcal{C}(4, 11, 21, 1/15)$	$23/16$	15	300
8_7	$\mathcal{C}(3, 10, 70, 1/47)$	$-23/18$	9	621	$\mathcal{C}(3, 13, 18, 1/42)$	$23/18$	12	204
	$\mathcal{C}(4, 7, 13, 0)$	$23/18$	9	108	$\mathcal{C}(4, 7, 13, 0)$	$23/18$	9	108
8_8	$\mathcal{C}(3, 10, 14, 1/60)$	$-25/16$	9	117	$\mathcal{C}(3, 10, 14, 1/60)$	$-25/16$	9	117
	$\mathcal{C}(4, 7, 24, 1/236)$	$25/14$	9	207	$\mathcal{C}(4, 7, 24, 1/236)$	$25/14$	9	207
8_9	$\mathcal{C}(3, 11, 110, 1/168)$	$25/7$	10	1090	$\mathcal{C}(3, 17, 25, 1/96)$	$-25/57$	16	384
	$\mathcal{C}(4, 7, 16, 1/54)$	$25/18$	9	135	$\mathcal{C}(4, 7, 16, 1/54)$	$25/18$	9	135
8_{11}	$\mathcal{C}(3, 11, 100, 1/84)$	$27/17$	10	990	$\mathcal{C}(3, 14, 38, 1/66)$	$-27/17$	13	481
	$\mathcal{C}(4, 9, 30, 1/82)$	$-27/8$	12	348	$\mathcal{C}(4, 11, 13, 2/25)$	$27/8$	15	180
8_{12}	$\mathcal{C}(3, 11, 54, 1/152)$	$29/17$	10	530	$\mathcal{C}(3, 14, 41, 1/19)$	$29/17$	13	520
	$\mathcal{C}(4, 9, 103, 1/68)$	$-29/12$	12	1224	$\mathcal{C}(4, 15, 18, 4/29)$	$29/12$	21	357

Chebyshev parametrizations of the first rational knots

K	minimal b	α/β	DP	deg	min. $(b-1)(c-1)$	α/β	DP	deg
8_{13}	$\mathcal{C}(3, 10, 17, 1/62)$ $\mathcal{C}(4, 9, 104, 1/66)$	−29/8 −29/18	9 12	144 1236	$\mathcal{C}(3, 10, 17, 1/62)$ $\mathcal{C}(4, 13, 18, 17/45)$	−29/8 29/8	9 18	144 306
8_{14}	$\mathcal{C}(3, 11, 93, 1/86)$ $\mathcal{C}(4, 9, 47, 1/200)$	31/13 31/44	10 12	920 552	$\mathcal{C}(3, 17, 22, 1/26)$ $\mathcal{C}(4, 13, 18, 1/20)$	−31/19 31/18	16 18	336 306
9_1	$\mathcal{C}(3, 13, 14, 0)$ $\mathcal{C}(4, 9, 87, 1/59)$	−9/8 9/8	12 12	156 1032	$\mathcal{C}(3, 13, 14, 0)$ $\mathcal{C}(4, 13, 14, 3/43)$	−9/8 −9/8	12 18	156 234
9_2	$\mathcal{C}(3, 13, 37, 1/114)$ $\mathcal{C}(4, 9, 325, 1/380)$	−15/2 −15/8	12 12	432 3888	$\mathcal{C}(3, 19, 24, 3/82)$ $\mathcal{C}(4, 17, 20, 7/23)$	15/8 15/2	18 24	414 456
9_3	$\mathcal{C}(3, 13, 143, 1/98)$ $\mathcal{C}(4, 9, 159, 1/108)$	−19/6 19/16	12 12	1704 1896	$\mathcal{C}(3, 16, 46, 1/54)$ $\mathcal{C}(4, 17, 18, 1/23)$	19/6 −19/44	15 24	675 408
9_4	$\mathcal{C}(3, 13, 115, 1/164)$ $\mathcal{C}(4, 9, 106, 3/301)$	−21/4 21/16	12 12	1368 1260	$\mathcal{C}(3, 19, 24, 1/40)$ $\mathcal{C}(4, 17, 18, 3/17)$	21/16 21/4	18 24	414 408
