An Algebraic Method to Solve the Minimal Partial Realization Problem for Scalar Sequences

M. Van Barel and A. Bultheel
Department of Computer Science
Katholieke Universiteit Leuven
Celestijnenlaan 200A
B-3030 Leuven (Heverlee), Belgium

Submitted by Richard A. Brualdi

ABSTRACT

We give an algebraic method to construct all minimal partial realizations of a finite sequence of scalar numbers. We use the minimal number of parameters to characterize all these minimal partial realizations. Three examples are worked out to illustrate the method.

1. INTRODUCTION

It is a well-known result that the Hankel matrix $H_{\infty,\infty} = [h_{p+q+1}]_{p, q=0}^{\infty}$ has rank $n$ iff $h(z) = h_1 z^{-1} + h_2 z^{-2} + \cdots$ is a proper rational function of degree $n$. This result is used in many places and goes back to Kronecker [5]. See also Gantmacher [2]. When this result is given a matrix interpretation, it gives a decomposition of the infinite Hankel matrix $H_{\infty,\infty}$ into the product of a matrix $C \in \mathbb{C}^{\infty \times n}$ and a Hankel matrix $H_{n,\infty} \in \mathbb{C}^{n \times \infty}$ which consists of the first $n$ rows of $H_{\infty,\infty}$. Such a decomposition is found in the work of Pták [6]. $C$ is called there an infinite companion matrix for the denominator of $h(z)$.

In this paper we give a simple (matrix) proof of this result and give some properties of the infinite companion matrix. We shall use this result to give an algebraic method to compute the rational function based on this decomposition. It can also be used to compute all solutions of the minimal partial realization problem [3,4] with a minimal number of parameters. This application of the decomposition seems to be new.


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2. THE MINIMAL PARTIAL REALIZATION PROBLEM

Given a sequence of complex numbers \( h_1, h_2, \ldots \), we call the rational function \( \frac{g(z)}{f(z)} \) a realization of this sequence iff

\[
\frac{g(z)}{f(z)} = h_1 z^{-1} + h_2 z^{-2} + \cdots, \quad z \to \infty.
\]

Note that the degree of \( g(z) \) is less than the degree of \( f(z) \). If \( N \) is finite and

\[
\frac{g(z)}{f(z)} = h_1 z^{-1} + h_2 z^{-2} + \cdots + h_N z^{-N} + O(z^{-N-1}), \quad z \to \infty,
\]

then \( \frac{g(z)}{f(z)} \) is called a partial realization of order \( N \). The degree \( n \) of the denominator \( f(z) \) when \( f(z) \) and \( g(z) \) are coprime is called the degree of the (partial) realization.

A (partial) realization is called minimal if there is no other (partial) realization of lower degree. If we set \( f(z) = f_0 + f_1 z + \cdots + f_n z^n (f_n \neq 0) \) and \( g(z) = g_0 + g_1 z + \cdots + g_{n-1} z^{n-1} \), then (2) can be written in matrix notation as

\[
\begin{bmatrix}
0 & 0 & \ldots & h_1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & h_1 & \ldots & h_n \\
h_1 & h_2 & \ldots & h_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
h_{N-n} & h_{N-n+1} & \ldots & h_N
\end{bmatrix}
\begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
f_n
\end{bmatrix}
= \begin{bmatrix}
g_{n-1} \\
g_0 \\
g_n
\end{bmatrix}, \quad f_n \neq 0.
\]

The minimal partial realization problem, as it is stated here, is closely connected to Padé approximation (see [1]).

It is clear that if \( g(z) \) and \( f(z) \) are as before, and \( h = \{h_i\}_{i=1}^\infty \), then the following two statements are equivalent:

1. \( \frac{g(z)}{f(z)} \) is a realization of the sequence \( h \).
2. \( h \) is a solution of the difference equation

\[
f_0 h_r + f_1 h_{r+1} + \cdots + f_n h_{r+n} = 0, \quad r = 1, 2, \ldots
\]
with initial conditions $h_1, \ldots, h_n$, satisfying

$$
\begin{bmatrix}
  f_n & 0 & \ldots & 0 \\
  f_{n-1} & f_n & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  f_1 & f_2 & \ldots & f_{n-1}
\end{bmatrix}
\begin{bmatrix}
  h_1 \\
  h_2 \\
  \vdots \\
  h_n
\end{bmatrix}
= 
\begin{bmatrix}
  g_{n-1} \\
  g_{n-2} \\
  \vdots \\
  g_0
\end{bmatrix}.
\tag{4b}
$$

In the following we shall represent the set of solutions for (4a) by $R(f)$.

