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KP hierarchy for Hodge integrals

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Abstract

Starting from the ELSV formula, we derive a number of new equations on the generating functions for Hodge integrals over the moduli space of complex curves. This gives a new simple and uniform treatment of certain known results on Hodge integrals like Witten's conjecture, Virasoro constrains, Faber's λ_g-conjecture, etc. Among other results we show that a properly arranged generating function for Hodge integrals satisfies the equations of the KP hierarchy.

Keywords: Witten's conjecture; Moduli spaces; KP hierarchy; Hurwitz numbers; ELSV formula

1. Introduction

By Hodge integrals we mean intersection numbers of the form

$$\langle \lambda_j \tau_{k_1} \dots \tau_{k_n} \rangle = \int\limits_{\overline{\mathcal{M}}_{g,n}} \lambda_j \psi_1^{k_1} \dots \psi_n^{k_n},$$

where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of complex stable curves with n ordered marked points, ψ_i is the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,n}$ formed by the cotangent lines at the ith marked point and λ_j is the jth Chern class of the rank g Hodge vector bundle whose fibers are the spaces of holomorphic one-forms. These numbers are well defined whenever the equality

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$j+\sum k_i = 3g-3+n$ $(=\dim \overline{\mathcal{M}}_{g,n})$ holds. They play an important role in various problems related to the Gromov–Witten theory.

There are several approaches to intersection theory on moduli spaces. Among those approaches the one that seems to be the most simple and the most straightforward is that based on the Ekedahl–Lando–Shapiro–Vainshtein (ELSV) formula [3]. This formula expresses Hurwitz numbers enumerating ramified coverings of the sphere as linear combinations of Hodge integrals. The ELSV formula can be inverted in order to get some information about Hodge integrals from known facts about Hurwitz numbers.

The ELSV formula was applied successfully in many papers, see, e.g. [1,10,11,17,18]. In this note, we undertake a revision of the methods developed in those papers. One of our main goals is to give a complete and clear description of the relationship between various known equations for Hurwitz numbers on one side and Hodge integrals on the other side: equations of integrable hierarchies, the cut-and-join equation, Virasoro constrains, λ_g-conjecture etc.

The method used in the present paper is close to that of [11]. The main difference is in a new change of variables inverting the ELSV formula. The advantage of the new change is that it induces an automorphism of the KP hierarchy. This permits one to derive the whole hierarchy of PDE's at once for the generating function of Hodge integrals, and, as a consequence, for the Witten's potential participating in his conjecture.

The change of variables explored in the present paper is motivated by that of Goulden–Jackson–Vakil used in [10] in their proof of the λ_g-conjecture and also in the paper [1] devoted to the new derivation of the Virasoro constrains for intersection numbers of ψ classes. In both papers the GJV change is done via the so called symmetrization operation. The symmetrization is used as an intermediate step of computations and is not used in the formulation of the final result. It is natural, therefore, to try to skip the symmetrization operation and to apply the GJV change directly to the original generating function. The main difficulty appearing in this approach is technical: one has to make a change of variables in a differential operator containing infinitely many summands and involving infinitely many variables. To overcome this difficulty, we apply here the machinery of the boson–fermion correspondence. It allows one to reduce the manipulation with differential operators in an infinite dimensional space to those in just one variable and containing finitely many terms. This makes all computations quite elementary and free of combinatorial difficulties (no infinite sums are involved).

2. Main results

Let us collect Hodge integrals into the following formal series in an infinite set of formal commuting variables $u, T_0, T_1, \ldots$:

$$\sum_{j,k_0,k_1,\ldots} (-1)^j \langle \lambda_j \tau_0^{k_0} \tau_1^{k_1} \ldots \rangle u^{2j} \frac{T_0{}^{k_0}}{k_0!} \frac{T_1{}^{k_1}}{k_1!} \cdots, \tag{1}$$

where the summation is taken over all possible monomials in the symbols τ_i and over all possible values $j \geqslant 0$. Denote by $G(u; q_1, q_2, \ldots)$ the series obtained from (1) by the following linear substitution of variables

$$T_0 = q_1,$$
$$T_1 = u^2 q_1 + 2u q_2 + q_3,$$

$$T_2 = u^4 q_1 + 6u^3 q_2 + 12u^2 q_3 + 10u q_4 + 3q_5,$$
$$T_3 = u^6 q_1 + 14u^5 q_2 + 61u^4 q_3 + 124u^3 q_4 + 131u^2 q_5 + 70u q_6 + 15q_7,$$
$$\ldots.$$

In general, the linear combination T_k of the variables q_i is defined as follows (the meaning of this change will be explained in the next section). Consider the sequence of polynomials $\varphi_k(u,z)$, $k = 0, 1, 2, \ldots$, defined by

$$\varphi_0(u,z) = z, \qquad \varphi_{k+1}(u,z) = D\varphi_k(u,z) = D^{k+1}\varphi_0(u,z), \quad \text{where } D = (u+z)^2 z\frac{\partial}{\partial z},$$
$$\varphi_0 = z, \quad \varphi_1 = u^2 z + 2uz^2 + z^3, \quad \varphi_2 = u^4 z + 6u^3 z^2 + 12u^2 z^3 + 10uz^4 + 3z^5, \quad \ldots.$$

Then T_k is obtained form φ_k by replacing z^m by q_m in each monomial. Equivalently, T_k is given by the following recursive equation

$$T_{k+1} = \sum_{m\geqslant 1} m\big(u^2 q_m + 2u q_{m+1} + q_{m+2}\big)\frac{\partial}{\partial q_m} T_k. \tag{2}$$

By construction, T_k is a linear combination of variables q_s, besides, the variable with the maximal index $s = 2k+1$ enters with the coefficient $(2k-1)!!$, and the coefficients of the variables with smaller indices contain positive powers of the parameter u.

Remark that the result of substitution $u = 0$ to G only depends on variables q_k with odd k and it turns into the Witten's potential F for the intersection numbers of ψ classes after rescaling $q_{2d+1} = \frac{t_d}{(2d-1)!!}$.

Theorem 2.1. *The series G is a solution of the KP hierarchy with respect to the variables q_i (identically in u).*

Witten's conjecture (now Kontsevich's theorem, see [13,19]) claims that F is a solution of the KdV hierarchy. This statement is an obvious specialization of the previous theorem. Indeed, the equations of the KdV hierarchy are obtained from the equations of the KP hierarchy by an additional requirement that the function is independent of even variables.

As it was shown by C. Faber [4] (based on earlier result of Mumford [15], see Section 9 below), the computation of Hodge integrals can be reduced to the computation of the intersection indices of ψ classes. In other words, all coefficients of the series G are determined by the coefficients of F. Therefore, one can try to derive Theorem 2.1 from the statement of Witten's conjecture. However, our direct arguments are based on the application of the ELSV formula relating Hodge integrals to the Hurwitz numbers. The Hurwitz numbers participating in this formula are discussed in the next section. Here we only remark that they are relatively simple combinatorial objects, in particular, the generating series $H(\beta; p_1, p_2, \ldots)$ for these numbers can be given by the following explicit closed formula

$$e^H = e^{\beta M_0} e^{p_1}, \tag{3}$$

where M_0 is the so-called *cut-and-join operator*,

$$M_0 = \frac{1}{2}\sum_{i,j}\left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\right).$$

As we shall see in Section 4, the very existence of a formula like (3) implies immediately the following statement.

Theorem 2.2. *(Cf. [11,16].) The generating function H for Hurwitz numbers is a solution of the KP hierarchy with respect to the variables p_i (identically in the formal parameter β).*

The ELSV formula expresses the Hurwitz numbers (the coefficients of H) in terms of the Hodge integrals (the coefficients of G). By formal manipulations this formula can be reduced to the following one.

