A smallest irregular oriented graph containing a given diregular one

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Abstract

A digraph is called irregular if its vertices have mutually distinct ordered pairs of semi-degrees. Let D be any diregular oriented graph (without loops or 2-dicycles). A smallest irregular oriented graph F, F = F(D), is constructed such that F includes D as an induced subdigraph, the smallest digraph being one with smallest possible order and with smallest possible size. If the digraph D is arcless then V(D) is an independent set of F(D) comprising almost all vertices of F(D) as |V(D)| → ∞. The number of irregular oriented graphs is proved to be superexponential in their order. We could not show that almost all oriented graphs are are not irregular.

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1. Introduction

Only finite graphs/digraphs without loops and without multiple edges/arcs are considered. For undefined terminology and notation we refer to [9].

The order and size of a digraph are the number of vertices and the number of arcs, respectively. The number of arcs incident with a vertex v in a digraph D is called the degree of v in D and is denoted by deg v (v). Hence the degree of v is the sum of the outdegree, od (v), and the indegree, id (v), i.e., deg (v) = od (v) + id (v). The ordered pair (od (v), id (v)) of semi-degrees of a vertex v (the outdegree followed by the indegree) is called the degree pair of v.

A loopless digraph without any dicycle C2 (on two vertices) is called an oriented graph. Let ρ ∈ {0, 1, 2, . . .}. A digraph D is called ρ-diregular if every vertex of D has the degree pair (ρ, ρ). Hence, if a ρ-diregular oriented graph has n vertices then ρ ≤ n−1. Moreover, a digraph is called diregular if it is ρ-diregular for some ρ.

It is well known, due to Pigeonhole Principle, that each nontrivial component of an undirected graph has two vertices of the same degree. So does each nontrivial graph. In contrast, a digraph of any order can have all vertices with mutually distinct degree pairs. Such digraphs, called irregular, are discovered and studied in a paper by Gargano et al. [11] published already in 1990. We are obliged to Prof. Louis Quintas for sending us a reprint of [11]. Our investigations into irregular digraphs, which resulted in publications [16,15], were prompted and influenced by papers on all-local irregularity of graphs [4] or digraphs [5]. Namely, in Alavi et al. [4] a graph G is called highly irregular if G is connected and each vertex of G is adjacent to vertices with distinct degrees only. Similarly, in Alavi et al. [5] a digraph D is called highly irregular if D is connected and the vertices in the out-neighborhood of any vertex have mutually distinct outdegrees. We can say that this is the definition of highly out–out irregularity. The corresponding “in–in”, “in–out” and “out–in” definitions are possible. For example, a connected digraph D is called highly out–in irregular if the vertices in the out-neighborhood

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of any vertex have mutually distinct indegrees. Due to the conversing operation it is enough to consider only “out–out” and “out–in” irregularities. On the other hand, Majcher and Michael in [14] define HI-digraphs as a specialization of highly irregular digraphs. Namely, a digraph is called a HI-digraph if, for each vertex, the vertices of the in-neighborhood have different out-degrees and vertices of the out-neighborhood have different in-degrees. All these definitions of highly irregular structures are all-local, i.e., globally local.

Recall that a digraph $D$ is said to be irregular if distinct vertices of $D$ have distinct degree pairs. There is no connectivity requirement in this definition. In contrast, the connectivity requirement appears in the above definitions of highly irregular graphs and digraphs in order to exclude graphs/digraphs with repeating (for instance, 1- or 2-vertices) components.

It is noted in [4] that, for every positive integer $n \neq 3, 5, 7$, there exists a highly irregular graph of order $n$. For every $n$, there exists a highly irregular oriented graph of order $n$, for example, the $n$-vertex transitive tournament $T_n$ is an oriented graph which is not only highly irregular but also irregular.

Any irregular oriented tree has at most two degree-1 and three or less degree-2 vertices. So, there exist only seven irregular oriented trees. Their list comprises the trivial graph $K_1$, directed paths $P_2$ and $P_3$, semipath obtained from the directed path $P_3$ by reversing the orientation of the innermost arc and three semipaths obtained from the directed path $P_3$ by reversing the orientation of one or two of the innermost arcs.

In this paper, we are interested in deregularization, in fact, in constructive irregularization of any irregular oriented graph. Note that there is an extensive literature on irregularization of simple graphs achievable by multiplying edges. Optimality is then measured by so-called irregularity strength (minimizing the maximum among resulting multiplicities) or irregularity cost (which is the minimum possible number of new edges), cf. [1,8,10,13]. Another optimality criterion of such irregularizations, the minimum of the number of distinct multiplicities, considered in [1] is identified there with what is called later the vertex-distinguishing edge-coloring number, which resulted in another natural method of irregularizing. This method or rather the corresponding edge-coloring parameters concerning nonproper [1–3] or proper edge-colorings [6,7] is a subject of a series of interesting publications.