3. THE INFINITE COMPANION MATRIX

From now on $h$ will denote a column matrix in $\mathbb{C}^{\infty \times 1}$ with entries $h_1, h_2, \ldots$. $S$ will denote the shift matrix $S_{ij} = \delta_{i+1, j}$, so that

$$Sh = \begin{bmatrix} h_2 & h_3 & \ldots \end{bmatrix}^T.$$

We associate with $h$ the Hankel matrices

$$H_{p,q} = \begin{bmatrix} h_1 & \ldots & h_q \\
  \vdots & & \vdots \\
  h_p & \ldots & h_{p+q-1} \end{bmatrix}, \quad p, q = 1, 2, \ldots.$$

So $h = H^{T}_{1,1} = H^{T}_{1,\infty}$ and $H_{\infty,n} = [h \, Sh \, \ldots \, S^{n-1}h] = H^{T}_{n,\infty}$. Now let $h \in R(f)$, $f_n \neq 0$ (take $f_n = 1$ for simplicity). Then also $Sh \in R(f)$, and by induction $S^kh \in R(f)$ for all $k \geq 0$. Hence, all the columns of $H_{\infty,\infty}$ are in $R(f)$. Now $R(f)$ is $n$ dimensional, and a simple choice for a basis is given by the $n$ solutions $C^{(i)} \in R(f)$, $i = 1, 2, \ldots, n$, which take as initial conditions

$$C^{(i)}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n.$$

All these solutions can be arranged in a matrix

$$C_{\infty,n} = \begin{bmatrix} C^{(1)} & C^{(2)} & \ldots & C^{(n)} \end{bmatrix}.$$

Note that if $h \in R(f)$ with initial conditions $[h_1, \ldots, h_m] = m^T$, then $h = C_{\infty,n}m$. Thus, since the initial conditions of the columns of $H_{\infty,\infty}$ are given
by $H_{n, \infty}$, we have

$$H_{\infty, \infty} = C_{\infty, n} H_{n, \infty}.$$  \hfill (5)

Given a monic polynomial $f(z) = f_0 + f_1 z + \cdots + f_{n-1} z^{n-1} + z^n$, its companion matrix is given by

$$C(f) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ -f_0 & -f_1 & \cdots & \cdots & -f_{n-1} \end{bmatrix} \in \mathbb{C}^{n \times n}.$$  

$C_{\infty, n}$ is called the infinite companion, since it is an extension of $C(f)$. It can be verified that $C_{\infty, n}$ has the form

$$C_{\infty, n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ C(f) & \cdot & \cdots & \cdot \end{bmatrix} \in \mathbb{C}^{\infty \times n}.$$  

It is completely defined in terms of $f$. The top $n$ rows form the $n \times n$ unit matrix (the initial conditions for the $C^{(i)}$), and the other elements can be generated columnwise from the recursion (4). There is also a rowwise recursion for the elements of $C_{\infty, n}$ which can be found as follows: Let $c_{k,*}$ represent the $k$th row of $C_{\infty, n}$; then (5) says that for $k \geq n + 1$

$$(S^{k-1}h)^T = c_{k,*} H_{n, \infty} = c_{k,1} h^T + c_{k,2} (Sh)^T + \cdots + c_{k,n} (S^{n-1}h)^T$$

and

$$(S^k h)^T = c_{k+1,1} h^T + c_{k+1,2} (Sh)^T + \cdots + c_{k+1,n} (S^{n-1}h)^T = c_{k,1} (Sh)^T + c_{k,2} (S^2 h)^T + \cdots + c_{k,n} (S^n h)^T$$

$$= c_{k,1} (Sh)^T + c_{k,2} (S^2 h)^T + \cdots + c_{k,n} (S^n h)^T$$

$$+ c_{k,n} \left[ c_{n+1,1} h^T + c_{n+1,2} (Sh)^T + \cdots + c_{n+1,n} (S^{n-1}h)^T \right].$$