Consider two variables x and z related to one another by the changes of variables

$$x = \frac{z}{1+\beta z}e^{-\frac{\beta z}{1+\beta z}} = z - 2\beta z^2 + \frac{7}{2}\beta^2 z^3 - \frac{17}{3}\beta^3 z^4 + \cdots,$$
$$z = \sum_{b\geqslant 1}\frac{b^b}{b!}\beta^{b-1}x^b = x + 2\beta x^2 + \frac{9}{2}\beta^2 x^3 + \frac{32}{3}\beta^3 x^4 + \cdots. \tag{4}$$

The fact that these changes of variables are inverse to one another follows from the Lagrange inversion theorem, see [7,9]. These changes provide a linear isomorphism (depending on the parameter β) of the spaces of formal power series in the variables x and z. If we identify the linear span of the variables p_i with the space of formal power series in x and the linear span of the variables q_i with the space of series in z by means of the correspondence

$$p_b \leftrightarrow x^b, \qquad q_k \leftrightarrow z^k, \tag{5}$$

then the isomorphism above provides a linear change of variables (depending on the parameter β) between variables p_i and q_j. More explicitly, this change is given by

$$p_b = \sum_{k\geqslant b} c_k^b \beta^{k-b} q_k, \tag{6}$$

where the rational coefficients c_k^b are determined by the expansion

$$x^b = \sum_{k\geqslant b} c_k^b \beta^{k-b} z^k.$$

Let us set also

$$H_{0,1} = \sum_{b=1}^{\infty}\frac{b^{b-2}}{b!}p_b\beta^{b-1}, \qquad H_{0,2} = \frac{1}{2}\sum_{b_1,b_2=1}^{\infty}\frac{b_1^{b_1}b_2^{b_2}}{(b_1+b_2)b_1!b_2!}p_{b_1}p_{b_2}\beta^{b_1+b_2}. \tag{7}$$

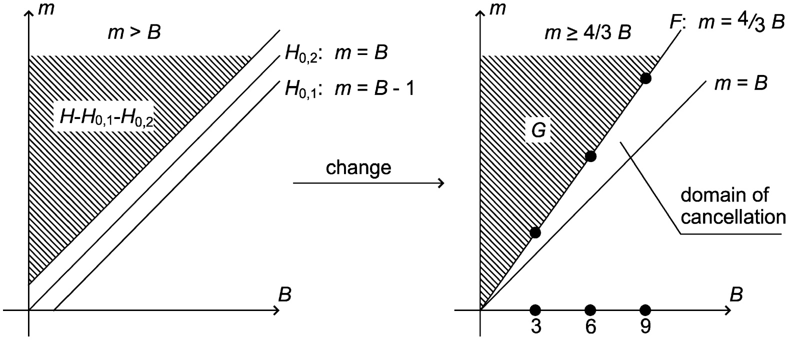

Fig. 1.

Theorem 2.3. *The change* (6) *in $H - H_{0,1} - H_{0,2}$ leads to the series G of Theorem* 2.1*, up to the rescaling $q_k \mapsto \beta^{\frac{4}{3}k} q_k$ and $u \mapsto \beta^{\frac{1}{3}}$, where $H = H(\beta; p_1, p_2, \ldots)$ is the generating function for Hurwitz numbers:*

$$(H - H_{0,1} - H_{0,2})|_{p=p(\beta;q)} = G\bigl(\beta^{\frac{1}{3}}; \beta^{\frac{4}{3}} q_1, \beta^{\frac{8}{3}} q_2, \beta^{\frac{12}{3}} q_3, \ldots\bigr).$$

Remark 2.4. The action of the change (6) described by Theorem 2.3 can be characterized as follows. Consider the plane of coordinates (m, B) and mark all points of this plane corresponding to non-trivial terms of the form const$p_{b_1} \ldots p_{b_n} \beta^m$ in the series H, where $B = \sum b_i$ (see Fig. 1). There are no marked points below the line $m = B - 1$. The points lying on the lines $m = B - 1$ and $m = B$ correspond to the contributions of $H_{0,1}$ and $H_{0,2}$ to H, respectively. The points, corresponding to the contribution of the remaining terms are situated above the diagonal $m = B$. The change (6) determines a linear transformation of the coefficients in H. Furthermore, a coefficient of the original series may contribute to some coefficient of the resulting series only if the corresponding points lie on the same diagonal $m - B =$ const and the point corresponding to the original series has smaller coordinates m and B. After the application of this change to the series $H - H_{0,1} - H_{0,2}$, we get, after some 'magic cancellations', a series having no non-trivial coefficients in the domain $m < \frac{4}{3}B$. The terms of the resulting series lying on the line $m = \frac{4}{3}B$ form the generating series F for intersection numbers of ψ classes and the other terms correspond to other summands of the series G of Theorem 2.1 (after the rescaling $q_i \mapsto \beta^{-\frac{4}{3}i} q_i$, $\beta^{\frac{1}{3}} \mapsto u$).

Consider now an arbitrary invertible formal series $x(z)$ and associate to this series a linear change of variables $p \mapsto p(q)$ given by

$$p_b = \sum_{k \geqslant b} c_k^b q_k, \tag{8}$$

whose coefficients c_k^b are determined by the expansion $x^b = \sum_{k \geqslant b} c_k^b z^k$.

Theorem 2.5. *There is a quadratic function $Q(p_1, p_2, \ldots)$ such that the transformation sending an arbitrary series $\Phi(p_1, p_2, \ldots)$ to the series $\Psi(q_1, q_2, \ldots) = (\Phi + Q)|_{p \to p(q)}$ is an authomorphism of the KP hierarchy: it sends solutions to solutions.*

In the case when $x(z) = \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$ this quadratic function is $Q = -H_{0,2}$.

Theorem 2.1 is an obvious corollary of Theorems 2.2, 2.3, and 2.5. Indeed, the transformation of Theorem 2.3 differs from that of Theorem 2.5 by linear terms $H_{0,1}$ that do not affect the equations of the KP hierarchy. The rescaling $q_k \mapsto \beta^{\frac{4}{3}k} q_k$ is also an automorphism of the hierarchy since its equations are quasihomogeneous. Therefore, the validity of these equations for the series G is equivalent to their validity for H, which is guaranteed, in turn, by the assertion of Theorem 2.2.

Remark 2.6. The definition for the change (6) looks unmotivated. Implicitly, the same change was used in [9,10]. The only motivation that we can provide here is that 'it works'. In fact, there is a freedom in the choice of a change inverting the ELSV formula. One of the possible changes was used in [11]. That change did not preserve the whole KP hierarchy but was sufficient to derive the KdV equation for the Witten's potential F.

3. The ELSV formula

This section is devoted to the proof of Theorem 2.3.

Consider a ramified covering of the sphere $S^2 = \mathbb{C}P^1$ by a smooth surface of genus g such that the point $\infty \in \mathbb{C}P^1$ has n marked preimages of multiplicities $b_1, \ldots, b_n$ and all other critical points are simple (with ramification of the second order each) and have pairwise different critical values. The number m of simple critical values is determined by the Riemann–Hurwitz formula:

$$m = 2g - 2 + n + \sum_{i=1}^{n} b_i .$$

The Hurwitz number $h_{g;b_1,\ldots,b_n}$ is defined as the number of such coverings with a fixed position of the critical values; the coverings are counted with their weights inverse to the order of the automorphism group of the covering. The celebrated ELSV formula [3] expresses these numbers via Hodge integrals:

$$\frac{h_{g;b_1,\ldots,b_n}}{m!} = \prod_{i=1}^{n} \frac{b_i^{b_i}}{b_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{\prod_{i=1}^{n}(1 - b_i \psi_i)} .$$

It is convenient to pack Hurwitz numbers into the generating series

$$H(\beta; p_1, p_2, \ldots) = \sum_{n \geqslant 1} \frac{1}{n!} \sum_{g, b_1, \ldots, b_n} h_{g;b_1,\ldots,b_n} \frac{\beta^m}{m!} p_{b_1} \ldots p_{b_n} .$$