9_5	$\mathcal{C}(3, 13, 326, 1/85)$ $\mathcal{C}(4, 11, 152, 1/44)$	23/4 −23/6	12 15	3900 2265	$\mathcal{C}(3, 17, 45, 1/182)$ $\mathcal{C}(4, 19, 22, 1/20)$	23/29 −23/6	16 27	704 567
9_6	$\mathcal{C}(3, 13, 64, 1/102)$ $\mathcal{C}(4, 9, 11, 1/55)$	−27/22 27/16	12 12	756 120	$\mathcal{C}(3, 16, 17, 1/34)$ $\mathcal{C}(4, 9, 11, 1/55)$	27/16 27/16	15 12	240 120
9_7	$\mathcal{C}(3, 13, 116, 1/80)$ $\mathcal{C}(4, 9, 201, 1/131)$	−29/20 29/16	12 12	1380 2400	$\mathcal{C}(3, 22, 24, 3/26)$ $\mathcal{C}(4, 11, 17, 1/65)$	−29/16 29/16	21 15	483 240
9_8	$\mathcal{C}(3, 13, 121, 1/170)$ $\mathcal{C}(4, 11, 187, 1/147)$	−31/20 −31/14	12 15	1440 2790	$\mathcal{C}(3, 16, 25, 3/97)$ $\mathcal{C}(4, 17, 20, 6/19)$	31/14 31/14	15 24	360 456
9_9	$\mathcal{C}(3, 13, 123, 1/80)$ $\mathcal{C}(4, 9, 31, 1/66)$	−31/22 31/22	12 12	1464 360	$\mathcal{C}(3, 19, 52, 2/29)$ $\mathcal{C}(4, 9, 31, 1/66)$	−31/24 31/22	18 12	918 360
9_{10}	$\mathcal{C}(3, 13, 246, 1/110)$ $\mathcal{C}(4, 11, 29, 1/45)$	−33/10 −33/10	12 15	2940 420	$\mathcal{C}(3, 19, 53, 1/32)$ $\mathcal{C}(4, 11, 29, 1/45)$	33/10 −33/10	18 15	936 420
9_{11}	$\mathcal{C}(3, 13, 114, 1/106)$ $\mathcal{C}(4, 9, 33, 1/75)$	−33/26 33/26	12 12	1356 384	$\mathcal{C}(3, 23, 24, 1/10)$ $\mathcal{C}(4, 9, 33, 1/75)$	−33/19 33/26	22 12	506 384
9_{12}	$\mathcal{C}(3, 13, 36, 1/80)$ $\mathcal{C}(4, 11, 68, 1/77)$	−35/22 −35/22	12 15	420 1005	$\mathcal{C}(3, 13, 36, 1/80)$ $\mathcal{C}(4, 15, 24, 1/17)$	−35/22 35/22	12 21	420 483
9_{13}	$\mathcal{C}(3, 13, 53, 1/78)$ $\mathcal{C}(4, 9, 41, 1/91)$	−37/26 37/26	12 12	624 480	$\mathcal{C}(3, 16, 31, 1/42)$ $\mathcal{C}(4, 17, 18, 1/6)$	37/10 37/26	15 24	450 408
9_{14}	$\mathcal{C}(3, 11, 83, 1/74)$ $\mathcal{C}(4, 11, 176, 1/108)$	−37/23 −37/14	10 15	820 2625	$\mathcal{C}(3, 17, 18, 1/9)$ $\mathcal{C}(4, 15, 22, 4/11)$	−37/29 −37/8	16 21	272 441
9_{15}	$\mathcal{C}(3, 13, 144, 1/310)$ $\mathcal{C}(4, 9, 39, 1/85)$	39/22 39/22	12 12	1716 456	$\mathcal{C}(3, 17, 18, 1/34)$ $\mathcal{C}(4, 13, 15, 7/45)$	39/17 −39/16	16 18	272 252
