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Mathieu series and associated sums involving the Zeta functions

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ABSTRACT

Almost twelve decades ago, Mathieu investigated an interesting series $S(r)$ in the study of elasticity of solid bodies. Since then many authors have studied various problems arising from the Mathieu series $S(r)$ in various diverse ways. In this paper, we present a relationship between the Mathieu series $S(r)$ and certain series involving the Zeta functions. By means of this relationship, we then express the Mathieu series $S(r)$ in terms of the Trigamma function $\psi'(z)$ or (equivalently) the Hurwitz (or generalized) Zeta function $\zeta(s, a)$. Accordingly, various interesting properties of $S(r)$ can be obtained from those of $\psi'(z)$ and $\zeta(s, a)$. Among other results, certain integral representations of $S(r)$ are deduced here by using the aforementioned relationships among $S(r)$, $\psi'(z)$ and $\zeta(s, a)$.

1. Introduction and preliminaries

Émile Leonard Mathieu (1835–1890) [1] investigated the following infinite series:

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^2}\qquad\left(r\in\mathbb{R}^+\right)\tag{1.1}$$

in the study of elasticity of solid bodies (see also [2]), $\mathbb{R}^+$ being (as usual) the set of positive real numbers. Pogány et al. [3] introduced an *alternating* version of the Mathieu series (1.1) as follows:

$$\tilde{S}(r)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{\left(n^2+r^2\right)^2}\qquad\left(r\in\mathbb{R}^+\right).\tag{1.2}$$

Since the time of Mathieu, many authors (see, for example, [4–7,2,8–12,3,13–23,36]; see also the references cited in each of these works) have investigated various problems arising from the Mathieu series (1.1) and its extensions and generalizations in various diverse ways. In particular, Pogány et al. [3] presented the following integral representations of the Mathieu series (1.1) and the alternating Mathieu series (1.2):

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{t\,\sin(rt)}{e^t-1}\,dt\tag{1.3}$$

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and

$$\tilde{S}(r) = \frac{1}{r}\int_0^\infty \frac{t\,\sin(rt)}{e^t+1}\,dt. \tag{1.4}$$

Srivastava and Tomovski [16], on the other hand, defined the following five-parameter family of generalized Mathieu series:

$$S_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = S_\mu^{(\alpha,\beta)}(r;\{a_k\}) = \sum_{n=1}^\infty \frac{2\,a_n^\beta}{\left(a_n^\alpha + r^2\right)^\mu} \quad \left(r,\alpha,\beta,\mu \in \mathbb{R}^+\right), \tag{1.5}$$

where it is *tacitly* assumed that the *positive* sequence

$$\mathbf{a} := \{a_k\}_{k=1}^\infty \quad \left(\lim_{k\to\infty} a_k = \infty\right) \tag{1.6}$$

is so chosen (and then the positive parameters α, β and μ are so constrained) that the infinite series in (1.5) converges, that is, that the following auxiliary series:

$$\sum_{n=1}^\infty \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent.

Several authors have considered some interesting variants and special cases of the generalized Mathieu series (1.5) (see, for example, [8,10,3,15]). And, very recently, Elezović et al. [7] derived several results involving the Laplace, Fourier and Mellin transforms of various functions belonging to the family of generalized Mathieu series (1.5) including, for example, the Laplace transforms of $S(r)$ and $\tilde{S}(r)$ as given below:

$$\mathcal{L}\big(S(r)\big)(x) = \int_0^\infty \left(\frac{t}{e^t-1}\right)\arctan\left(\frac{t}{x}\right)dt \tag{1.7}$$

and

$$\mathcal{L}\big(\tilde{S}(r)\big)(x) = \int_0^\infty \left(\frac{t}{e^t+1}\right)\arctan\left(\frac{t}{x}\right)dt. \tag{1.8}$$

Here, in our present investigation, we aim first at expressing the Mathieu series (1.1) and the alternating Mathieu series (1.2) as sums involving the Riemann Zeta function $\zeta(s)$ and sums involving the Dirichlet Eta function $\eta(s)$, respectively, which are then used to give certain useful relationships among $S(r)$, $\tilde{S}(r)$ and $\psi'(z)$ $\big($or $\zeta(s,a)\big)$. We also derive integral representations of $S(r)$ and $\tilde{S}(r)$ by making use of some known integral expressions for $\psi'(z)$ and $\zeta(s,a)$.