The ELSV formula is not applicable to coverings of genus 0 with 1 or 2 preimages at infinity (since the corresponding moduli spaces $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ do not exist). The summands $H_{0,1}$ and $H_{0,2}$, respectively, corresponding to such coverings are presented in Eq. (7) of the previous section. Let us represent the remaining summands in the form $H - H_{0,1} - H_{0,2} = \sum_{n \geqslant 1} \frac{1}{n!} H_n$, where H_n contains all terms corresponding to the coverings with exactly n preimages at infinity. Then, since

$$m = 2g - 2 + n + \sum b_i = \sum \left(b_i + \frac{1}{3}\right) + \frac{2}{3}(3g - 3 + n) = \sum \left(b_i + \frac{1}{3}\right) + \frac{2}{3} \dim \overline{\mathcal{M}}_{g,n},$$

we get

$$H_n = \sum_{g,b_i,\dots,b_n} \prod_{i=1}^{n} \frac{b_i^{b_1} \beta^{b_i+\frac{1}{3}}}{b_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\beta^{\frac{2}{3}}\lambda_1+\beta^{\frac{4}{3}}\lambda_2-\cdots}{\prod_{i=1}^{n}(1-b_i\beta^{\frac{2}{3}}\psi_i)} p_{b_1}\dots p_{b_n}$$

$$= \left\langle \left(1-\beta^{\frac{2}{3}}\lambda_1+\beta^{\frac{4}{3}}\lambda_2-\cdots\right) \prod_{i=1}^{n} \sum_{b\geqslant 1} \frac{b^b}{b!} \frac{\beta^{b+\frac{1}{3}} p_b}{(1-b\beta^{\frac{2}{3}}\psi_i)} \right\rangle$$

$$= \left\langle \left(1-\beta^{\frac{2}{3}}\lambda_1+\beta^{\frac{4}{3}}\lambda_2-\cdots\right) \prod_{i=1}^{n} \sum_{d\geqslant 0} T_d \psi_i^d \right\rangle,$$

where

$$T_d = \sum_{b\geqslant 1} \frac{b^{b+d}}{b!} \beta^{b+\frac{1}{3}+\frac{2}{3}d} p_b,$$

and where for each monomial in the classes ψ_i and λ_j we denote by angular brackets its integral over the space $\overline{\mathcal{M}}_{g,n}$, $g \geqslant 0$, whose dimension is equal to the degree of the monomial. Setting $u = \beta^{\frac{1}{3}}$ we can rewrite the ELSV formula in the following form (a similar form of the ELSV formula was observed in [9])

$$H - H_{0,1} - H_{0,2} = \sum_{j,k_0,k_1,\dots} (-1)^j \langle \lambda_j \tau_0^{k_0} \tau_1^{k_1} \dots \rangle u^{2j} \frac{T_0{}^{k_0}}{k_0!} \frac{T_1{}^{k_1}}{k_1!} \cdots.$$

It follows that Theorem 2.3 is a corollary of the following lemma.

Lemma 3.1. *The defined above series T_d in the variables p_i turn under the change* (6) *into polynomials in variables q_i participating in the statement of Theorem* 2.1*, up to the rescaling $q_i \mapsto \beta^{\frac{4}{3}i} q_i$, $u \mapsto \beta^{\frac{1}{3}}$.*

Proof. Under the correspondence (5), the series T_d corresponds to the following function in x:

$$T_d \leftrightarrow \beta^{\frac{4}{3}+\frac{2}{3}d} \sum_{b\geqslant 1} \frac{b^{b+d}}{b!} \beta^{b-1} x^b = \left(\beta^{\frac{2}{3}} D\right)^d \beta^{\frac{4}{3}} z(x),$$

where $D = x\frac{\partial}{\partial x}$ and where the series $z(x) = \sum_{b\geqslant 1} \frac{b^b}{b!} \beta^{b-1} x^b$ is inverse to the series

$$x(z) = \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}.$$

It is easy to check that the differential operator $D = x\frac{\partial}{\partial x}$ takes in terms of the coordinate z the form

$$D = (1+\beta z)^2 z \frac{\partial}{\partial z}.$$

Under the correspondence (5) the function $\beta^{\frac{4}{3}}z \leftrightarrow T_0$ turns into $\beta^{\frac{4}{3}}q_1$, and the operator $\beta^{\frac{2}{3}}D$ turns, respectively, into

$$\beta^{\frac{2}{3}}D = \sum_{m\geqslant 1} m\bigl(\beta^{\frac{2}{3}}q_m + 2\beta^{\frac{5}{3}}q_{m+1} + \beta^{\frac{8}{3}}q_{m+2}\bigr)\frac{\partial}{\partial q_m}$$
$$= \sum_{m\geqslant 1} m\bigl(\beta^{\frac{2}{3}+\frac{4}{3}m}q_m + 2\beta^{\frac{1}{3}+\frac{4}{3}(m+1)}q_{m+1} + \beta^{\frac{4}{3}(m+2)}q_{m+2}\bigr)\frac{1}{\beta^{\frac{4}{3}m}}\frac{\partial}{\partial q_m}.$$

After rescaling $\beta^{\frac{4}{3}m}q_m \mapsto q_m$, $\beta^{\frac{1}{3}} \mapsto u$ this operator coincides with the operator participating in the recursive relation (2). This proves Lemma 3.1, and hence, Theorem 2.3. $\square$

4. KP hierarchy

In this section we recall several well known facts about the KP hierarchy that are sufficient for the proof of Theorems 2.2 and 2.5. For a more detailed exposition of the theory we refer to the papers [2,14], and to Section 6 of the present paper. The KP (Kadomtsev–Petviashvili) hierarchy is a particular system of partial differential equations on the unknown function (power series) F in infinite set of variables $p_1, p_2, \ldots$. Here are several first equations of the hierarchy

$$\begin{aligned}
F_{2,2} &= -\frac{1}{2}F_{1,1}^2 + F_{3,1} - \frac{1}{12}F_{1,1,1,1},\\
F_{3,2} &= -F_{1,1}F_{2,1} + F_{4,1} - \frac{1}{6}F_{2,1,1,1},\\
F_{4,2} &= -\frac{1}{2}F_{2,1}^2 - F_{1,1}F_{3,1} + F_{5,1} + \frac{1}{8}F_{1,1,1}^2 + \frac{1}{12}F_{1,1}F_{1,1,1,1} - \frac{1}{4}F_{3,1,1,1} + \frac{1}{120}F_{1,1,1,1,1,1},\\
F_{3,3} &= \frac{1}{3}F_{1,1}^3 - F_{2,1}^2 - F_{1,1}F_{3,1} + F_{5,1} + \frac{1}{4}F_{1,1,1}^2 + \frac{1}{3}F_{1,1}F_{1,1,1,1} - \frac{1}{3}F_{3,1,1,1}\\
&\quad + \frac{1}{45}F_{1,1,1,1,1,1}.
\end{aligned} \tag{9}$$

The exponent $\tau = e^F$ of any solution is called the *τ-function* of the hierarchy. It is known that the space of solutions (or the space of τ-functions) is homogeneous: there is a Lie algebra $\widehat{\mathfrak{gl}(\infty)}$ acting on the space of solutions and the action of the corresponding Lie group is transitive. In other words, any solution can be obtained from any other solution (say, from the solution $\tau = 1$) by the action of an appropriate transformation from this group. The actual definition of the Lie algebra $\widehat{\mathfrak{gl}(\infty)}$ and of its action is given in Section 6. Here we only present several sample operators (acting on τ-functions) belonging to this algebra.

Example 4.1. The scalar operator (the operator of multiplication by a constant) belongs to $\widehat{\mathfrak{gl}(\infty)}$. It follows that the τ-function is defined up to a multiplicative constant. This constant is usually chosen in such a way that $\tau(0) = 1$. In this case the logarithm $F = \log(\tau)$ is a correctly defined power series if the function τ is.