Let $D$ be any irregular oriented graph with $n$ vertices. In this paper, a smallest irregular oriented graph $F$, $F = F(D)$, is constructed such that $F$ includes $D$ as an induced subdigraph, the smallest digraph $F$ being one with smallest possible order and with smallest possible size. Our proof, however, is quite long. But the result is that $F$ is of order as small as $n + \lfloor \sqrt{2n - \frac{1}{2}} \rfloor$ for $n \geq 4$. On the other hand, an easy construction in [16, Theorem 1] yields a 1–1 embedding of nonisomorphic $n$-vertex digraphs (oriented graphs) into nonisomorphic irregular digraphs (irregular oriented graphs) on $n + 2 \lfloor \sqrt{n} \rfloor$ vertices only. We also present an easy alternative construction which requires $2n$ vertices. In the pioneering paper [11], however, an irregular superdigraph inducing $D$ can have up to $n \cdot 6^{(n+2)/2}$ vertices.

Additionally, we study the independence number and note that if the digraph $D$ is arcless and of arbitrarily large order, then almost all vertices of $F(D)$ are in $V(D)$ which is an independent set. The total number of irregular oriented graphs is proved to be superexponential in their order.

2. Preliminaries

**Proposition 1.** There are exactly $i + 1$ distinct degree pairs possible for a vertex of degree $i$.

**Proof.** The degree pairs in question are $(i, 0), (i - 1, 1), \ldots, (0, i)$. □

**Corollary 2.** The complete graphs are the only regular graphs that admit of an irregular orientation. In fact, each irregular orientation of a complete graph $K_n$ is the transitive tournament $T_n$.

Let

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\}. $$

Let $\nu : \mathbb{N}_0 \to \mathbb{N}_0$ be defined so that

$$\nu(k) = \sum_{i=0}^{k} i = \frac{1}{2} k(k + 1). \tag{1}$$

Define $\tau : \mathbb{N} \to \mathbb{N}_0$ so that

$$\tau(n) = k \quad \text{if} \quad \nu(k) < n \leq \nu(k + 1), \; n \in \mathbb{N}, \; k \in \mathbb{N}_0.$$  

The function $\tau$ is well defined because $\nu$ is a function strictly increasing to infinity. In particular, $\tau(1) = 0$.  

Proposition 3. Let \( n, r \in \mathbb{N} \) and let \( t \in \mathbb{N}_0 \). Then the statements
\[
t = \tau(n)
\]
and
\[
n = \nu(t) + r \quad \text{where} \quad 1 \leq r \leq t + 1
\]
are equivalent. Furthermore,
\[
\tau(n) = \left\lfloor \sqrt{2n} - \frac{1}{2} \right\rfloor.
\]
Equivalently,
\[
\tau(n) = \left\lfloor \sqrt{2n + 2l_1 - \frac{7}{4} - \frac{1}{2}} \right\rfloor = \left\lfloor \sqrt{2n + \frac{1}{4} - 2l_2 - \frac{3}{2}} \right\rfloor
\]
for any nonnegative reals \( l_1, l_2 < 1 \).

Proof. The definitions of the functions \( \nu \) and \( \tau \) imply the first equivalence. Then \( t = \tau(n) \) implies that
\[
\frac{1}{2} t (t + 1) < n \leq \frac{1}{2} (t + 1)(t + 2)
\]
and equivalently
\[
\frac{1}{2} t (t + 1) + 1 - l_1 \leq n \leq \frac{1}{2} (t + 1)(t + 2) + l_2
\]
for \( 0 \leq l_1, l_2 < 1 \). Therefore \( \tau(n) \) is the largest (smallest) integer \( t, t \geq 0 \), such that the first (second) inequality in (3) [or in (4)] holds. Hence, the concluding equalities can be obtained.

3. Main result

Theorem 4. Let \( D \) be an \( n \)-vertex \( \rho \)-diregular oriented graph \( (\rho \leq \frac{n - 1}{2}) \). Let \( \hat{F} \) be an irregular oriented graph of order \( n + t \) which includes \( D \) as an induced subdigraph. Then \( t \geq \tau(n) \) (cf. (2)) unless \( n = 3 \) and \( \rho = 1 \), and then \( t \geq 2 \). Moreover, the lower bound, \( \tau(n) \) or 2, on \( t \) is attainable. In fact, one of the smallest \( \hat{F} \)s (of the smallest possible order \( n + t \) and with smallest possible size), being denoted by \( F \), \( F = F(D) = F(n, \rho) \), is constructed.

Proof. Let \( V = V(D) \) and let \( U = V(\hat{F}) - V(D) \). Hence \( |V| = n \) and \( |U| = t \). Consider a fixed ordering of vertices in \( V \) and \( U \),
\[
V = \{v_i; \ i = 1, \ldots, n\} \quad \text{and} \quad U = \{u_i; \ i = 1, \ldots, t\}.
\]

Claim 1. One has \( t \geq \tau(n) \).