So, if we set $c_{k,0} = 0 \forall k$, then

$$c_{k+1,j} = c_{n+1,j} c_{k,n} + c_{k,j-1}, \quad j = 1, 2, \ldots, n, \quad k = n + 1, n + 2, \ldots$$
(recall that \( c_{n+1,j} = -f_{j-1} \)), or

\[ c_{k+1} = c_k C(f) \]

If \( g(z)/f(z) \) is a minimal realization of \( h \), then \( h \) will satisfy a difference equation like (4) and not of lower order. Hence, the columns of \( H_{\infty,n} \), which are solutions, cannot be linearly dependent. Since different solutions of the same difference equation are linearly independent iff their initial conditions are linearly independent, \( H_{\infty,n} \) must have linearly independent columns. Thus \( H_{\infty,n} \) is invertible and we find from (5) another characterization of the infinite companion matrix:

\[ C_{\infty,n} = H_{\infty,n} H_{n,n}^{-1}. \]

The \((n+1)\)th row of this relation can also be formulated as

\[
H_{n,n} \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} = - \begin{bmatrix} h_{n+1} \\ \vdots \\ h_{2n} \end{bmatrix},
\]

which is a well-known relation used e.g. to find the denominator of a Padé approximant of \( h(z) = \sum_k h_k z^{-k} \) [1].

Yet another characterization of the infinite companion matrix can be given in terms of the poles of \( g(z)/f(z) \), which we suppose to be simple. Let

\[
\frac{g(z)}{f(z)} = \sum_{k=1}^{\infty} h_k z^{-k} = \sum_{l=1}^{n} \frac{w_l}{z - \lambda_l}.
\]

Then a simple computation yields

\[ h_k = \sum_{l=1}^{n} w_l \lambda_l^{k-1}. \]

Hence

\[ h = V_{\infty,n} W \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \]
with \( W = \text{diag}(w_1, \ldots, w_n) \) and \( V_{\infty, n} \) the infinite Vandermonde matrix
\[
(V_{\infty, n})_{ij} = [\lambda_j^{i-1}]_{i=1,2, \ldots}.
\]

It then simply follows that
\[
H_{\infty, n} = \begin{bmatrix} h & Sh & \cdots & S^{n-1}h \end{bmatrix} = V_{\infty, n}WV_{\infty, n}^T,
\]
(7)

Where \( V_{n, n} \) consists of the first \( n \) rows of \( V_{\infty, n} \). When (7) is restricted to the first \( n \) rows, we get
\[
H_{n, n} = V_{n, n}WV_{n, n}^T,
\]
(8)

so that from (6), (7), and (8) we get
\[
C_{\infty, n} = H_{\infty, n}H_{n, n}^{-1} = V_{\infty, n}V_{n, n}^{-1}.
\]

As a special case we have
\[
C(f) = V_{n, n}A V_{n, n}^{-1},
\]

with \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \); or more generally, \( n \) consecutive rows of \( C_{\infty, n} \) starting from the \( r + 1 \)th row are given by
\[
V_{n, n} A^r V_{n, n}^{-1} = C(f)^r,
\]

a property which is also given by Pták [6].

4. FUNDAMENTAL THEOREM

In the previous section, we rediscovered in fact a part of Kronecker's result, viz.: If
\[
\frac{g(z)}{f(z)} = \sum_{k=1}^{\infty} h_k z^{-k}, \quad z \to \infty,
\]
is a rational function with \( f(z) \) of (minimal) degree \( n \), then

\[
H_{\infty, \infty} = \begin{bmatrix} h & Sh & S^2h & \cdots \end{bmatrix}
\]

will be of finite rank \( n \). This was expressed by the decomposition \( H_{\infty, \infty} = C_{\infty, n} H_{n, \infty} \) with \( H_{n, n} \) nonsingular. This expression is somewhat explicit in that it relates the denominator coefficients (the \( n + 1 \)st row of \( C_{\infty, n} \)) with \( H_{\infty, \infty} \). The numerator can be easily found from (3). Kronecker's result includes also the converse, which is formulated in the next theorem.