Example 4.2. Set

$$a_k = \begin{cases} p_k & k > 0, \\ 0 & k = 0, \\ (-k)\frac{\partial}{\partial p_{-k}} & k < 0. \end{cases}$$

Then for all k the operator a_k belongs to $\widehat{\mathfrak{gl}(\infty)}$. For positive k this means that addition of a linear function preserves the space of solutions of the hierarchy. For negative k this means that a shift of arguments also preserves the hierarchy. Both assertions are obvious. Indeed, the equations of the hierarchy have constant coefficients and the partial derivatives have order at least two.

Example 4.3. The following operator also belongs to $\widehat{\mathfrak{gl}(\infty)}$ for any integer $m \neq 0$:

$$\Lambda_m = \frac{1}{2}\sum_{i=-\infty}^{\infty} a_i a_{m-i}.$$

The right-hand side is well defined for any $m \neq 0$ since a_i and a_{m-i} commute. For $m = 0$ the formula should be corrected:

$$\Lambda_0 = \sum_{i=1}^{\infty} a_i a_{-i} = \sum_{i=1}^{\infty} i p_i \frac{\partial}{\partial p_i}.$$

For positive m these operators have the following form

$$\Lambda_m = \sum_{i \geqslant 1} i p_{i+m} \frac{\partial}{\partial p_i} + \frac{1}{2}\sum_{j=1}^{m-2} p_j p_{m-1-j}, \quad m > 0. \tag{10}$$

For negative m the operators Λ_m involve second order partial derivatives. The operators Λ_{-2m}, $m \geqslant -1$, participate in the Virasoro equations for the Witten's potential of intersection indices of ψ classes (see Section 5).

Example 4.4. The following operators belong to $\widehat{\mathfrak{gl}(\infty)}$ for any integer m,

$$M_m = \frac{1}{6}\sum_{i,j=-\infty}^{\infty} {:}a_i a_j a_{m-i-j}{:},$$

where we use notation ${:}a_{i_1} \dots a_{i_k}{:} = a_{\sigma(i_1)} \dots a_{\sigma(i_k)}$, where σ is a permutation of the indices $i_1, \dots, i_k$ such that $\sigma(i_1) \geqslant \dots \geqslant \sigma(i_k)$. The operator M_0 is called also the *cut-and-join* operator

$$M_0 = \frac{1}{2}\sum_{i,j=1}^{\infty}\left((i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}\right).$$

It is known (see, e.g. [8]) that the exponent of the generating function for Hurwitz numbers satisfies the cut-and-join equation

$$\frac{\partial e^H}{\partial\beta} = M_0 e^H. \tag{11}$$

Since for $\beta = 0$ the function $H|_{\beta=0} = p_1$ satisfies the KP hierarchy by trivial reason and since the infinitesimal transformation of H with a change of the parameter β belongs to $\widehat{\mathfrak{gl}(\infty)}$, we conclude that H satisfies the KP hierarchy for all parameter values β. This proves, in particular, Theorem 2.2.

More explicitly, the solution of the cut-and-join equation is given by

$$e^H = e^{\beta M_0} e^{p_1}.$$

This expression shows that the τ-function e^H is obtained from the trivial one $1 = e^0$ by the action of the composition of the operators e^{a_1} and $e^{\beta M_0}$ both of which belong to the Lie group of the algebra $\widehat{\mathfrak{gl}(\infty)}$.

We are ready now to fulfill the computations leading to the proof of Theorem 2.5. An infinitesimal version of the change $z \mapsto x(z)$ is a vector field on the line of the coordinate z which can be written as a linear combination of the basic fields $z^{m+1}\frac{\partial}{\partial z}$, $m \geqslant 0$. These fields can be viewed as linear transformations of the space of power series in z:

$$z^{m+1}\frac{\partial}{\partial z} : z^i \mapsto i z^{i+m}.$$

Under the correspondence (5) this operator sending p_i to $i p_{i+m}$ can be written as a linear differential operator $\sum i p_{i+m}\frac{\partial}{\partial p_i}$ which is nothing but the 'differential' part of the operator Λ_m (10). Integrating such infinitesimal transformations we obtain a global linear change of variables of the form (8). The 'polynomial' part of Λ_m is a quadratic form. Since the space of quadratic functions is invariant under linear changes of variables, the first statement of Theorem 2.5 follows.

To prove the second statement, we observe that the change $x(z) = \frac{z}{1+\beta z}e^{-\frac{\beta z}{1+\beta z}}$ can be obtained from the identity as the time β flow of the non-autonomous vector field $-(2z + \beta z^2)z\frac{\partial}{\partial z}$. This assertion is a reformulation of the following easily verified identity

$$\frac{\partial x(z)}{\partial\beta} = -\bigl(2z + \beta z^2\bigr)z\frac{\partial x(z)}{\partial z}.$$

The field $-(2z + \beta z^2)z\frac{\partial}{\partial z}$ corresponds to the operator $-(2\Lambda_1 + \beta\Lambda_2) \in \widehat{\mathfrak{gl}(\infty)}$. It remains to check that the quadratic function $Q = -H_{0,2}$ agrees with the 'polynomial' part of this operator.

Consider an arbitrary series $\Phi(p_1, p_2, \ldots)$ and set $Z(\beta; q) = \exp(\Phi - H_{0,2})|_{p\to p(\beta;q)}$, where the linear change of coordinates $p \to p(\beta; q)$ corresponds to our choice of $x(z)$.

Lemma 4.5. *The series Z is subject to the differential equation*

$$\frac{\partial Z}{\partial\beta} = -(2\Lambda_1 + \beta\Lambda_2)Z.$$

Proof. Differentiating the function $Z = e^{\Phi - H_{0,2}}$ we get

$$\frac{\partial Z}{\partial \beta} = \left(-\frac{\partial H_{0,2}}{\partial \beta}\bigg|_{p=\text{const}} + \sum_{b\geqslant 1} \frac{\partial p_b}{\partial \beta}\frac{\partial}{\partial p_b}\right) Z. \tag{12}$$

We wish to rewrite the right-hand side in q-coordinates. The arguments above show that the second summand is the 'differential' part of the operator $-(2\Lambda_1 + \beta\Lambda_2)$:

$$\sum_{b\geqslant 1} \frac{\partial p_b}{\partial \beta}\frac{\partial}{\partial p_b} = -\sum_{i\geqslant 1} i(2q_{i+1} + \beta q_{i+2})\frac{\partial}{\partial q_i}.$$

(The reader can check this equality by straightforward computations.) For the first summand, we have

$$-\frac{\partial H_{0,2}}{\partial \beta}\bigg|_{p=\text{const}} = -\frac{1}{2}\sum_{i,j\geqslant 1} \frac{i^i}{i!}\frac{j^j}{j!}\beta^{i+j-1} p_i p_j = -\frac{\beta}{2}\left(\sum_{i\geqslant 1} \frac{i^i}{i!}\beta^{i-1} p_i\right)^2 = -\frac{\beta}{2} q_1^2,$$

which coincides with the 'polynomial' part of the operator $-(\beta\Lambda_2 + 2\Lambda_1)$. The lemma is proved.

The lemma shows that Z is a τ-function for any parameter value β if it is for the initial parameter value $\beta = 0$. This completes the proof of Theorem 2.5. □

5. Cut-and-join and the Virasoro constrains

Consider the cut-and-join equation (11) for the generating function of Hurwitz numbers. The change of Theorem 2.3 transforming the series H to the series G of Theorem 2.1 acts on differential equations as well. The cut-and-join equation is transformed under this change into the following one.