Proof. Due to Proposition 1, for any nonnegative integer \( i \), the set \( V \) contains at most \( i + 1 \) vertices of degree \( i \) in the bipartite subdigraph of \( \hat{F} \) induced by all \( V - U \) arcs. Hence, if \( n = \nu(k) + r \) where \( k, r \in \mathbb{N}_0 \) and \( 1 \leq r \leq k + 1 \), then \( V \) contains a vertex adjacent to \( k \) or more vertices in the set \( U \). Therefore, \( t = |U| \geq k = \tau(n) \) due to Proposition 3.

To complete the proof it is enough to construct a (smallest) oriented graph \( F \). If \( n = 1 \) then clearly \( D = K_1 = F \). For \( n \geq 2 \), we are going to show that \( F \) is the edge-disjoint union of oriented graphs \( D \) and \( B \), where \( B \) is a bipartite digraph induced by \( V - U \) arcs unless \( (n, \rho) \in \{(5,1),(6,1)\} \) and then a smallest \( F \) has to include an additional \( U - V \) arc.

Let \( n = 2 \) or 3. Then \( \tau(n) = 1 \). However, if \( n = 3 \), \( \rho = 1 \), and \( |U| = 1 \) then there is no required digraph \( B \).

Claim 2. If \( n = 3 \) and \( \rho = 1 \) then \( D \) is the dicycle \( C_5 \) and \( t \geq 2 \). For \( t = 2 \), there is a required digraph \( F \) on 5 vertices, e.g., \( F \) is the union of two dipaths \( u_1 \rightarrow v_2 \rightarrow v_3 \rightarrow u_2 \) and \( v_3 \rightarrow v_1 \rightarrow v_2 \), \( F \) being uniquely determined up to isomorphism.

In remaining cases \( \rho = 0 \) if \( n = 2, 3 \) and then the smallest possible \( B \) has \( |U| = 1 \) and \( n - 1 \) arcs whence such a \( B \) can be a dipath which includes all vertices except \( v_1 \).

Therefore in what follows \( n \geq 4 \) and \( t = \tau(n) \geq 2 \). Moreover, due to Proposition 3,
\[
r = n - t(t + 1)/2 \geq 1.
\]
Using Proposition 1, design mutually distinct degree pairs in \( B \) of vertices belonging to \( V \). To this end, partition \( V \) into \( t + 1 \) sets \( I_0, I_1, \ldots, I_r \) such that \( |I_i| = r \) and for each remaining subscript \( i = 0, 1, \ldots, t - 1 \), \( |V_i| = i + 1 \) and \( \bigcup_{j=0}^{t} I_j \) comprises only initial vertices from \( V \), e.g., \( I_0 = \{v_1\}, I_1 = \{v_2, v_3\} \) etc. Assume that each \( I_j \) comprises all vertices from \( V \) of degree exactly \( j \) in the subgraph \( B \). Thus, the vertex \( v_1 \) is left nonadjacent to the set \( U \). Moreover, let \((i, 0), (i - 1, 1), \ldots, (0, i)\) be the degree pairs in \( B \) of consecutive vertices in \( V_i, i = 0, 1, \ldots, t - 1 \). For \( i = t \), only \( r \) degree pairs are selected out of \( t + 1 \) pairs \((t, 0), \ldots, (0, t)\). Pairs are selected so that the sum of their first components and that of the second components either coincide or the former is 1 greater than the latter. The sums clearly must differ whenever the numbers \( t \) and \( r \) are both odd.

Define the distribution of \( V \cup U \) arcs among vertices in \( U \) by constructing a \( 0, 1, t \) matrix \( A \) which represents \( V \cup U \) adjacency in the bipartite digraph \( B \) wherein the entries \( 0, 1, t \) stand for \((0, 0)\), \((1, 0)\) and \((0, 1)\), respectively. Assume therefore that \( A = [a_{ij}]_{n \times t} \) where, for \( v_i \in V \) and \( u_j \in U \),

\[
a_{ij} = \begin{cases} 
1 & \text{if the arc } (v_i, u_j) \text{ is in } B, \\
i & \text{if the arc } (u_j, v_i) \text{ is in } B, \\
0 & \text{otherwise.} 
\end{cases}
\]

Hence,

\[
\sum_j a_{ij} = (od_\theta(v_i), id_\theta(v_i)), \quad v_i \in V,
\]

which is to be the above-designed degree pair of the vertex \( v_i \). Moreover,

\[
\sum_i a_{ij} = (id_\theta(u_j), od_\theta(u_j)), \quad u_j \in U.
\]

We first construct an auxiliary matrix, still denoted by \( A \), whose columns other than \( 1, \lfloor \frac{i+1}{t} \rfloor, \lceil \frac{i+1}{t} \rceil, t \) are in the final form (i.e., will appear in the final matrix \( A \)). In order to simplify notation the corresponding bipartite graph is still denoted by \( B \). Assume that all 1s and \( t \)s make up the initial and terminal segments, respectively, of each row both in \( A \) and in another auxiliary \((t + 1) \times t\) matrix \( \tilde{A} \) to be defined below. Assume that the partition \( V_0, V_1, \ldots, V_t \) of the vertex set \( V \) induces the partition of the matrix \( A \) into submatrices \( A_0, A_1, \ldots, A_t \), where \( A_i \) is the matrix which represents \( V_i \cup U \) adjacency. Hence, \( A_j \) is an \((i + 1) \times t\) matrix for \( i < t \). Moreover, if \( 1 \leq j \leq i + 1 \), the row \( j \) of \( A_i \), which sums up to \((i - j + 1, j - 1) \), has \( i - j + 1 \) entries 1 as an initial segment which is followed by \( t - i \) zeros and next exclusively \( s \) follow.