We begin by recalling the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{1-2^{-s}}\sum_{n=1}^\infty \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\ \displaystyle\frac{1}{1-2^{1-s}}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\quad s \neq 1\big). \end{cases} \tag{1.9}$$

The Riemann Zeta function $\zeta(s)$ is a special case ($a = 1$) of the Hurwitz (or *generalized*) Zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^\infty \frac{1}{(k+a)^s} \quad \big(\Re(s) > 1;\quad a \in \mathbb{C}\setminus\mathbb{Z}_0^-\big), \tag{1.10}$$

where, as usual, $\mathbb{C}$ denotes the complex plane and

$$\mathbb{Z}_0^- := \{0,-1,-2,\ldots\}.$$

Both of the Zeta functions $\zeta(s)$ and $\zeta(s,a)$ can be continued meromorphically, in many different ways, to the whole complex s-plane except for a simple pole just at $s = 1$ with their respective residues 1. The Dirichlet Eta function (or the alternating Riemann Zeta function) $\eta(s)$ is defined by

$$\eta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} \quad \big(\Re(s) > 0\big). \tag{1.11}$$

It is easy to find from (1.9) and (1.11) that

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s) \quad \big(\Re(s) > 1\big). \tag{1.12}$$

The Psi (or Digamma) function $\psi(z)$ is defined by

$$\psi(z) := \frac{\mathrm{d}}{\mathrm{d}z}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t, \tag{1.13}$$

where $\Gamma(z)$ is the well-known Gamma function. The Polygamma functions $\psi^{(n)}(z)$ $(n \in \mathbb{N})$ are defined by

$$\psi^{(n)}(z) := \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\{\log\Gamma(z)\} = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\psi(z)\} \tag{1.14}$$
$$(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\, z \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

where $\mathbb{N}$ denotes the set of *positive* integers.

It is noted that $\psi'(z)$ is often called the Trigamma function. Moreover, in terms of the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ in (1.10), we can write

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\sum_{k=0}^{\infty}\frac{1}{(k+z)^{n+1}} = (-1)^{n+1}\,n!\,\zeta(n+1,z) \tag{1.15}$$
$$(n \in \mathbb{N}; \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

2. Series involving the Zeta functions

An interesting historical introduction to the remarkably widely investigated subject of *closed-form evaluation of series involving the Zeta functions* has been presented (see [24] and [25]; see also [26]). The following formula:

$$\sum_{k=2}^{\infty}[\zeta(k) - 1] = 1 \tag{2.1}$$

is presumably the origin of this subject (see [27] and [24]). A considerable number of summation formulas have been derived, by using various methods and techniques, in the vast literature on this subject (see, e.g., [26, Chapter 3]; see also [28–31,24, 25,32]). For a simple example, we recall here the following sum:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)\cdot 2^{2k}} = \log 2 - \gamma, \tag{2.2}$$

which, as noted in [24], is contained in a memoir of 1781 by Leonhard Euler (1707–1783) (see also [31, p. 28, Eq. (8)]; it was rederived by Wilton [32, p. 92]). A rather extensive collection of closed-form sums of series involving the Zeta functions was given in [26].

In this section, we express the Mathieu series in (1.1) and the alternating Mathieu series in (1.2) as sums involving the Riemann Zeta function, which can also be evaluated in terms of the Trigamma function $\psi'(z)$, and so the Hurwitz Zeta function $\zeta(s,a)$, as follows.

Theorem 1. *Each of the following series representations holds true:*

$$S(r) = 2\sum_{k=1}^{\infty}(-1)^{k-1}\,k\,\zeta(2k+1)\,r^{2(k-1)} \quad (|r| < 1) \tag{2.3}$$

and

$$\tilde{S}(r) = 2\sum_{k=1}^{\infty}(-1)^{k-1}\,k\,\eta(2k+1)\,r^{2(k-1)} \quad (|r| < 1), \tag{2.4}$$

where $\zeta(s)$ and $\eta(s)$ are the Riemann Zeta function and the Dirichlet Eta function given in (1.9) and (1.11), respectively.

Proof. By applying the following elementary identity:

$$\frac{1}{(1+z)^2} = \sum_{k=0}^{\infty}(-1)^k\,(k+1)\,z^k \quad (|z| < 1) \tag{2.5}$$

to (1.1), we obtain

$$\begin{aligned} S(r) &= \sum_{n=1}^{\infty} \frac{2n}{n^4} \frac{1}{\left[1+(r/n)^2\right]^2} \\ &= \sum_{n=1}^{\infty} \frac{2n}{n^4} \sum_{k=0}^{\infty} (-1)^k (k+1) \frac{r^{2k}}{n^{2k}} \quad \left(\left|\frac{r}{n}\right| < 1; \quad n \in \mathbb{N}\right) \\ &= 2 \sum_{k=0}^{\infty} (-1)^k (k+1) r^{2k} \sum_{n=1}^{\infty} \frac{1}{n^{2k+3}} \\ &= 2 \sum_{k=0}^{\infty} (-1)^k (k+1) \zeta(2k+3) r^{2k}, \end{aligned}$$

where the rearrangement of the sums in the penultimate step is guaranteed by the absolute convergence of the double series involved.