Theorem 5.1. *The exponent of the function G of Theorem 2.1 is subject to the differential equation*

$$\frac{1}{3}u^{-2}\frac{\partial e^G}{\partial u} = \left(M_0 + 4u^{-1}M_1 + 6u^{-2}M_2 + 4u^{-3}M_3 + u^{-4}M_4 - \frac{4}{3}u^{-3}\Lambda_0 - u^{-4}\Lambda_1 + \frac{1}{4}u^{-2}a_2 + \frac{1}{3}u^{-3}a_3 + \frac{1}{8}u^{-4}a_4\right)e^G,$$

where the operators a_m, Λ_m, and M_m are defined in the previous section.

One of the possible proofs is the direct substitution. The reader can try to compute himself the coefficients of the equation starting from the definition of the transformation sending H to G. More elementary computations leading to the same equation are explained in Section 7.

Each term in the equation of the theorem has degree greater than or equal to -4 with respect to the variable u. Since the series $F = G|_{u=0}$ only depends on odd variables, we obtain from the explicit form of M_4 that the variables p_{2m} enter the coefficient of u^{-4} at most linearly. Extracting the coefficient of $p_{2m+4}u^{-4}$ in the equation we get the equalities

$$(2m+3)\frac{\partial e^F}{\partial p_{2m+3}} = \left(\Lambda_{-2m} + \frac{1}{8}\delta_{m,0}\right)e^F, \quad m \geqslant -1.$$

These equations known as *Virasoro constrains* for the generating function F of the intersection numbers of ψ-classes form an equivalent reformulation of Witten's conjecture. The derivation of the Virasoro constrains presented above is parallel to that from [1]. I hope, however, that in the presented form the computations look more clear and natural.

6. Boson–fermion correspondence

In this section we review the basics of the boson–fermion correspondence with application to Sato Grassmannian and the KP hierarchy. The basic references are [2] and [14].

Consider the space $\mathbb{C}[[p_1, p_2, \dots]]$ of formal power series and consider the additive basis in this space formed by Schur functions

$$\mathbb{C}[[p_1, p_2, \dots]] \simeq \overline{\bigoplus_\lambda \mathbb{C}s_\lambda(p)}.$$

Schur functions are certain polynomials labelled by partitions (Young diagrams). One of the possible their definitions is given below. Here are several of them

$$s_0 = 1, \qquad s_1 = p_1, \qquad s_2 = \frac{1}{2}\left(p_1^2 + p_2\right), \qquad s_3 = \frac{1}{6}\left(p_1^3 + 3p_1p_2 + 2p_3\right),$$
$$s_{1,1} = \frac{1}{2}\left(p_1^2 - p_2\right), \qquad s_{2,1} = \frac{1}{3}\left(p_1^3 - p_3\right), \qquad s_{1,1,1} = \frac{1}{6}\left(p_1^3 - 3p_1p_2 + 2p_3\right).$$

There is another space with the basis naturally labelled by Young diagrams. Namely, consider first an auxiliary space $V = \mathbb{C}[z^{-1}][[z]]$ of formal Laurent series. The *semi-infinite wedge space* $\Lambda^{\frac{\infty}{2}}V$ is, by definition, (the completion of) the vector space whose basic vectors are semi-infinite formal wedge products of the form

$$v_\lambda = z^{k_1} \wedge z^{k_2} \wedge \cdots, \quad k_i = \lambda_i - i.$$

The sequences $(k_1, k_2, \dots)$ appearing in these products are just arbitrary strictly decreasing sequences of integers such that $k_i = -i$ for sufficiently large i. The elements of the space $\Lambda^{\frac{\infty}{2}}V$ can be represented as linear combinations of infinite wedge products of the form $\varphi_1(z) \wedge \varphi_2(z) \wedge \cdots$, $\varphi_i \in V$, such that $\varphi_i = z^{-i} +$ (terms of higher order in z) for sufficiently large i. Using polylinearity and skew-symmetry of the wedge product one can represent such a wedge product as a (possibly infinite) linear combination of basic ones.

The *boson–fermion correspondence* is the coordinate-wise isomorphism of vector spaces

$$\mathbb{C}[p_1, p_2, \dots] \simeq \Lambda^{\frac{\infty}{2}}V, \quad s_\lambda \leftrightarrow v_\lambda.$$

The spaces on the left- and the right-hand sides of the isomorphism are called *bosonic* and *fermionic Fock spaces* (of zero charge), respectively. The vector $v_\varnothing = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge \cdots$ corresponding to the function $s_\varnothing = 1$ is called the *vacuum vector*.

The geometric viewpoint to the theory of KP hierarchy is formulated as follows.

Theorem 6.1. *The function $\tau \in \mathbb{C}[[p_1, p_2, \ldots]]$ is the exponent of a solution of the KP hierarchy if and only if its image under boson–fermion correspondence can be represented by a decomposable wedge product*

$$\tau \leftrightarrow \varphi_1(z) \wedge \varphi_2(z) \wedge \cdots.$$

Decomposable wedge products are characterized uniquely up to a multiplicative constant by the linear span of the vectors φ_i. Therefore, the theorem has geometric reformulation that τ-functions form the cone over the Grassmannian $G_{\frac{\infty}{2}}(V)$ of half-infinite subspaces Plücker embedded to the projective space $P\Lambda^{\frac{\infty}{2}}(V)$. It is known in algebraic geometry that the Plücker embedding of the Grassmannian is given by quadratic equations. These algebraic equations on the Taylor coefficients of the series τ are called *Hirota bilinear equations*. They can be represented in a form of partial differential equations on τ. These equations rewritten in terms of the logarithm $F = \log(\tau)$ are exactly the equations of the KP hierarchy.

Example 6.2. The wedge product $(z^{-1} + z^2) \wedge z^{-2} \wedge z^{-3} \wedge \cdots = v_0 + v_3$ corresponds to the function

$$s_0 + s_3 = 1 + \frac{1}{6}\big(p_1^3 + 3p_1p_2 + 2p_3\big).$$

It follows that its logarithm $\log(1 + \frac{1}{6}(p_1^3 + 3p_1p_2 + 2p_3))$ satisfies the equations of the KP hierarchy. The reader may check this fact by substituting to the first of Eq. (9). These computations being elementary occupy several pages and are quite laborious when made by hand.

The Grassmannian is a homogeneous space; every subspace can be obtained from any other by a linear transformation. The infinite dimension of the spaces under consideration implies some additional phenomena that we describe now. Denote by $\mathfrak{gl}(\infty)$ the Lie algebra of differential operators in one variable z with Laurent coefficients (completed in a suitable way that we do not discuss here). Every element of this algebra can be treated as a linear operator acting on the space V. This operator is represented in the basis z^i, $i \in \mathbb{Z}$, by an infinite matrix $a_{i,j}$, $z^j \mapsto \sum_i a_{ij} z^i$. This algebra is graded by the agreement $\deg(z^i(\partial/\partial z)^j) = i - j$. For the operators of degree k all non-zero components $a_{i,j}$ are situated on the diagonal $i - j = k$, moreover, the component $a_{i,i-k}$ has polynomial dependence on i.