Thus, \( A_i \) is a row of zeros if \( i = 0 \). If \( 0 < i < t \), then the entries 1 on one hand and \( s \)s on the other hand make up two disjoint triangular sections of the submatrix \( A_i \), the upper left corner of \( A_i \), being filled with 1s, the lower right corner with \( s \). Assume that such is the structure of the matrix \( A_i \). In this case the triangle of 1s fits to the triangle of \( s \)s so that no entry of \( A_i \) is 0.

The last submatrix \( A_t \) of \( A \) has \( r \) rows and \( t \) columns. Define \( \tilde{A}_t \) to be a submatrix of the auxiliary matrix \( \tilde{A} \). Then, let \( A_t \) comprise \( r \) central rows of \( \tilde{A}_t \) if the sum \( t + r \) is odd, else let \( A_t \) comprise \( r \) central rows of the matrix obtained from \( \tilde{A}_t \) by deleting the row \( \lfloor \frac{i+1}{t} \rfloor + 1 \). Thus, \( A_t \) is obtained by removal of the same number of rows from the top and the bottom of the respective matrix (\( \tilde{A}_t \) or the row-deleted submatrix of \( \tilde{A}_t \)).

For example, let \( 7 \leq n \leq 10 \). Then \( t = 3, r = n - 6, \) and

\[
A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & i \end{bmatrix}, \quad \tilde{A}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & i \\ 1 & i & i \\ t & i & i \end{bmatrix}.
\]

Moreover, the submatrix \( A_3 = \tilde{A}_3 \) if \( n = 10 \), else \( A_3 \) is

\[
\begin{bmatrix} 1 & 1 & i \\ 1 & 1 & i \\ 1 & i & i \\ 1 & i & i \end{bmatrix}
\]

for \( n = 7, 8, 9 \), respectively. Therefore, the following degree pairs for \( u_j \) if \( n = 7, 8, 9, 10 \) are the reversed sums (6) of the column \( j \) of the matrix \( A, j = 1, 2, 3, \)

\[
n = 7: (0, 4) (1, 2) (4, 0) \quad n = 9: (1, 5) (2, 3) (5, 1),
n = 8: (0, 5) (2, 2) (5, 0) \quad n = 10: (1, 6) (3, 3) (6, 1).
\]
Then, due to (5) and (6), the matrix $A$ determines a required digraph $B$ (and $F$) unless

$$(n, \rho) \in \{(7, 1), (8, 1), (8, 2), (9, 2), (10, 2), (10, 3)\}$$

because precisely then the vertex $u_2$ and one in $V$ have the same degree pair.

We now construct $F$ if (7) holds. If either $n = 7$ and $\rho = 1$ or $n = 9, 10$ and $\rho = 2$, in order to get $F$ we only replace the submatrix $A_1$ of the matrix $A$ by

$$A'_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & i & 0 \end{bmatrix}.$$  

If either $n = 8$ and $\rho = 2$ or $n = 10$ and $\rho = 3$, in order to get $F$ we only replace the submatrix $A_2$ of the matrix $A$ by

$$A'_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & i \\ i & 0 & i \end{bmatrix}.$$  

Let $4 \leq n \leq 6$. Then $t = 2$, $r = n - 3$, and

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix}.$$  

Moreover, $A_2 = \tilde{A}_2$ if $n = 6$, else $A_2$ is

$$[1 \ i], \quad \begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix}$$  

for $n = 4, 5$, respectively. Therefore, due to (6), the columns of the matrix $A$ determine the following degree pairs for $u_j$ if $n = 4, 5, 6$.

- $n = 4$: $(2, 3)$, $(0, 2)$, $(2, 0)$,
- $n = 5$: $(1, 2)$, $(2, 1)$,
- $n = 6$: $(1, 3)$, $(3, 1)$.

Then, due to (5) and (6), the matrix $A$ determines a required digraph $B$ (and $F$) unless

$$(n, \rho) \in \{(5, 1), (6, 1)\}$$

because then either vertex in $U$ has the same degree pair as a vertex in $V$.

The following statement can be easily seen.

**Claim 3.** Assume (8) holds. Then $t = 2$. There is no required digraph $F$ without any $U-U$ arc. However, if a $U-U$ arc is allowed then, for each $D$, there are two distinct sets of degree pairs of vertices in a required minimal digraph $F$ for each $n$, $n = 5, 6$. Moreover, deleting arcs $A(D)$ of $D$ from $F$ gives two and four nonisomorphic $n$-vertex digraphs $F-A(D)$ for $n = 5, 6$, respectively.