9_{17}	$\mathcal{C}(3, 11, 16, 0)$ $\mathcal{C}(4, 9, 92, 1/92)$	−39/25 39/14	10 12	150 1092	$\mathcal{C}(3, 11, 16, 0)$ $\mathcal{C}(4, 11, 24, 1/53)$	−39/25 −39/14	10 15	150 345
9_{18}	$\mathcal{C}(3, 13, 194, 1/144)$ $\mathcal{C}(4, 9, 11, 0)$	−41/12 41/24	12 12	2316 120	$\mathcal{C}(3, 16, 43, 1/36)$ $\mathcal{C}(4, 9, 11, 0)$	−41/24 41/24	15 12	630 120
9_{19}	$\mathcal{C}(3, 11, 83, 1/82)$ $\mathcal{C}(4, 9, 32, 1/416)$	−41/25 −41/64	10 12	820 372	$\mathcal{C}(3, 14, 22, 1/110)$ $\mathcal{C}(4, 9, 32, 1/416)$	41/105 −41/64	13 12	273 372
9_{20}	$\mathcal{C}(3, 13, 275, 1/86)$ $\mathcal{C}(4, 7, 17, 0)$	−41/26 −41/30	12 9	3288 144	$\mathcal{C}(3, 16, 46, 1/34)$ $\mathcal{C}(4, 7, 17, 0)$	41/26 −41/30	15 9	675 144

Chebyshev parametrizations of the first rational knots

K	minimal b	α/β	DP	deg	min. $(b-1)(c-1)$	α/β	DP	deg
9_{21}	$\mathcal{C}(3, 13, 179, 1/106)$	−43/18	12	2136	$\mathcal{C}(3, 16, 29, 9/172)$	−43/18	15	420
	$\mathcal{C}(4, 9, 94, 1/105)$	−43/68	12	1116	$\mathcal{C}(4, 11, 23, 1/23)$	−43/18	15	330
9_{23}	$\mathcal{C}(3, 13, 44, 1/98)$	−45/26	12	516	$\mathcal{C}(3, 13, 44, 1/98)$	−45/26	12	516
	$\mathcal{C}(4, 11, 370, 1/118)$	45/26	15	5535	$\mathcal{C}(4, 13, 16, 1/20)$	45/26	18	270
9_{26}	$\mathcal{C}(3, 11, 25, 1/92)$	−47/29	10	240	$\mathcal{C}(3, 11, 25, 1/92)$	−47/29	10	240
	$\mathcal{C}(4, 9, 184, 1/79)$	47/34	12	2196	$\mathcal{C}(4, 13, 17, 1/25)$	−47/18	18	288
9_{27}	$\mathcal{C}(3, 13, 180, 1/84)$	−49/18	12	2148	$\mathcal{C}(3, 17, 51, 1/30)$	49/31	16	800
	$\mathcal{C}(4, 9, 39, 1/66)$	49/30	12	456	$\mathcal{C}(4, 15, 20, 1/28)$	−49/80	21	399
9_{31}	$\mathcal{C}(3, 10, 17, 0)$	55/34	9	144	$\mathcal{C}(3, 10, 17, 0)$	55/34	9	144
	$\mathcal{C}(4, 11, 68, 1/100)$	−55/34	15	1005	$\mathcal{C}(4, 13, 24, 1/43)$	55/34	18	414
10_{1}	$\mathcal{C}(3, 14, 38, 1/102)$	−17/9	13	481	$\mathcal{C}(3, 17, 19, 1/44)$	17/9	16	288
	$\mathcal{C}(4, 11, 141, 1/44)$	17/8	15	2100	$\mathcal{C}(4, 19, 23, 1/33)$	−17/8	27	594
10_{2}	$\mathcal{C}(3, 14, 43, 1/176)$	−23/3	13	546	$\mathcal{C}(3, 20, 25, 9/314)$	23/15	19	456
	$\mathcal{C}(4, 9, 278, 1/85)$	−23/20	12	3324	$\mathcal{C}(4, 17, 20, 4/61)$	23/38	24	456