Finally, upon replacing the summation index k by $k-1$ in the last expression, we are led to the assertion (2.3) of Theorem 1. Similarly, we can prove the assertion (2.4) of Theorem 1. □

By using (2.3) and (2.4), we obtain Theorem 2 below.

Theorem 2. *Each of the following relationships holds true:*

$$S(r) = \frac{\mathrm{i}}{2r} \left[\psi'(1+\mathrm{i}r) - \psi'(1-\mathrm{i}r)\right] \quad (0 < |r| < 1; \quad \mathrm{i} = \sqrt{-1}), \tag{2.6}$$

$$S(r) = \frac{\mathrm{i}}{2r} \left[\zeta(2, 1+\mathrm{i}r) - \zeta(2, 1-\mathrm{i}r)\right] \quad (0 < |r| < 1; \quad \mathrm{i} = \sqrt{-1}) \tag{2.7}$$

and

$$\tilde{S}(r) = S(r) - \frac{1}{4} S\left(\frac{r}{2}\right) \quad (|r| < 1), \tag{2.8}$$

where $\psi'(z)$ *and* $\zeta(s, a)$ *are the Trigamma function and the Hurwitz (or generalized) Zeta function given in* (1.14) *and* (1.10), *respectively.*

Proof. We recall the following known identity (see [26, p. 160, Eq. (16)]):

$$\sum_{k=1}^{\infty} \zeta(2k+1) t^{2k} = -\frac{1}{2} \left[\psi(1+t) + \psi(1-t)\right] - \gamma \quad (|t| < 1), \tag{2.9}$$

where γ denotes the Euler–Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \cong 0.57721\,56649\,01532\,8606\,0651\,2090\,082402431042\ldots. \tag{2.10}$$

Differentiating each side of (2.9) with respect to t and dividing the resulting identity by t, we get

$$2 \sum_{k=1}^{\infty} k\, \zeta(2k+1) t^{2(k-1)} = -\frac{1}{2t} \left[\psi'(1+t) - \psi'(1-t)\right] \quad (0 < |t| < 1). \tag{2.11}$$

By setting $t = \mathrm{i}\, r$ ($\mathrm{i} = \sqrt{-1}$) in (2.11), and making use of (2.3), we prove the assertion (2.6) of Theorem 2. Similarly, by applying (1.15) to (2.6), we obtain (2.7). Furthermore, by combining (2.3), (2.4) and (1.12), we can prove (2.8). □

Remark 1. In view of the relationship in (2.8), several properties of the alternating Mathieu series $\tilde{S}(r)$ can easily be obtained from the corresponding properties of the Mathieu series $S(r)$.

3. Integral representations of $S(r)$ and $\tilde{S}(r)$

First of all, we analytically continue the Mathieu series $S(r)$ from $r \in \mathbb{R}^+$ to the complex plane $\mathbb{C}$ as follows.

Lemma 1. *The Mathieu series* $S(r)$ *is analytic on the punctured complex plane*

$$\mathbb{C} \setminus \{z : z = \pm n\,\mathrm{i} \quad (n \in \mathbb{N}; \quad \mathrm{i} = \sqrt{-1})\}.$$

Proof. It is obvious that each term $2n/(n^2+r^2)$ in the series (1.1) defining the $S(r)$ is analytic on

$$\mathbb{C}\setminus\{z : z = \pm n\,\mathrm{i} \quad (n\in\mathbb{N};\quad \mathrm{i}=\sqrt{-1})\}.$$

Since the series for $S(r)$ is seen to be uniformly convergent on any compact subset $\mathbb{D}$ in

$$\mathbb{C}\setminus\{z : z = \pm n\,\mathrm{i} \quad (n\in\mathbb{N};\quad \mathrm{i}=\sqrt{-1})\},$$

$S(r)$ is analytic on $\mathbb{D}$. Now, by the well-known principle due to Weierstrass (see, e.g., [33, p. 88, Theorem]), the Mathieu series $S(r)$ is analytic on

$$\mathbb{C}\setminus\{z : z = \pm n\,\mathrm{i} \quad (n\in\mathbb{N};\quad \mathrm{i}=\sqrt{-1})\}. \qquad \square$$

By means of the relationships asserted by Theorem 2, we can now present various integral representations of $S(r)$ and $\tilde{S}(r)$, including (1.3) and (1.4). For this purpose, we recall several known integral representations of $\psi(z)$ and $\zeta(s,a)$ as Lemma 2 below.