To every operator $A \in \mathfrak{gl}(\infty)$ we associate the operator $\widehat{A}$ acting on the fermionic space by the following rule. If the matrix of A has no non-trivial diagonal elements, then the action of $\widehat{A}$ is determined by the Leibnitz rule:

$$\begin{aligned}\widehat{A}\big(z^{k_1} \wedge z^{k_2} \wedge \cdots\big) = {} & A\big(z^{k_1}\big) \wedge z^{k_2} \wedge z^{k_3} \wedge \cdots \\ & + z^{k_1} \wedge A\big(z^{k_2}\big) \wedge z^{k_3} \wedge \cdots \\ & + z^{k_1} \wedge z^{k_2} \wedge A\big(z^{k_3}\big) \wedge \cdots + \cdots.\end{aligned}$$

In the case when A has zero grading, that is, when $A = \mathrm{Diag}(\dots, a_{-1}, a_0, a_1, \dots)$ is diagonal, the application of the above formula may lead to divergence. Therefore, the action of $\widehat{A}$ should be regularized and we set, by definition,

$$\widehat{A}\bigl(z^{k_1} \wedge z^{k_2} \wedge \cdots\bigr) = \sum_{i=1}^{\infty} (a_{k_i} - a_{-i}) z^{k_1} \wedge z^{k_2} \wedge \cdots.$$

Example 6.3. Consider the operator $z\frac{\partial}{\partial z} \in \mathfrak{gl}(\infty)$, $z^k \mapsto kz^k$. Its image $\widehat{z\partial/\partial z}$ is called the *energy operator*. Every basic vector v_λ is an eigenvector for this operator with the corresponding eigenvalue $|\lambda| = \sum \lambda_i$. The eigenvalues of the energy operator give rise to the grading on the space $\Lambda^{\frac{\infty}{2}} V$. Under boson–fermion correspondence this grading corresponds to the quasihomogeneous grading in the space of power series in the variables p_i with $\deg p_i = i$. In other words, $\widehat{z\partial/\partial z} = \sum_{i=1}^{\infty} i p_i \frac{\partial}{\partial p_i}$.

The correspondence $A \mapsto \widehat{A}$ is *not* a Lie algebra homomorphism. For example, the operators of the multiplication by x^m commute for different m, but one can easily compute that $[\widehat{x^m}, \widehat{x^n}] = n\delta_{m,-n}$. In fact, one has $[\widehat{A}, \widehat{B}] = \widehat{[A, B]} +$ (scalar operator). Therefore, we obtain not a linear but a *projective* representation of the Lie algebra $\mathfrak{gl}(\infty)$. More exactly, the value of the scalar correction forms a cocycle. Therefore, we obtain a linear representation not of the algebra $\mathfrak{gl}(\infty)$ itself but of its one-dimensional central extension denoted by $\widehat{\mathfrak{gl}(\infty)}$. This algebra is generated by the operators of the form $\widehat{A}$ and by the scalar operators.

Remark, however, that the most part of the operators used in the present paper lie in the subalgebra of upper triangular matrices (that is, the operators of non-negative grading). The cocycle is trivial on this subalgebra and the correspondence $A \mapsto \widehat{A}$ is a Lie algebra homomorphism.

Proposition 6.4. *The action of the Lie group of the algebra* $\widehat{\mathfrak{gl}(\infty)}$ *preserves the set of decomposable vectors.*

Indeed, if the matrix of A has zero diagonal entries, then

$$e^{t\widehat{A}}\bigl(\varphi_1 \wedge \varphi_2 \wedge \cdots\bigr) = e^{tA}\varphi_1 \wedge e^{tA}\varphi_2 \wedge \cdots.$$

Similarly, if $A = \mathrm{Diag}(\dots, a_{-1}, a_0, a_1, \dots)$, then

$$e^{t\widehat{A}}\bigl(\varphi_1 \wedge \varphi_2 \wedge \cdots\bigr) = e^{t(A-a_1)}\varphi_1 \wedge e^{t(A-a_2)}\varphi_2 \wedge \cdots.$$

The proposition suggests a natural way to construct solutions of the KP hierarchy. It is sufficient to pick any transformation from the corresponding Lie group and to apply it to the vacuum vector. The resulting vector corresponds under boson–fermion correspondence to the τ-function of some solution. In order to apply this procedure in practice, it is useful to have an explicit description of the action of the Lie algebra $\widehat{\mathfrak{gl}(\infty)}$ in terms of the variables p_i.

Proposition 6.5. *The correspondence between the differential operators in* z *of order at most two and the action of these operators in* $\mathbb{C}[[p_1, p_2, \dots]]$ *is presented in Table* 1.

Table 1

Notation	Action in $V=\mathbb{C}[[z]][z^{-1}]$	Action in $\mathbb{C}[[p_1,p_2,\dots]]$
$a_m,\ m>0$	z^m	p_m
$a_{-m},\ m>0$	z^{-m}	$m\frac{\partial}{\partial p_m}$
a_0	1	0
Λ_m	$z^m(z\frac{\partial}{\partial z}+\frac{m+1}{2})$	$\frac{1}{2}\sum_{i=-\infty}^{\infty}{:}a_i a_{m-i}{:}$
M_m	$z^m(\frac{1}{2}(z\frac{\partial}{\partial z})^2+\frac{m+1}{2}z\frac{\partial}{\partial z}+\frac{(m+1)(m+2)}{12})$	$\frac{1}{6}\sum_{i,j=-\infty}^{\infty}{:}a_i a_j a_{m-i-j}{:}$

The equality $\widehat{z^m}=a_m=p_m$ can be used for an independent invariant definition of the boson–fermion correspondence. Namely, the polynomial (or the formal series) $P(p_1,p_2,\dots)$ corresponds to the vector $P(a_1,a_2,\dots)v_\varnothing$ of the semi-infinite wedge space. Conversely, any function can be recovered from its partial derivatives by the Taylor formula

$$P(p_1,p_2,\dots)=e^{\sum p_m\frac{\partial}{\partial q_m}}P(q_1,q_2,\dots)\big|_{q=0}.$$

Therefore the equality $\widehat{z^{-m}}=a_{-m}=m\frac{\partial}{\partial p_m}$ can be used for the inverse homomorphism of the boson–fermion correspondence: the vector $v\in\Lambda^{\frac{\infty}{2}}V$ corresponds to the series

$$v\leftrightarrow\left\langle e^{\sum\frac{p_m a_{-m}}{m}}v\right\rangle_0,$$

where $\langle\cdot\rangle_0$ denotes the coefficient of the vacuum vector. If these equalities are taken as the definition of the boson–fermion correspondence, then its coordinate presentation given in the beginning of this section can serve as the definition of Schur functions.

For the operators Λ_m the correspondence of the table is proved, for example, in [12] (in a slightly different normalization). The case of the cut-and-join operator M_0 is treated in [11] in the relationship with the Hurwitz theory: this operator is diagonal and the correspondence is established by comparison of the eigenvalues. Finally, for the operators M_n, $n\neq 0$, the correspondence follows from the commutating relations

$$2nM_n=[M_0,\Lambda_n]-\frac{n^3-n}{12}a_n,$$

that can be checked independently both in the space of differential operators in one variable z and in the space of differential operators in the variables $p_1,p_2,\dots$.

Example 6.6. Since $\widehat{1}=0$, the cut-and-join operator can be represented in the form $M_0=\widehat{\frac{1}{2}(z\frac{\partial}{\partial z})^2+\frac{1}{2}z\frac{\partial}{\partial z}+\frac{1}{6}}=\frac{1}{2}(\widehat{z\frac{\partial}{\partial z}+\frac{1}{2}})^2$. Therefore the exponent e^H of the generating function for the Hurwitz numbers (see Example 4.4) corresponds to the infinite wedge product

$$e^{\beta M_0}e^{a_{-1}}v_\varnothing=\varphi_1\wedge\varphi_2\wedge\varphi_3\wedge\cdots,$$

where

$$\varphi_k=e^{\frac{\beta}{2}[(z\frac{\partial}{\partial z}+\frac{1}{2})^2-(\frac{1}{2}-k)^2]}e^z z^{-k}=\sum_{i=0}^{\infty}e^{\frac{\beta}{2}[(i-k+\frac{1}{2})^2-(\frac{1}{2}-k)^2]}\frac{z^{-k+i}}{i!}.$$

7. Once again about the change of coordinates in the ELSV

In the framework of the boson–fermion correspondence, the proof of Theorem 2.5 takes the following form. Consider the transformation Ξ sending any function $\Phi(p)$ to the function $\Psi(q) = (\Phi - H_{0,2})|_{p\to p(q)}$. This transformation belongs to the Lie group of the Lie algebra $\widehat{\mathfrak{gl}(\infty)}$, therefore, it can be considered, via the boson–fermion correspondence, as a linear transformation Ξ of the space L of Laurent series in z.