10_{3}	$\mathcal{C}(3, 14, 16, 1/94)$	−25/19	13	195	$\mathcal{C}(3, 14, 16, 1/94)$	−25/19	13	195
	$\mathcal{C}(4, 13, 348, 1/138)$	−25/4	18	6246	$\mathcal{C}(4, 19, 20, 2/17)$	−25/6	27	513
10_{4}	$\mathcal{C}(3, 14, 101, 1/130)$	−27/23	13	1300	$\mathcal{C}(3, 23, 26, 1/38)$	27/23	22	550
	$\mathcal{C}(4, 9, 257, 1/145)$	−27/20	12	3072	$\mathcal{C}(4, 11, 34, 2/17)$	−27/20	15	495
10_{5}	$\mathcal{C}(3, 13, 169, 9/1034)$	33/28	12	2016	$\mathcal{C}(3, 19, 24, 1/116)$	33/28	18	414
	$\mathcal{C}(4, 11, 194, 1/110)$	33/28	15	2895	$\mathcal{C}(4, 13, 20, 2/35)$	−33/28	18	342
10_{6}	$\mathcal{C}(3, 14, 128, 1/92)$	−37/7	13	1651	$\mathcal{C}(3, 20, 25, 1/42)$	37/21	19	456
	$\mathcal{C}(4, 11, 488, 5/499)$	−37/16	15	7305	$\mathcal{C}(4, 17, 20, 1/15)$	37/58	24	456
10_{7}	$\mathcal{C}(3, 14, 127, 1/128)$	−43/27	13	1638	$\mathcal{C}(3, 23, 33, 1/24)$	43/27	22	704
	$\mathcal{C}(4, 11, 229, 1/70)$	43/16	15	3420	$\mathcal{C}(4, 19, 21, 1/17)$	43/8	27	540
10_{8}	$\mathcal{C}(3, 14, 37, 1/144)$	−29/23	13	468	$\mathcal{C}(3, 14, 37, 1/144)$	−29/23	13	468
	$\mathcal{C}(4, 11, 77, 1/66)$	29/24	15	1140	$\mathcal{C}(4, 13, 36, 2/45)$	−29/24	18	630
10_{9}	$\mathcal{C}(3, 14, 281, 1/232)$	−39/11	13	3640	$\mathcal{C}(3, 17, 43, 1/186)$	39/89	16	672
	$\mathcal{C}(4, 9, 35, 1/133)$	−39/28	12	408	$\mathcal{C}(4, 13, 17, 17/120)$	−39/28	18	288
10_{10}	$\mathcal{C}(3, 13, 253, 1/250)$	45/28	12	3024	$\mathcal{C}(3, 19, 31, 1/31)$	−45/118	18	540
	$\mathcal{C}(4, 11, 102, 1/45)$	45/28	15	1515	$\mathcal{C}(4, 15, 20, 1/10)$	−45/28	21	399
10_{11}	$\mathcal{C}(3, 14, 101, 1/116)$	−43/33	13	1300	$\mathcal{C}(3, 20, 25, 1/196)$	−43/185	19	456
	$\mathcal{C}(4, 11, 126, 1/97)$	43/30	15	1875	$\mathcal{C}(4, 21, 24, 2/31)$	43/76	30	690
10_{12}	$\mathcal{C}(3, 13, 61, 1/178)$	47/36	12	720	$\mathcal{C}(3, 19, 24, 1/44)$	47/36	18	414
	$\mathcal{C}(4, 11, 115, 1/393)$	47/58	15	1710	$\mathcal{C}(4, 13, 20, 1/18)$	−47/36	18	342
10_{13}	$\mathcal{C}(3, 14, 211, 3/322)$	−53/41	13	2730	$\mathcal{C}(3, 23, 26, 1/26)$	53/31	22	550
	$\mathcal{C}(4, 11, 147, 1/84)$	53/22	15	2190	$\mathcal{C}(4, 15, 18, 3/22)$	53/22	21	357
10_{14}	$\mathcal{C}(3, 14, 139, 1/180)$	−57/13	13	1794	$\mathcal{C}(3, 17, 44, 2/43)$	−57/13	16	688