**Proof.** Let $n = 6$ (and $\rho = 1$). Then, the set of degree pairs for $V (=V(D))$ in $F$ is unique. It comprises all (six) ordered pairs of positive integers which sum up to 2, 3 or 4 in each pair. Thus no semi-degree in $U$ can be zero. Moreover, because $\rho = 1$, the number of $V-U$ arcs is 8. Therefore both vertices in $U$ must have the same number, four, of neighbors in $V$ so that a single $U-U$ arc could make $F$ irregular. It can be seen, once degrees of vertices in $U$ are made four, that any $U-U$ arc can appear in $F$. Therefore there are two sets of degree pairs for $U$ in $F$, namely, $\{(2, 3), (3, 2)\}$ and $\{(1, 4), (4, 1)\}$. However, one can see that there are four mutually nonisomorphic oriented graphs $F-A(D)$.  

Let $n = 5$. Then one of the six degree pairs for $V$ in $F$ is missing. If the missing pair has sum 4 then the vertices in $U$ either have the same number, three, of neighbors in $V$ or one of them has two and another four neighbors in $V$. Moreover, only one vertex in $U$ (with two or three neighbors in $V$) can have empty either in- or out-neighborhood in $F$ but then the other vertex in $U$ and a vertex in $V$ have the same degree pair. Therefore, only on adding a $U$–$V$ arc so that the semi-neighborhood in question is kept empty, one can get an irregular $F$. However, such $F$ cannot arise if both vertices in $U$ have three neighbors in $V$. Therefore, there are two sets of degree pairs for $F$ in $U$, namely, $\{(0, 3), (3, 2)\}$ and $\{(3, 0), (2, 3)\}$, the missing degree pairs being $(1, 3)$ and $(3, 1)$, respectively. Then there are two nonisomorphic oriented graphs $F' = A(D)$.

If the missing pair does not have sum 4 then it must have sum 3 so that a minimal $F$ could arise. But then vertices in $U$ have, respectively, three and four neighbors in $V$. Furthermore, each of them has nonempty both in- and out-neighborhood in $V$. Therefore minimal irregular $F$ does not exist. 

**Remark.** If $(n, \rho) = (5, 1)$ or $(6, 1)$ then $D$ is, respectively, a dicycle $C_5$ or one of two oriented graphs: a dicycle $C_6$ or a disjoint union, $2C_3$, of two dicycles $\tilde{C}_3$. Moreover, if $c_n := (n - 1)!$ then the numbers of labeled digraphs $\tilde{C}_n$ and $2\tilde{C}_n$ are $c_n$ and $c_{2n}/n$, respectively. Hence, the number of mutually nonisomorphic minimal irregular oriented graphs $F$ containing $D$ as an induced subdigraph is $2c_5 = 48$ if $n = 5$ and $4(c_6 + c_6/3) = 640$ if $n = 6$.

It remains to consider the following case.

\[ n \geq 11 \quad \text{and} \quad t = \tau(n) \geq 4. \] (9)

Recall that the outdegree $od_{\tilde{g}}(u_j)$ of the vertex $u_j \in U$ is the number of entries $i$ in column $j$ of $A$. The number of entries 1 in the same column $j$ equals the indegree $id_{\tilde{g}}(u_j)$. One can easily see the following.

**Claim 4.** Under assumption (9) the indegrees $id_{\tilde{g}}(u_j)$ strictly decrease if $j$ increases while the outdegrees $od_{\tilde{g}}(u_j)$ strictly increase then.

Moreover, the sequence of degree pairs is skew-symmetric in the sense that $od_{\tilde{g}}(u_j) = id_{\tilde{g}}(u_{n-j+1})$ for $j = 1, 2, \ldots, t$ with the exception of $j = \lfloor \frac{t+1}{2} \rfloor$ for odd $t$ and odd $r$ because then $od_{\tilde{g}}(u_j) = -1 + id_{\tilde{g}}(u_j)$ can be seen.

Define the *irregularity* of a vertex $x$ in a digraph $F$, in symbols $\text{irr}(x)$ or $\text{irr}_F(x)$, to be the absolute value of the difference between the indegree and the outdegree of $x$ in $F$, $\text{irr}_F(x) = |\text{id}_F(x) - \text{id}_F(x)|$. Let $\Theta$ be the maximum irregularity in the digraph $B$ among vertices in the set $V$. Then, due to the definition of matrices $\tilde{A}_t$ and $A_t$,

\[
\Theta = \begin{cases} 
\lfloor \frac{t-1}{2} \rfloor & \text{if } r \leq t - 1, \\
1 & \text{otherwise.}
\end{cases}
\] (10)

Let $t \geq 5$. We are going to show first that $\text{irr}_{\tilde{g}}(u_j) > \Theta$ for any vertex $u_j$ such that $j \notin \{1, \lfloor \frac{t+1}{2} \rfloor, \lfloor \frac{t+2}{2} \rfloor, t\}$. Therefore the auxiliary matrix $A$ can be used. Due to monotonicity of both outdegrees and indegrees (Claim 4) and skew-symmetry of the sequence of degree pairs, it is enough to show that

\[
\theta := \text{irr}_{\tilde{g}}(u_{\lfloor \frac{t-1}{2} \rfloor}) > \Theta.
\] (11)