Lemma 2. *Each of the following representations holds true:*

$$\zeta(s,a) = a^{-s} + \sum_{k=0}^{n} \frac{\Gamma(k+s-1)}{\Gamma(s)}\,\frac{B_k}{k!}\,a^{-k-s+1} + \frac{1}{\Gamma(s)}\int_0^\infty \left(\frac{1}{e^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}\,t^{k-1}\right) e^{-at}\,t^{s-1}\,dt \tag{3.1}$$

$$\big(\Re(s) > -(2n-1);\quad \Re(a)>0;\quad n\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}\big),$$

$$\Gamma(s)\,\zeta(s,a) = \int_0^\infty \frac{t^{s-1}\,e^{-at}}{1-e^{-t}}\,dt = \int_0^\infty \frac{t^{s-1}\,e^{-(a-1)t}}{e^t-1}\,dt$$

$$= \int_0^1 \frac{t^{a-1}}{1-t}\left(\log\frac{1}{t}\right)^{s-1} dt \quad \big(\Re(s)>1;\quad \Re(a)>0\big), \tag{3.2}$$

$$\psi(z) = \int_0^\infty \left(e^{-t} - \frac{1}{(1+t)^z}\right)\frac{dt}{t} \quad \big(\Re(z)>0\big), \tag{3.3}$$

$$\psi(z) = \log z - \int_0^1 \frac{1-t+\log t}{(1-t)\,\log t}\,t^{z-1}\,dt \quad \big(\Re(z)>0\big) \tag{3.4}$$

and

$$\psi(z) = \log z - \frac{1}{2z} - 2\int_0^\infty \frac{t\,dt}{(t^2+z^2)\,(e^{2\pi t}-1)} \quad \big(\Re(z)>0\big). \tag{3.5}$$

Proof. We need only to provide a suitable reference for each of the integral formulas asserted by Lemma 2. We refer the reader to [26, p. 93, Eq. (25)] for (3.1), [26, p. 89, Eq. (2)] for (3.2), [26, p. 15, Eq. (18)] for (3.3), [34, p. 261, Example 19] for (3.4), and [34, p. 251, Example] for (3.5). $\square$

Remark 2. Formula (1.3) can be derived by employing the first integral representation of (3.2) in the identity (2.7) and the application of (1.3) to the relationship (2.8) yields (1.4).

Theorem 3. *Each of the following representations holds true:*

$$\begin{aligned} S(r) \;=\; & \frac{1}{(1+r^2)^2} + \frac{1}{1+r^2} + \frac{\mathrm{i}}{2r}\sum_{k=2}^{n}\left(\frac{1}{(1+\mathrm{i}r)^{k+1}} - \frac{1}{(1-\mathrm{i}r)^{k+1}}\right)B_k \\ & + \frac{1}{r}\int_0^\infty \left(\frac{1}{e^t-1} - \sum_{k=0}^{n}\frac{B_k}{k!}\,t^{k-1}\right) t\,e^{-t}\,\sin(rt)\,dt \end{aligned} \tag{3.6}$$

$$\big(\Re(r)>0;\quad n\in\mathbb{N}_0;\quad \mathrm{i}=\sqrt{-1}\big),$$

where B_k denotes the Bernoulli numbers and an empty sum is (as usual) understood to be nil;

$$S(r) = \frac{1}{r}\int_0^1 \frac{\log t}{1-t}\,\sin(r\,\log t)\,dt \quad \big(\Re(r)>0\big); \tag{3.7}$$

$$S(r) = \frac{1}{r}\int_0^\infty \frac{\log(1+t)}{t(1+t)}\,\sin[r\,\log(1+t)]\,dt \quad \big(\Re(r)>0\big); \tag{3.8}$$

$$S(r) = \frac{1}{1+r^2} + \frac{1}{r} \int_0^1 \frac{1-t+\log t}{1-t} \sin(r \log t)\,\mathrm{d}t \quad \big(\Re(r) > 0\big); \tag{3.9}$$

$$S(r) = \frac{1}{\left(1+r^2\right)^2} + \frac{1}{1+r^2} + 4 \int_0^\infty \frac{1+r^2-t^2}{\left(1+t^2-r^2\right)^2 - 4r^2} \frac{t\,\mathrm{d}t}{\mathrm{e}^{2\pi t}-1} \quad \big(\Re(r) > 0\big). \tag{3.10}$$

Proof. By applying (3.1) and the third integral representation in (3.2) to (2.7), we obtain (3.6) and (3.7), respectively. Differentiating the right-hand side of each of the integral representations (3.3) to (3.5) with respect to z under the sign of integration (which can be validated by means of a known result [34, p. 74, Corollary]), and applying each of the resulting integral representations of the Trigamma function $\psi'(z)$ to (2.6), we obtain (3.8) to (3.10), respectively.