**Theorem.** Let \( h = [h_1, h_2, \ldots]^T \in \mathbb{C}^{\infty \times 1} \).

**A.** The following two statements are equivalent:

1. \( h \) has a minimal realization of degree \( n \);  
2. \( H_{\infty, \infty} = [h \ Sh \ S^2h \ \cdots] \) has finite rank \( n \).

**B.** If moreover \( f(z) = f_0 + f_1 z + \cdots + f_{n-1} z^{n-1} + f_n z^n \), \( f_n = 1 \), and \( g(z) = g_0 + g_1 z + \cdots + g_{n-1} z^{n-1} \), then the following are equivalent:

1. \( g(z)/f(z) \) is a minimal realization of \( h \);  
2. \( H_{\infty, \infty} = C_{\infty, n} H_{n, \infty} \) with \( H_{n, n} \) nonsingular and \( C_{\infty, n} \) the infinite companion matrix of \( f(z) \). The numerator can be found from

\[
\begin{bmatrix} 0 & 0 & \cdots & h_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & h_1 & \cdots & h_n \\ f_0 & f_1 & \cdots & f_n \\ g_0 & g_1 & \cdots & g_{n-1} \end{bmatrix} = \begin{bmatrix} g_{n-1} \\ \vdots \\ g_0 \end{bmatrix}.
\]  

(9)

**Proof.**  
**A.1 \(\Rightarrow\) 2:** See the previous section.  
**A.2 \(\Rightarrow\) 1:** Suppose \( H_{\infty, \infty} \) has finite rank \( n \), and let \((S^m h)^T\) be the first row of \( H_{\infty, \infty} \) which is linearly dependent on the previous ones \( (m \leq n) \). Then there exists some \( f = [-f_0 \ \cdots \ -f_{m-1}], \ f_m = 1, \) such that

\[
(S^m h)^T = fH_{m, \infty}.
\]  

(10)

This shows that \( h \) has a realization of degree \( m \). Repeated application of \( S \) on (10) will show that all rows of \( H_{\infty, \infty} \) can be expressed as linear combinations of the first \( m \) rows. It follows that \( n = \text{rank } H_{\infty, \infty} \leq m \). Thus \( n = m \) and part A is proved.

**B.1 \(\Rightarrow\) 2:** Follows from the previous section.  
**B.2 \(\Rightarrow\) 1:** The \( (n + 1) \)th row of \( H_{\infty, \infty} = C_{\infty, n} H_{n, \infty} \) expresses that \( h \) satisfies the difference equation (4a) with \( f_n = 1 \), while (9) is equivalent with (4b).
Hence \( g(z)/f(z) \) is a partial realization of \( h \) of degree \( n \). This realization is minimal, since if a lower degree realization existed, then by part A of this theorem, the degree of \( H_{\infty, \infty} \) would be lower than \( n \). This proves B. 

5. AN ALGEBRAIC METHOD TO SOLVE THE MINIMAL PARTIAL REALIZATION PROBLEM

In the previous section, we assumed that the complete sequence \( \{ h_1, h_2, \ldots \} \) was known. Now we shall assume that only the first \( N \) numbers \( \{ h_1, \ldots, h_N \} \) are specified. The unspecified values are denoted by \( h_{N+1}^*, h_{N+2}^*, \ldots \). The sequence \( \{ h_1, \ldots, h_N, h_{N+1}^*, \ldots \} \) is called an extension of \( \{ h_1, \ldots, h_N \} \). The problem of minimal partial realization can then be stated as: find an extension of \( \{ h_1, \ldots, h_N \} \) such that the Hankel matrix \( H_{\infty, \infty} \) for this extension has minimal rank. Given such a minimal extension, suppose \( n = \text{rank } H_{\infty, \infty} = \text{rank } H_{N, N} \leq N \). Then we can write

\[
H_{N, N} = \begin{bmatrix} I_{n, n} & 0_{n, N-n} \\ F_{N-n, n} & 0_{N-n, N-n} \end{bmatrix} D_{N, N},
\]

with \( H_{n, n} \) nonsingular. The first row of \( F_{N-n, n} \) will give the denominator coefficients.