Then $\theta = id_{\tilde{g}}(u_{\lfloor \frac{t-1}{2} \rfloor}) - od_{\tilde{g}}(u_{\lfloor \frac{t-1}{2} \rfloor})$ which is the difference between the number of 1s and that of 0s in the column $\lfloor \frac{t-1}{2} \rfloor$ of the matrix $A$. We evaluate $\theta$ as the sum of contributions of consecutive matrices $A_i$. Recall the triangular sections of $A_i$ filled in by 1s and 0s. First of all, those contributions are all 0 if $i < \lfloor \frac{t-1}{2} \rfloor$. Next, if $i$ increases up to $i = t - 1$, the consecutive contributions are a single 1 and next 2s if $i$ is odd, else they are a single 1, single 2 and next 3s. Due to the above definition of the matrix $A_i$, the contribution of $A_i$ is

1. if $r = 1$ or $t$ and $r$ are both odd,
2. if $r$ is even,
3. if $r \geq 3$ is odd and $t$ is even.

Therefore, to find a lower bound on $\theta$ we replace the contribution of $A_i$ by the summand 1. Hence

\[
\theta \geq \begin{cases} 
1 + 2(t - 1 - \lfloor \frac{t-1}{2} \rfloor) + 1 = t + 1 & \text{if } t \text{ is odd}, \\
1 + 2 + 3(t - 1 - \frac{t}{2}) + 1 = \frac{3}{2} t + 1 & \text{otherwise},
\end{cases}
\]

which together with (10) implies (11).
Consider the following two cases in order to finish the construction of the matrix $A$. The final form of $A$ we denote by $A'$.

**Case I:** The number $t$ is even and $t \geq 4$.

Let $A'$ be the matrix obtained from the auxiliary matrix $A$ by interchanging $a_{ij}$ and $a_{ji}$ for each $j$ such that $a_{ij} = 1$ and $a_{ji} = 0$ and by interchanging $a_{ij}$ and $a_{ji}$ for each $j$ such that $a_{ij} = 0$ and $a_{ji} = 1$. Let the matrix obtained this way from the submatrix $A_t$ be denoted by $A'_t$. Then clearly $A'_t = A_t$. Notice that for $i \leq \frac{t}{2}$ the column $\frac{t}{2}$ of the submatrix $A_t$ is the column of zeros. Hence, all $i$ entries 1 from the first column of $A_t$ are moved to column $\frac{t}{2}$. For $i = \frac{t}{2}$, the matrix $A'_t$ has $\frac{t}{2}$ entries 1 and one 0 in the column $\frac{t}{2}$ because $\frac{t}{2} - 1$ entries 1 are moved from the first column of $A_t$ to the column $\frac{t}{2}$. Moreover, a single 1 remains in the first column of $A'_t$. For $\frac{t}{2} < i < t$ and any row of $A_t$, if zero occurs in the column $\frac{t}{2}$ then 1 occurs in the first column. Then $t-i$ entries 1 are moved from and $2i-t$ entries 1 are left in the column 1 of $A_t$. Therefore the matrix $A'_t$ has $1 + \frac{t}{2}$ entries 1 at the top of the column $\frac{t}{2}$ which are followed by $i - \frac{t}{2}$ entries 1 at the bottom. The matrix $A_t$ contributes to $\text{irr}_{\theta}(u_{i})$ either 1 if $r$ is odd or 0 if $r$ is even. Hence

$$\text{irr}_{\theta}(u_{i}) \geq 1 + 2 + \cdots + \frac{t}{2} + \frac{t}{2} + \left(\frac{t}{2} - 1\right) + \left(\frac{t}{2} - 2\right) + \cdots + 2$$

$$= \frac{1}{4} t^2 + \frac{1}{2} t - 1 > \theta \quad \text{by (10)}.$$

Moreover, because the column 1 of the $r \times t$ submatrix $A_t$ comprises exclusively entries 1 unless $r = t, t + 1$ and then 1s are followed by a single 1,

$$\text{irr}_{\theta}(u_{t+1}) = 1 + 2 + 4 + 6 + \cdots + (t - 2) + \begin{cases} r & \text{for } r < t, \\ r - 2 & \text{for } r = t, t + 1 \end{cases}$$

$$= \frac{1}{4} t^2 - \frac{1}{2} t + 1 + \begin{cases} r & \text{for } r < t, \\ r - 2 & \text{for } r = t, t + 1 \end{cases} > \theta \quad \text{by (10)}.$$

Hence, because the transformation $A \mapsto A'$ does not spoil the skew-symmetry of the sequence of degree pairs,

$$\text{irr}_{\theta}(u_{i}) > \theta.$$

Thus, degree pairs in $U$ are away from those in $V$.