Proposition 7.1. *The transformation Ξ is given explicitly by*

$$\Xi : \varphi(z) \mapsto \psi(\beta, z) = (1+\beta z)^{-\frac{3}{2}} e^{-\frac{\beta z}{2(1+\beta z)}} \varphi\left(\frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}\right). \tag{13}$$

Proof. The transformation Ξ is obtained by integrating the differential equation of Lemma 4.5. Using the correspondence from the table of Proposition 6.5 one can rewrite this equation as the following one:

$$\begin{aligned} \frac{\partial\psi}{\partial\beta} &= -(2\Lambda_1 + \beta\Lambda_2)\psi \\ &= -2\left(z^2\frac{\partial\psi}{\partial z} + z\psi\right) - \beta\left(z^3\frac{\partial\psi}{\partial z} + \frac{3}{2}z^2\psi\right) \\ &= -\left(2z^2 + \beta z^3\right)\frac{\partial\psi}{\partial z} - \left(2z + \frac{3}{2}\beta z^2\right)\psi. \end{aligned} \tag{14}$$

This linear PDE can be solved explicitly using, for example, the method of characteristics. The solution is presented in the formula of the proposition (as long as the formula is presented, there is no difficulty to check its validity by the direct substitution to the equation). The proposition is proved. □

The statement of the proposition is applicable also if the initial function φ depends on additional parameters. Consider the transformation $e^{\gamma M_0} e^{a_1}$ sending the vacuum vector to the exponent $e^{H(\gamma;p)}$ of the generating function for Hurwitz numbers (see Example 4.4; we changed temporarily from β to γ the notation for the parameter in this transformation). The Laurent series obtained by the action of this transformation satisfy the cut-and-join equation

$$\frac{\partial\varphi}{\partial\gamma} = M_0\varphi = \left(\frac{1}{2}\left(z\frac{\partial}{\partial z}\right)^2 + \frac{1}{2}z\frac{\partial}{\partial z} + \frac{1}{12}\right)\varphi.$$

The change (13) acts on this equation and the direct substitution shows that for the transformed function ψ the equation takes the form

$$\frac{\partial\psi}{\partial\gamma} = \left(M_0 + 4\beta M_1 + 6\beta^2 M_2 + 4\beta^3 M_3 + \beta^4 M_4 + \frac{1}{4}\beta^2 z^2 + \frac{1}{3}\beta^3 z^3 + \frac{1}{8}\beta^4 z^4\right)\psi. \tag{15}$$

Since $\gamma = \beta$, we get that the total derivative over the parameter is given by the sum of the operators on the right-hand side of (14) and (15). Taking into account the change $\gamma = \beta = u^3$

and the corrections arisen from addition of linear terms $H_{0,1}$ and from the rescaling $q_i \mapsto u^{-4i} q_i$ that can be written in terms of Laurent series as the rescaling $z \mapsto u^{-4} z$, we obtain finally the equation of Theorem 5.1.

All these computations being slightly cumbersome are nevertheless absolutely elementary; they are fulfilled in the space of differential operators (of order not greater than 2) in one variable z. In particular, we have finite number of summands at every step of computation.

8. Remark on the λ_g conjecture

The conjecture of C. Faber (proved by now in several different ways) asserts the equality

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi_1^{d_1} \dots \psi_n^{d_n} = \binom{2g-3+n}{d_1, \dots, d_n} c_g, \tag{16}$$

where $\sum d_i = 2g - 3 + n$ and where the constant c_g depends only on g but is independent of n and of the exponents d_i. One of the simplest proofs of this equality is given in [10]. Using the language of the present paper this proof becomes even more transparent and is reduced to the following.

Theorem 8.1. *Let F^{top} denote the generating series for the top Hodge integrals,*

$$F^{\mathrm{top}} = \sum (-1)^g \langle \lambda_g \tau_0^{k_0} \tau_1^{k_1} \dots \rangle_g \frac{t_0^{k_0}}{k_0!} \frac{t_1^{k_1}}{k_1!} \cdots .$$

Then this series is subject to the equation

$$F^{\mathrm{top}} - \sum_{i \geqslant 0} t_i \frac{\partial F^{\mathrm{top}}}{\partial t_i} + \frac{1}{2} \sum_{i,j \geqslant 0} \binom{i+j}{i} t_i t_j \frac{\partial F^{\mathrm{top}}}{\partial t_{i+i-1}} + \frac{t_0^3}{3} = 0.$$

Denoting by $F^{(m)}$ the homogeneous summand of degree m in F^{top} we can rewrite the equation of the theorem in the form

$$F^{(m+1)} = \frac{1}{m} A F^{(m)}, \qquad A = \frac{1}{2} \sum_{i,j \geqslant 0} \binom{i+j}{i} t_i t_j \frac{\partial}{\partial t_{i+i-1}}.$$

This recursive equation allows one to recover the whole series by induction from its linear part $F^{(1)} = \sum c_g t_{2g-2}$, where $c_g = \langle \lambda_g \tau_{2g-2} \rangle_g$. The fact that this procedure leads to the equality of Faber's conjecture can be proved inductively by elementary considerations. Set

$$P_{m,d} = \frac{1}{m!} \sum_{d_1 + \dots + d_m = d} \binom{d}{d_1, \dots, d_m} t_{d_1} \dots t_{d_m}.$$

Then the equality (16) would follow from the relation

$$A P_{m,d} = m P_{m+1,d+1}.$$

Consider the linear operation Σ_m defined by

$$\Sigma_m t_{d_1} \dots t_{d_m} = \mathrm{Sym}_m x_1^{d_1} \dots x_m^{d_m},$$

where $\mathrm{Sym}_m f(x_1, \dots, x_m) = \sum_{\sigma \in S(m)} f(x_{\sigma(1)}, \dots, x_{\sigma(m)})$. The operation Σ_m provides an isomorphism between the space of homogeneous of degree m polynomials in $t_0, t_1, \dots$ and the space of all symmetric polynomials in $x_1, \dots, x_m$. Besides, we have, by definition, $\Sigma_m P_{m,d} = (x_1 + \dots + x_m)^d$. Now we compute

$$\Sigma_{m+1} A t_{d_1} \dots t_{d_m} = \frac{1}{2} \mathrm{Sym}_{m+1} \sum_{k=1}^{m} x_1^{d_1} \dots (x_k + x_{m+1})^{d_k+1} \dots x_m^{d_m}.$$

Therefore,

$$\begin{aligned}
\Sigma_{n+1} A P_{m,d} &= \frac{1}{2m!} \mathrm{Sym}_{m+1} \sum_{k=1}^{m} (x_k + x_{m+1}) \bigl(x_1 + \dots + (x_k + x_{m+1}) + \dots + x_m\bigr)^d \\
&= \frac{1}{2m!} (x_1 + \dots + x_{m+1})^m \mathrm{Sym}_{m+1} (x_1 + \dots + x_m + m x_{m+1}) \\
&= m (x_1 + \dots + x_{m+1})^{m+1} \\
&= m \Sigma_{m+1} P_{m+1,d+1}.
\end{aligned}$$

This proves the relation $A P_{m,d} = m P_{m+1,d+1}$, and hence, the equality (16) of the λ_g-conjecture.