	$\mathcal{C}(4, 9, 35, 2/135)$	−57/44	12	408	$\mathcal{C}(4, 9, 35, 2/135)$	−57/44	12	408
10_{15}	$\mathcal{C}(3, 13, 17, 1/80)$	43/24	12	192	$\mathcal{C}(3, 13, 17, 1/80)$	43/24	12	192
	$\mathcal{C}(4, 9, 105, 1/40)$	−43/24	12	1248	$\mathcal{C}(4, 17, 27, 2/35)$	−43/24	24	624
10_{16}	$\mathcal{C}(3, 14, 127, 1/104)$	−47/37	13	1638	$\mathcal{C}(3, 23, 25, 1/9)$	−47/37	22	528
	$\mathcal{C}(4, 11, 37, 1/57)$	47/14	15	540	$\mathcal{C}(4, 19, 21, 1/20)$	47/14	27	540

Chebyshev parametrizations of the first rational knots

K	minimal b	α/β	DP	deg	min. $(b-1)(c-1)$	α/β	DP	deg
10_{17}	$\mathcal{C}(3, 13, 194, 1/79)$	41/32	12	2316	$\mathcal{C}(3, 19, 24, 1/220)$	41/32	18	414
	$\mathcal{C}(4, 9, 16, 1/69)$	−41/32	12	180	$\mathcal{C}(4, 9, 16, 1/69)$	−41/32	12	180
10_{18}	$\mathcal{C}(3, 14, 37, 1/148)$	−55/43	13	468	$\mathcal{C}(3, 14, 37, 1/148)$	−55/43	13	468
	$\mathcal{C}(4, 11, 211, 1/52)$	55/32	15	3150	$\mathcal{C}(4, 19, 21, 2/11)$	−55/32	27	540
10_{19}	$\mathcal{C}(3, 13, 128, 1/158)$	51/14	12	1524	$\mathcal{C}(3, 22, 33, 1/48)$	51/142	21	672
	$\mathcal{C}(4, 9, 162, 1/181)$	−51/40	12	1932	$\mathcal{C}(4, 13, 23, 1/148)$	51/40	18	396
10_{20}	$\mathcal{C}(3, 14, 292, 1/94)$	−35/11	13	3783	$\mathcal{C}(3, 23, 29, 1/22)$	−35/19	22	616
	$\mathcal{C}(4, 11, 298, 1/133)$	35/16	15	4455	$\mathcal{C}(4, 19, 28, 1/12)$	−35/54	27	729
10_{21}	$\mathcal{C}(3, 14, 133, 1/108)$	−45/31	13	1716	$\mathcal{C}(3, 20, 46, 5/72)$	−45/29	19	855
	$\mathcal{C}(4, 11, 193, 1/60)$	45/16	15	2880	$\mathcal{C}(4, 19, 23, 1/36)$	−45/16	27	594
10_{22}	$\mathcal{C}(3, 14, 230, 1/554)$	49/15	13	2977	$\mathcal{C}(3, 20, 27, 1/126)$	−49/15	19	494
	$\mathcal{C}(4, 9, 96, 1/52)$	−49/36	12	1140	$\mathcal{C}(4, 11, 25, 2/31)$	−49/36	15	360
10_{23}	$\mathcal{C}(3, 13, 124, 1/362)$	−59/18	12	1476	$\mathcal{C}(3, 19, 26, 1/64)$	59/18	18	450
	$\mathcal{C}(4, 11, 38, 1/105)$	59/18	15	555	$\mathcal{C}(4, 19, 20, 2/13)$	−59/36	27	513
10_{24}	$\mathcal{C}(3, 14, 127, 3/319)$	−55/31	13	1638	$\mathcal{C}(3, 23, 36, 5/82)$	55/31	22	770
	$\mathcal{C}(4, 11, 247, 1/74)$	55/24	15	3690	$\mathcal{C}(4, 19, 23, 1/37)$	−55/24	27	594