Furthermore, the transformation $A \mapsto A'$ changes no more than the indegrees of $u_0, u_1, \ldots, u_{\frac{t}{2}}$ only and outdegrees of $u_{\frac{t}{2}+1}$ and $u_t$ only. Hence, due to Claim 4, degree pairs of vertices in $U$ remain mutually distinct. Therefore, the matrix $A'$ determines both a required $B$ and a smallest irregular oriented graph $F$.

**Case II:** The number $t$ is odd and $t \geq 5$.

Let $A'$ be the matrix obtained from $A$ by interchanging $a_{ij}$ and $a_{ji}$ for each $j$ such that $a_{ij} = 1$ and $a_{ji} = 0$. Notice that for $i \leq \frac{t-1}{2}$ the central column $\frac{t+1}{2}$ of the submatrix $A_t$ of $A$ is the column of zeros. Hence, all $i$ entries 1 which are in the first column of $A_t$ are moved to the central column in order to make up the corresponding submatrix $A'_t$ of $A'$. Let $\frac{t-1}{2} < i < t$. Then, for any row of $A_t$, if zero occurs in the central column $\frac{t+1}{2}$ then 1 occurs in the first column. Hence, $t-i$ entries 1 are moved from and $2i-t$ entries 1 are left in the column 1 of $A_t$. Therefore the submatrix $A'_t$ has $\frac{t+1}{2}$ entries 1 at the top of the column $\frac{t+1}{2}$ which are followed by $i - \frac{t-1}{2}$ entries 1 at the bottom. Finally, $A'_t = A_t$ and the matrix $A_t$ contributes to $\text{irr}_{\theta}(u_{i})$ either 1 if $r$ is odd or 0 if $r$ is even. Hence

$$\text{irr}_{\theta}(u_{i}) \geq 1 + 2 + \cdots + \frac{t-1}{2} + \frac{t-1}{2} + \left(\frac{t-1}{2} - 1\right) + \cdots + 1$$

$$= \frac{1}{4} (t^2 - 1) > \theta \quad \text{by (10)}.$$

Similarly

$$\text{irr}_{\theta}(u_{t+1}) = 1 + 3 + \cdots + (t - 2) + \begin{cases} r & \text{for } r < t, \\ r - 2 & \text{for } r = t, t + 1 \end{cases}$$

$$= \frac{1}{4} (t - 1)^2 + \begin{cases} r & \text{for } r < t, \\ r - 2 & \text{for } r = t, t + 1 > \theta \quad \text{by (10)}.$$

Then the matrix $A'$ determines what is required.
4. The independence number

Notice that the number of vertices of the graph \( F = F(s, 0) \) constructed in the proof of Theorem 4 for \( \rho = 0 \) is different from all natural numbers

\[
a_k := \nu(k) + k \quad \text{for } k = 1, 2, \ldots,
\]

whence, by (1), \( a_k = 2, 5, 9, \ldots \), i.e., \( a_k = \frac{1}{2}k(k + 3) \).

**Corollary 5.** If \( N \) is the smallest order of an irregular oriented graph, say \( F \), with \( s \) independent vertices (e.g., \( F = F(s, 0) \) in Theorem 4) then \( N = s + \tau(s) \) whence, due to Proposition 3, \( N \) is a natural number such that \( N \neq a_k \) (see (12)) where \( k \in \mathbb{N} \). Moreover, if \( F \) is nontrivial and connected then the order, \( N' \), of \( F \) is \( s + \tau(s + 1) \) whence \( N' 
eq a_k - 1 = 1, 4, 8, \ldots \) (for \( k \in \mathbb{N} \)).

**Proof.** To get the smallest among connected irregular oriented graphs with \( s \) independent vertices we delete one isolated vertex from the irregular oriented graph \( F(s + 1, 0) \) constructed in the proof of Theorem 4. \( \square \)

Recall that the *independence number* \( \alpha(D) \) of a digraph \( D \) denotes the maximum cardinality among independent sets in \( D \).

**Theorem 6.** Let \( F \) be an \( n \)-vertex irregular oriented graph. Then

\[
\alpha := \alpha(F) \leq \begin{cases} n - \left\lfloor \frac{\sqrt{2n + 1} - \frac{1}{2}}{} \right\rfloor + 1 & \text{for } n \neq a_k \text{ with } k \in \mathbb{N}, \\ n - \left\lfloor \frac{\sqrt{2n} - \frac{1}{2}}{} \right\rfloor & \text{otherwise}, \end{cases}
\]

the inequality being sharp.

**Proof.** Due to Corollary 5, \( \alpha + \tau(\alpha) \leq n \), \( \tau(\alpha) \) being a nondecreasing function of \( \alpha \). Assume that \( F \) is chosen so that \( \alpha \) is as large as possible.

*Case I:* The number \( n \neq a_k \) for all \( k \in \mathbb{N} \), cf. (12).