For the proof of Theorem 8.1 consider the generating series G of Theorem 2.1 for Hodge integrals. Set $v = u^{-3} = \beta^{-1}$. A simple counting of dimensions shows that the series G can be rewritten in the form

$$G(q) = v^{-1} \sum_{j, k_0, k_1, \dots} (-1)^{g-j} \bigl\langle \lambda_{g-j} \tau_0^{k_0} \tau_1^{k_1} \dots \bigr\rangle v^j \frac{\widetilde{T}_0^{k_0}}{k_0!} \frac{\widetilde{T}_1^{k_1}}{k_1!} \dots, \qquad \widetilde{T}_k = v^{\frac{1-k}{3}} T_k,$$

where T_k are linear functions in the variables q_i defined by (2). Let us make one more change of coordinates passing to the variables $r_1, r_2, \dots$ related to q_k by the equalities

$$q_k = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} v^{i - \frac{k}{3} - 1} r_i.$$

Under the identification $q_k \leftrightarrow z^k$ the variable r_k corresponds to the polynomial $r_k \leftrightarrow v(v^{-\frac{2}{3}} z + v^{-1})^k - v^{1-k}$. Up to a rescaling, passing to the new variables r_k is equivalent to the shift $z \mapsto z + 1$ in the space of polynomials in z. In terms of new variables the recursive equations (2) take the following form

$$\widetilde{T}_0 = r_0, \qquad \widetilde{T}_{k+1} = \sum_{m \geqslant 1} m (v r_{m+2} - r_{m+1}) \frac{\partial}{\partial r_m} \widetilde{T}_k.$$

It follows immediately by induction that the summand of the smallest degree in v in $\widetilde{T}_k$ is equal to $(-1)^k k! r_{k+1}$:

$$\begin{aligned}
\widetilde{T}_0 &= r_1, \\
\widetilde{T}_1 &= -r_2 + v r_3, \\
\widetilde{T}_2 &= 2r_3 - 5vq_4 + 3v^2 r_5, \\
\widetilde{T}_3 &= -6r_4 + 26vr_5 - 35v^2 r_6 + 15v^3 r_7, \\
&\ldots.
\end{aligned}$$

Denote by $\Psi = \Psi(v; r_1, r_2, \ldots)$ the series vG expressed in terms of the variables r_k. We conclude

Proposition 8.2. *The series Ψ is a power series with respect to the parameter v; for $v = 0$ this series turns into the generating function F^{top} for the top Hodge integrals after the substitution*

$$t_k = \widetilde{T}_k|_{v=0} = (-1)^k k! r_{k+1}.$$

We would like to apply the change of variables to the 'cat-and-join' equation of Theorem 5.1 for G in order to obtain the corresponding equation for Ψ. Remark that the change corresponding to the shift $z \mapsto z + 1$ in the space of polynomials in z preserves the KP hierarchy (in spite of the fact that the action of this shift is not defined on the space of Laurent series). However, the subsequent rescaling is not quasihomogeneous and destroys the hierarchy. Therefore, it is convenient to introduce 'intermediate' variables $\tilde{r}_i = r_i/v$ and to denote by $\widetilde{G}$ the series G expressed in terms of these variables. Then we have

$$\Psi(r_1, r_2, \ldots) = v\widetilde{G}(r_1/v, r_2/v, \ldots). \tag{17}$$

The transition from the variables q_i to $\tilde{r}_i$ corresponds to the linear non-homogeneous change $z \mapsto v^{-\frac{2}{3}} z + v^{-1}$ in the space of polynomials in z. Applying this change to the differential operator in z corresponding to the right-hand side of the equality of Theorem 5.1 we compute the action of this change on the bosonic side as well. As a result of these computations (that are absolutely elementary since they do not require consideration of infinite sums) we arrive at the following statement.

Proposition 8.3. *The series $\widetilde{G}$ in the variables $\tilde{r}_k$ is a solution of the KP hierarchy* (*identically in v*) *and its dependence in the parameter v is described by the differential equation*

$$\frac{\partial e^{\widetilde{G}}}{\partial v} = \left(-M_2 + 2vM_3 - M_4 + \Lambda_1 + \frac{v}{6}a_3 - \frac{v^2}{8}a_4\right) e^{\widetilde{G}}.$$

The final rescaling (17) exits the space of solutions of the KP hierarchy. Fortunately, this change is simple enough in order to be able to compute its action on differential operators in the variables $r_1, r_2, \ldots$ directly. We get the following.

Corollary 8.4. *The series Ψ of Proposition* 8.2 *satisfies the equation*

$$
\begin{aligned}
v\frac{\partial\Psi}{\partial v} &= \Psi + \sum_{i\geqslant 1}(v i r_{i+1} - r_i)\frac{\partial\Psi}{\partial r_i} \\
&\quad + \frac{1}{2}\sum_{\substack{i+j=k\\ i,j,k\geqslant 1}}\Bigg(ijv\big(-r_{k+2}+2vr_{k+3}-v^2r_{k+4}\big)\bigg(v\frac{\partial^2\Psi}{\partial r_i\partial r_j}+\frac{\partial\Psi}{\partial r_i}\frac{\partial\Psi}{\partial r_j}\bigg) \\
&\quad + r_ir_j\bigg(-(k-2)\frac{\partial\Psi}{\partial r_{k-2}}+2v(k-3)\frac{\partial\Psi}{\partial r_{k-3}}-v^2(k-4)\frac{\partial\Psi}{\partial r_{k-4}}\bigg)\Bigg) \\
&\quad + \frac{1}{3}r_1^3 - \frac{v}{2}r_1^2r_2 + \frac{v^2}{6}r_3 - \frac{v^3}{8}r_4.
\end{aligned}
$$

Setting $v=0$ in this equation and denoting $t_k=(-1)^k k! r_{k+1}$ we obtain the required equation of Theorem 8.1.

9. Reduction of Hodge integrals

For completeness, we review in this section a formula expressing the Hodge integrals in terms of intersection numbers of just ψ-classes. Denote by $\mathcal{F}(u, T_0, T_1, \ldots)$ the generating series (1) for Hodge integrals written in terms of T-variables. Then $F(T)=\mathcal{F}(0,T)$ is the Witten's potential for intersection numbers of ψ-classes. The algorithm outlined in [4] allowing one to recover $\mathcal{F}$ from F is essentially equivalent to the following formula (cf. [5,6]):

$$
e^{\mathcal{F}} = e^W e^F,
$$

$$
W = -\sum_{k=1}^{\infty}\frac{B_{2k}u^{2(2k-1)}}{2k(2k-1)}\bigg(\frac{\partial}{\partial t_{2k}} - \sum_{i=0}^{\infty} t_i\frac{\partial}{\partial t_{i+2k-1}} + \frac{1}{2}\sum_{i+j=2k-2}(-1)^i\frac{\partial^2}{\partial t_i\partial t_j}\bigg), \tag{18}
$$

where B_m's are the Bernoulli numbers.

The Chern classes λ_i of the Hodge bundle over $\overline{\mathcal{M}}_{g,n}$ can be expressed via the homogeneous components of its Chern character $\mathrm{ch} = g + \mathrm{ch}_1 + \mathrm{ch}_2 + \cdots$ by the identity

$$
1 - u^2\lambda_1 + u^4\lambda_2 - u^6\lambda_3 + \cdots = e^{-\sum_{m=1}^{\infty}(m-1)!\,\mathrm{ch}_m u^{2m}}.
$$

The classes ch_m can be computed from the Mumford's theorem [15]. It claims that the even components ch_{2k} vanish, and for odd $m=2k-1$ one has

$$
(m-1)!\,\mathrm{ch}_m = \frac{B_{m+1}}{m(m+1)}\bigg(\pi_*\big(\psi_{n+1}^{m+1}\big) - \sum_{i=1}^{n}\psi_i^m + \frac{1}{2}j_*\bigg(\sum_{i+j=m-1}(-1)^i\psi_{n+1}^i\psi_{n+2}^j\bigg)\bigg),
$$

where $\pi: \overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ is the natural forgetful map and $j: \Delta\to\overline{\mathcal{M}}_{g,n}$ is the double cover of the boundary divisor considered itself as a moduli space of curves with two additional markings corresponding to the two branches at the double point, the markings being numbered by $n+1$ and $n+2$, respectively.

It is straightforward to check that this relation for ch_m written in terms of intersection numbers is equivalent to (18).

Remark that the operator W participating in (18) does not belong to $\widehat{\mathfrak{gl}(\infty)}$. Therefore, its usage is not convenient in the context of the present paper.

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