10_{25}	$\mathcal{C}(3, 14, 148, 1/108)$	−65/41	13	1911	$\mathcal{C}(3, 17, 64, 1/46)$	−65/19	16	1008
	$\mathcal{C}(4, 9, 116, 2/135)$	−65/46	12	1380	$\mathcal{C}(4, 15, 20, 1/43)$	−65/106	21	399
10_{26}	$\mathcal{C}(3, 14, 110, 1/98)$	−61/17	13	1417	$\mathcal{C}(3, 17, 67, 1/86)$	−61/17	16	1056
	$\mathcal{C}(4, 9, 35, 1/77)$	−61/44	12	408	$\mathcal{C}(4, 9, 35, 1/77)$	−61/44	12	408
10_{27}	$\mathcal{C}(3, 13, 126, 1/218)$	71/50	12	1500	$\mathcal{C}(3, 22, 36, 1/50)$	71/44	21	735
	$\mathcal{C}(4, 11, 278, 1/115)$	71/44	15	4155	$\mathcal{C}(4, 15, 26, 2/71)$	−71/44	21	525
10_{28}	$\mathcal{C}(3, 13, 191, 1/112)$	53/14	12	2280	$\mathcal{C}(3, 19, 25, 5/138)$	53/34	18	432
	$\mathcal{C}(4, 11, 114, 1/139)$	−53/92	15	1695	$\mathcal{C}(4, 15, 38, 1/38)$	−53/92	21	777
10_{29}	$\mathcal{C}(3, 14, 292, 1/93)$	−63/17	13	3783	$\mathcal{C}(3, 17, 45, 1/42)$	−63/37	16	704
	$\mathcal{C}(4, 9, 168, 1/106)$	−63/46	12	2004	$\mathcal{C}(4, 15, 23, 1/8)$	−63/46	21	462
10_{30}	$\mathcal{C}(3, 14, 201, 1/96)$	−67/41	13	2600	$\mathcal{C}(3, 17, 39, 1/34)$	−67/49	16	608
	$\mathcal{C}(4, 13, 306, 1/738)$	−67/18	18	5490	$\mathcal{C}(4, 21, 25, 3/77)$	−67/18	30	720
10_{31}	$\mathcal{C}(3, 13, 103, 1/80)$	57/32	12	1224	$\mathcal{C}(3, 19, 27, 1/32)$	−57/32	18	468
	$\mathcal{C}(4, 9, 111, 1/66)$	−57/32	12	1320	$\mathcal{C}(4, 15, 20, 1/11)$	−57/32	21	399
10_{32}	$\mathcal{C}(3, 14, 148, 1/172)$	−69/19	13	1911	$\mathcal{C}(3, 16, 56, 1/166)$	−69/50	15	825
	$\mathcal{C}(4, 11, 134, 1/103)$	69/50	15	1995	$\mathcal{C}(4, 15, 22, 1/34)$	69/40	21	441
10_{33}	$\mathcal{C}(3, 13, 182, 1/105)$	65/18	12	2172	$\mathcal{C}(3, 22, 40, 1/38)$	65/148	21	819
	$\mathcal{C}(4, 13, 856, 1/328)$	65/18	18	15390	$\mathcal{C}(4, 25, 30, 5/17)$	−65/18	36	1044
10_{34}	$\mathcal{C}(3, 13, 41, 1/90)$	37/20	12	480	$\mathcal{C}(3, 16, 20, 1/44)$	−37/24	15	285
	$\mathcal{C}(4, 11, 142, 1/122)$	37/20	15	2115	$\mathcal{C}(4, 13, 19, 1/11)$	37/20	18	324
10_{35}	$\mathcal{C}(3, 14, 38, 1/108)$	−49/27	13	481	$\mathcal{C}(3, 14, 38, 1/108)$	−49/27	13	481
	$\mathcal{C}(4, 13, 273, 3/697)$	−49/20	18	4896	$\mathcal{C}(4, 19, 24, 1/9)$	49/78	27	621