Then \( n = \alpha + \tau(\alpha) \) (and \( F \) can equal \( F(s, 0) \), cf. Theorem 4). Let \( t = \tau(\alpha) \). Then, equivalently, \( \alpha = \nu(t) + r \) where \( 1 \leq r \leq t + 1 \) by Proposition 3. Moreover, \( n = \alpha + t \). Hence

\[
\nu(t) + t < n \leq \nu(t) + t + t + 1
\]

which means that \( \tau(\alpha) \) is the largest integer/(smallest integer) \( t \), \( t \geq 0 \), such that the first/second inequality therein holds. Because all sides of the two inequalities are integers, one can see that

\[
t = \tau(\alpha) = \left\lfloor \sqrt{2n + \frac{9}{4} - 2l_1} - \frac{3}{2} \right\rfloor = \left\lfloor \sqrt{2n + \frac{9}{4} + 2l_2} - \frac{5}{2} \right\rfloor
\]

for any reals \( l_1 \) and \( l_2 \) such that \( -1 < l_1, l_2 \leq 1 \). Because the largest possible value of \( \alpha \) is \( n - t \), this implies our Theorem in Case I.

*Case II:* \( n = a_k \) for some \( k \in \mathbb{N} \).

Then \( \alpha + \tau(\alpha) = n - 1 \). (Moreover, deleting the isolated vertex from \( F(s + 1, 0) \) gives an exemplary \( F \).) Hence, the required upper bound on \( \alpha \) can be obtained by substituting \( n \leftarrow n - 1 \) into one proved in Case I. This can be seen to complete the proof. \( \square \)

**Corollary 7.** Some irregular oriented graphs make up a family comprising digraphs of any order and with almost all vertices in an independent set.

5. Superexponential cardinality

**Lemma 8.** There exists an injection \( D \mapsto F \) which with any oriented graph \( D \) of order \( n \) associates irregular oriented graph \( F \) of order \( 2n \) such that \( D \) can easily be recognized as an induced subdigraph of \( F \).
Proof. Set $V = V(D)$. Let $U$ be another set of $n$ vertices, $U := \{u_i : i = 1, \ldots, n\}$, disjoint from $V$, $U \cap V = \emptyset$. Construct an irregular oriented graph $F$ as the edge-disjoint union of oriented graphs $D$ and $B$ where $B$ is a bipartite digraph induced by $V - U$ arcs. Order the vertices in $V$ so that their indegrees increase in $D$,

$$\text{id}_{D}(v_i) \leq \text{id}_{D}(v_{i+1}) \quad \text{for} \quad i = 1, \ldots, n - 1. \tag{13}$$

To get the bipartite oriented graph $B$, join $u_i$ to each $v_j$ with $i \leq j \leq n$ by arcs $(u_i,v_j)$ for $i=1,\ldots,n$. Then $(\text{od}_{B}(u_i),\text{id}_{B}(u_i)) = (n+1-i,0)$ for $i=1,\ldots,n$ and $(\text{od}_{B}(v_i),\text{id}_{B}(v_i)) = (0,i)$ for $i=1,\ldots,n$. This together with (13) implies that the indegrees of vertices $v_i$ strictly increase in $F$ if $i$ increases. Then the resulting $2n$-vertex oriented graph $F$ is an irregular oriented graph which contains $D$ as an induced subdigraph. Moreover, having constructed $F$ we can uniquely recover the original oriented graph $D$. Namely, $D$ is induced by all $n$ vertices of nonzero indegree in $F$. Thus, the injection exists.

Remark. Notice that the injection constructed in the above proof is not uniquely determined. Another injection is obtainable if $V$ is ordered so that outdegrees in $D$ increase and all arcs of $B$ go from $V$ to $U$.

Let $\text{or}(n)$ and $\text{io}(n)$ denote the numbers of nonisomorphic $n$-vertex digraphs which are oriented graphs and irregular oriented graphs, respectively.

Theorem 9. There are superexponentially many irregular oriented graphs on $n$ vertices.

Proof. It follows from the Lemma 8 that there are at least as many irregular oriented graphs of order $2n$ as there are oriented graphs of order $n$. Furthermore, if we add one isolated vertex to the oriented graph $F$ constructed in proof of Lemma 8, we obtain an irregular oriented graph of order $2n + 1$ containing $D$ as an induced subdigraph. Consequently, there are at least as many irregular oriented graphs of order $2n + 1$ as there are oriented graphs of order $n$. Therefore

$$\text{io}(n) \geq \text{or}\left(\binom{n+1}{2}\right).$$

From [17] (with errata), $\text{or}(n) = \frac{n^2}{2^\gamma(1+o(1))}$. Hence

$$\text{io}(n) \geq \frac{n^2}{2^\gamma(1+o(1))}$$

which proves the theorem.

Remark. The lower bound on $\text{io}(n)$ can be improved by using a strengthening [16, Theorem 1] of Lemma 8, with order $n + 2 \left\lceil \sqrt{n} \right\rceil$ in place of $2n$ in the Lemma. For instance, if $n = a^2 - 1$ for an integer $a \geq 2$ then

$$\text{io}(n) \geq \text{or}\left(n + 2 - 2\sqrt{n + 1}\right).$$

References