Conversely, if \( H_{N, N} \) has rank \( n \) and if there exists some \( N \times N \) nonsingular matrix \( D_{N, N} \) such that

\[
H_{N, N} = \begin{bmatrix} I_{n, n} & 0_{n, N-n} \\ F_{N-n, n} & 0_{N-n, N-n} \end{bmatrix} D_{N, N},
\]

then \( D_{n, N} = H_{n, N} \) and the first row of \( F_{N-n, n} \) still gives the denominator coefficients of the minimal partial realization. The algorithm will construct \( D_{N, N}^{-1} \) as successive transformations on the columns of \( H_{N, N} \).

Suppose after \( i-1 \) steps we have transformed \( H_{N, N} \) as

\[
H_{N, N} Q^{(i-1)} = \begin{bmatrix} I_{i-1, i-1} & 0_{i-1, N-i+1} \\ U_{N-i+1, i-1} & V_{N-i+1, N-i+1} \end{bmatrix}.
\]

If \( i-1 < n \), then \( V_{N-i+1, N-i+1} \) cannot be zero. In step \( i \) we further
transform this as
\[
H_{\nu, \nu}'Q^{(i)} = \begin{bmatrix}
I_{i, i} & 0_{i, N-i} \\
U_{N-i, i} & V_{N-i, N-i}
\end{bmatrix}
\]
with \(Q^{(i)} = Q^{(i-1)}Q_i\) and \(Q_i\) nonsingular. Now rank \(H_{\nu, \nu} = i + \text{rank } V_{N-i, N-i}\), because \(Q^{(i)}\) will not change the rank. Thus if the unspecified parameters \(h_{N+1}^*, h_{N+2}^*, \ldots, h_{2N}^*\) can be chosen such that \(V_{N-i, N-i}\) is the zero matrix, then \(i = n\), the rank of the minimal partial realization, and \(F_{N-n, n} = U_{N-n, n}\). If not, we have to proceed to step \(i + 1\).

6. EXAMPLES

**Example 1.** Consider the sequence \(\{0, 1, -1, 0, 1, -1, \ldots\}\). The \(6 \times 6\) Hankel matrix \(H_{6,6}\) looks as follows:
\[
H_{6,6} = \begin{bmatrix}
0 & 1 & -1 & 0 & 1 & -1 \\
1 & -1 & 0 & 1 & -1 & a \\
-1 & 0 & 1 & -1 & a & b \\
0 & 1 & -1 & a & b & c \\
1 & -1 & a & b & c & d \\
-1 & a & b & c & d & e
\end{bmatrix}.
\]

**Step 1.** Take
\[
Q_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & -1 & 0 & 1
\end{bmatrix}.
\]

Then
\[
H_{6,6}Q_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-a & 1 & a-1 & -a & 1 & a-1 \\
-b & -1 & b & -b+1 & -1 & b+a \\
-c & 0 & c+1 & -c-1 & a & b+c \\
-d & 1 & d-1 & a-d & b & c+d \\
-e & -1 & a+e & b-e & c & d+e
\end{bmatrix} = \begin{bmatrix}
I_{1, 1} & 0_{1, 5} \\
U_{5, 1} & V_{5, 5}
\end{bmatrix}.
\]

Clearly \(V_{5, 5} \neq 0\).
Step 2. Take

\[ Q_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
a & 1 & -a+1 & a & -1 & -a+1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]

Then

\[ H_{6,6} Q_1 Q_2 \]

\[ = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
-a-b & -1 & a+b-1 & -a-b+1 & 0 & 2a+b-1 \\
-c & 0 & c+1 & -c-1 & a & b+c \\
ar-d & 1 & -a+d & 2a-d & b-1 & -a+c+d+1 \\
-a-e & -1 & 2a+c-1 & -a+b-e & c+1 & a+d+e-1 \\
\end{bmatrix} \]

\[ = \begin{bmatrix}
I_{2,2} & 0_{2,4} \\
U_{4,2} & V_{4,4} \\
\end{bmatrix} \]

Now \( V_{4,4} = 0 \) has a unique solution for

\[ a = 0, \quad b = 1, \quad c = -1, \quad d = 0, \quad c = 1. \]

Thus the minimal partial realization is unique:

\[ \frac{g(z)}{f(z)} = \frac{1}{z^5 + z + 1} \]

\[ = 0z^{-1} + 1z^{-2} - 1z^{-3} + 0z^{-4} + 1z^{-5} - 1z^{-6} \]

\[ + 0z^{-7} + 1z^{-8} - 1z^{-9} + 0z^{-10} + 1z^{-11} + \ldots . \]