10_{36}	$\mathcal{C}(3, 14, 385, 1/146)$	51/20	13	4992	$\mathcal{C}(3, 20, 32, 1/44)$	51/31	19	589
	$\mathcal{C}(4, 9, 179, 1/222)$	−51/28	12	2136	$\mathcal{C}(4, 11, 17, 1/77)$	51/28	15	240
10_{37}	$\mathcal{C}(3, 13, 17, 0)$	53/30	12	192	$\mathcal{C}(3, 13, 17, 0)$	53/30	12	192
	$\mathcal{C}(4, 11, 468, 1/103)$	−53/30	15	7020	$\mathcal{C}(4, 13, 19, 5/59)$	53/30	18	324

Chebyshev parametrizations of the first rational knots

K	minimal b	α/β	DP	deg	min. $(b-1)(c-1)$	α/β	DP	deg
10_{38}	$\mathcal{C}(3, 14, 120, 1/134)$	$-59/25$	13	1547	$\mathcal{C}(3, 17, 23, 5/122)$	$-59/33$	16	352
	$\mathcal{C}(4, 11, 369, 1/109)$	$59/26$	15	5520	$\mathcal{C}(4, 23, 27, 2/19)$	$59/34$	33	858
10_{39}	$\mathcal{C}(3, 14, 373, 1/182)$	$61/22$	13	4836	$\mathcal{C}(3, 20, 25, 1/34)$	$61/39$	19	456
	$\mathcal{C}(4, 9, 277, 1/119)$	$-61/36$	12	3312	$\mathcal{C}(4, 11, 39, 1/18)$	$61/36$	15	570
10_{40}	$\mathcal{C}(3, 13, 190, 1/112)$	$75/44$	12	2268	$\mathcal{C}(3, 16, 34, 1/21)$	$75/46$	15	495
	$\mathcal{C}(4, 11, 57, 1/291)$	$-75/106$	15	840	$\mathcal{C}(4, 17, 23, 2/25)$	$75/44$	24	528
10_{41}	$\mathcal{C}(3, 14, 208, 1/110)$	$-71/41$	13	2691	$\mathcal{C}(3, 17, 55, 1/23)$	$-71/41$	16	864
	$\mathcal{C}(4, 9, 165, 2/141)$	$71/112$	12	1968	$\mathcal{C}(4, 15, 25, 6/47)$	$-71/30$	21	504
10_{42}	$\mathcal{C}(3, 13, 134, 1/166)$	$81/50$	12	1596	$\mathcal{C}(3, 16, 25, 1/22)$	$81/50$	15	360
	$\mathcal{C}(4, 11, 131, 1/132)$	$81/34$	15	1950	$\mathcal{C}(4, 13, 32, 5/47)$	$-81/34$	18	558
10_{43}	$\mathcal{C}(3, 13, 174, 1/114)$	$73/46$	12	2076	$\mathcal{C}(3, 16, 51, 15/976)$	$-73/46$	15	750
	$\mathcal{C}(4, 11, 277, 1/134)$	$73/46$	15	4140	$\mathcal{C}(4, 13, 36, 1/48)$	$-73/100$	18	630
10_{44}	$\mathcal{C}(3, 14, 132, 1/114)$	$-79/49$	13	1703	$\mathcal{C}(3, 20, 53, 5/148)$	$79/29$	19	988
	$\mathcal{C}(4, 11, 64, 1/119)$	$79/30$	15	945	$\mathcal{C}(4, 15, 24, 3/13)$	$79/50$	21	483
10_{45}	$\mathcal{C}(3, 11, 19, 0)$	$89/55$	10	180	$\mathcal{C}(3, 11, 19, 0)$	$89/55$	10	180
	$\mathcal{C}(4, 13, 132, 1/238)$	$-89/34$	18	2358	$\mathcal{C}(4, 15, 26, 2/21)$	$-89/144$	21	525

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