Remark 3 below would obviously suffice to complete the proof of Theorem 3.

Remark 3. Even though the identities (3.6) to (3.10) are proved under the condition that $0 < |r| < 1$, yet (in view of Lemma 1) each of these identities holds true, by the principle of analytic continuation, throughout the extended region given by $\Re(r) > 0$. □

As a by-product of our main results, we can obtain a variety of interesting identities, some of which are stated as Corollaries 1 and 2 below.

Corollary 1. *Each of the following results holds true:*

$$S(r) = \frac{1}{\left(1+r^2\right)^2} + \frac{1}{1+r^2} + \frac{1}{r} \int_0^\infty \left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t} + \frac{1}{2}\right) t\,\mathrm{e}^{-t} \sin(rt)\,\mathrm{d}t \tag{3.11}$$
$$\big(\Re(r) > 0\big),$$

$$S(r) = \frac{1}{1+r^2} + O\left(\frac{1}{r^4}\right) \quad (r \to \infty; \quad r \in \mathbb{R}^+) \tag{3.12}$$

and

$$\tilde{S}(r) = \frac{1}{2r} \int_0^1 \frac{\log t}{1-t} \sin\left(\frac{r}{2} \log t\right) \left[4 \cos\left(\frac{r}{2} \log t\right) - 1\right] \mathrm{d}t \quad \big(\Re(r) > 0\big). \tag{3.13}$$

Proof. Eq. (3.11) is a special case of (3.6) when $n = 1$. By making use of the generating function of Bernoulli numbers (see, e.g., [26, p. 57, Eq. 1.6(2)]):

$$\frac{t}{\mathrm{e}^t-1} - 1 + \frac{t}{2} = \sum_{k=1}^\infty B_{2k} \frac{t^{2k}}{(2k)!} \quad (|t| < 2\pi) \tag{3.14}$$

and the following entry in the readily available tables of Fourier sine transforms (see, e.g., [35, Vol. I, p. 72, Entry 2.4(3)]):

$$\int_0^\infty t^2\,\mathrm{e}^{-t} \sin(r\,t)\,\mathrm{d}t = 2r \left(\frac{1}{1+r^2}\right)^3 \left(3-r^2\right) \quad (r > 0), \tag{3.15}$$

in conjunction with (3.11), we obtain (3.12). Furthermore, if the various already-developed integral representations of $S(r)$ are applied to the relationship (2.8), several integral representations of $\tilde{S}(r)$ can be obtained. For example, by applying (3.7) to (2.8), we obtain the last assertion (3.13) of Corollary 1. □

Corollary 2. *Each of the following identities holds true:*

$$\begin{aligned} T(r) &:= \sum_{n=1}^\infty \frac{8n}{\left(n^2+r^2\right)^3} \\ &= \frac{\mathrm{i}}{2r^3} \left[\psi'(1+\mathrm{i}r) - \psi'(1-\mathrm{i}r)\right] + \frac{1}{2r^2} \left[\psi^{(2)}(1+\mathrm{i}r) + \psi^{(2)}(1-\mathrm{i}r)\right] \\ &= \frac{\mathrm{i}}{2r^3} \left[\zeta(2, 1+\mathrm{i}r) - \zeta(2, 1-\mathrm{i}r)\right] - \frac{1}{r^2} \left[\zeta(3, 1+\mathrm{i}r) + \zeta(3, 1-\mathrm{i}r)\right] \end{aligned} \tag{3.16}$$
$$(0 < |r| < 1; \quad i = \sqrt{-1})$$

and

$$T(r) = \frac{2\left(3+r^2\right)}{\left(1+r^2\right)^3} + \frac{1}{r^3} \int_0^\infty \left(\frac{1}{\mathrm{e}^t-1} - \frac{1}{t} + \frac{1}{2}\right) t\,\mathrm{e}^{-t} \left[\sin(rt) - rt \cos(rt)\right] \mathrm{d}t \quad \big(\Re(r) > 0\big). \tag{3.17}$$

Proof. Upon differentiating (1.1) and (2.6) with respect to r, if we combine the resulting identities and make use of (1.15), we can find a new series $T(r)$ which is expressed in terms of the Polygamma functions and the Hurwitz Zeta function as in (3.16). By applying (3.11) to (3.16), we obtain (3.17). □

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