**Example 2.** Given only the first three numbers of Example 1 \((0, 1, -1)\), \(H_{3,3}\) becomes

\[ H_{3,3} = \begin{bmatrix}
0 & 1 & -1 \\
1 & -1 & a \\
-1 & a & b \\
\end{bmatrix} \]
Step 1.

\[ Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow H_{3,3}Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ a & -1 & a+b \end{bmatrix} = \begin{bmatrix} I_{1,1} & 0_{1,2} \\ V_{2,1} & V_{2,2} \end{bmatrix}. \]

\[ V_{2,2} \neq 0; \text{ hence step 2.} \]

Step 2.

\[ Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -a+1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \]

\[ H_{3,3}Q_1Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a-1 & -1 & 2a+b-1 \end{bmatrix} = \begin{bmatrix} I_{2,2} & 0_{2,1} \\ U_{1,2} & V_{1,1} \end{bmatrix}. \]

\[ V_{1,1} = 0 \text{ for } b = 1 - 2a. \]

Thus the minimal partial realizations have a free parameter, and they are given by

\[ \frac{g(z)}{f(z)} = \frac{1}{z^2 + z + (1-a)} = 0z^{-1} + 1z^{-2} - 1z^{-3} + az^{-4} + (1 - 2a)z^{-5} + \ldots. \]

Note that in those two simple examples, the conditions on the unspecified values in \( V \) were always linear. This is a general phenomenon: the transformation \( Q_i \) can always be found such that the conditions \( V_{N-i, N-i} = 0 \) on the unspecified values are all linear. However, if some of the given numbers \( h_i, i \leq N, \) are missing and become also unspecified, linearity no longer holds, as will be shown in Example 3.

Example 3. Take the sequence 0, c, -1:

\[ H_{3,3} = \begin{bmatrix} 0 & c & -1 \\ c & -1 & a \\ -1 & a & b \end{bmatrix}. \]

Step 1.

\[ Q_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & c \end{bmatrix} \rightarrow H_{3,3}Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & c & ac -1 \\ -b & -1 & bc + a \end{bmatrix}. \]
Step 2. Now there are two possibilities:

(2a) $c = 0$:

$$H_{3,3}Q_1 = \begin{bmatrix}
1 & 0 & 0 \\
-a & 0 & -1 \\
-b & -1 & a
\end{bmatrix}.$$ 

Thus $H_{3,3}$ has rank 3 for any choice of $a$ and $b$, so each polynomial of degree 3 can be taken as a denominator of a minimal partial realization:

$$\frac{g(z)}{f(z)} = \frac{-1}{z^3 + dz^2 + ez + f} = 0z^{-1} + 0z^{-2} - z^{-3} + \ldots.$$ 

(2b) $c \neq 0$:

$$Q_2 = \begin{bmatrix}
1 & 0 & 0 \\
a & 1 & -\frac{ac-1}{c} \\
c & c & c
\end{bmatrix}$$

$$\rightarrow H_{3,3}Q_1Q_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-b - a & -\frac{1}{c} & (bc + a) + \frac{(ac - a)}{c}
\end{bmatrix}.$$ 

$V_{1,1}$ is zero when $bc^2 + 2ac = 1$, or $b = (1 - 2ac)/c^2$. The parameter $c$ is free as long as it is nonzero, while the parameter $a$ is completely free. The minimal partial realizations are

$$\frac{g(z)}{f(z)} = \frac{c}{z^2 + \frac{1}{c} + \left( b + \frac{a}{c} \right)} = \frac{c}{z^2 + \frac{1}{c} - \frac{ac}{c}}$$

$$= 0z^{-1} + cz^{-2} - 1z^{-3} + az^{-4} + \frac{1 - 2ac}{c^2}z^{-5} + \ldots.$$ 

With $c = 1$ we get the solution of Example 2.
7. CONCLUSION

The fundamental theorem, which is essentially a matrix formulation of Kronecker's result, gives a decomposition of an infinite Hankel matrix of finite rank such that the denominator of the minimal partial realization can be read off. If only the first $N$ entries of the Hankel sequence are given, we can derive an algorithm to find all minimal extensions of this sequence.

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