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Eigenspaces of the Laplacian on Hyperbolic Spaces: Composition Series and Integral Transforms

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Let X be a projective real, complex, or quaternion hyperbolic space, realized as the pseudo-Riemannian symmetric space $X \cong G/H$ with $G = O(p, q)$, $U(p, q)$, or $Sp(p, q)$ (these are the classical isotropic symmetric spaces). Let Δ be the G-invariant Laplace–Beltrami operator on X. A complete description (by K-types), for each $\chi \in \mathbb{C}$, of all closed G-invariant subspaces of the eigenspace $\{f \in C^\infty(X) \mid \Delta f = \chi f\}$, is given. The eigenspace representations are compared with principal series representations, using "Poisson-transformations". Similar results are obtained also for the exceptional isotropic symmetric space. The Langlands parameters of the spherical discrete series representations are determined. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let G/H be a homogeneous space of a Lie group G and a closed subgroup H, and let $\mathbb{D}(G/H)$ be the algebra of G-invariant differential operators on G/H. According to Helgason [12, p. 2], *harmonic analysis* on G/H consists of the following program:

A. *Decompose functions (e.g., compactly supported C^∞-functions) on G/H into joint eigenfunctions of $\mathbb{D}(G/H)$, that is, functions on G/H satisfying*

$$Df = \chi(D)\, f, \qquad \forall D \in \mathbb{D}(G/H) \tag{1.1}$$

for some map (necessarily a homomorphism) $\chi\colon \mathbb{D}(G/H) \to \mathbb{C}$.

B. *Describe for each χ as above the joint eigenspace $\mathscr{E}_\chi$ consisting of the C^∞-functions on G/H satisfying* (1.1).

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C. *Determine for which* χ *the space* $\mathscr{E}_\chi$ *(equipped with the Fréchet topology inherited from* $C^\infty(G/H)$*) is irreducible under the natural representation* π_χ *of* G.

For Riemannian symmetric spaces this program has been carried out completely (cf. [12] and the references given there). For general non-Riemannian semisimple symmetric spaces the program is however far from being complete (cf. [25] and the references given there). The best known examples of these spaces are the projective real, complex, and quaternion hyperbolic spaces, which are considered in the papers of Faraut [5], solving problem A, and Sekiguchi [26], related to problem B. The present paper also deals with these hyperbolic spaces, solving in particular the following stronger version of C:

C′. *Determine for each* χ *the composition series of* π_χ, *that is, determine all closed invariant subspaces of* $\mathscr{E}_\chi$.

The result (Theorem 6.1) may be viewed as the analog for the eigenspace representations of the results of Johnson–Wallach [14] for the spherical principal series of a classical real rank one semisimple Lie group G. In fact, in the special case where H equals K (a maximal compact subgroup of G) the solution to C′ can be obtained directly from [14], using the following result of Helgason's: By [10, Theorem IV.1.4] the Poisson transformation is an isomorphism of the K-finite vectors in a spherical principal series representation onto the K-finite vectors in the eigenspace. For the complex Riemannian hyperbolic spaces Problem C′ is also solved in [23].

The method used in this paper to solve Problem C′ is based on explicit calculations on the K-finite eigenfunctions involved. Each subrepresentation of π_χ is described by its K-types. In particular we determine the K-types in each discrete series representation for the hyperbolic space (Theorem 6.4).

The second result of this paper is a generalization to the hyperbolic spaces of the above mentioned result of Helgason's [10, Theorem IV.1.4]. Associated to the hyperbolic spaces there are also some "principal series" of representations and "Poisson transformations" from these to the eigenspaces. In general, and in contrast to the situation in [10], there are eigenspace representations which are not isomorphic to any of these principal series representations (on the level of $(\mathfrak{g}, K)$-modules). However, we prove (Corollary 7.5) that for each of these principal series representations, each irreducible subquotient is isomorphic (on the level of $(\mathfrak{g}, K)$-modules) to a subquotient of the corresponding eigenspace representation—and vice versa. The isomorphisms are provided by Poisson-"like" transformations.

In Section 8 similar results as the previous mentioned are proved (with fewer details) for the exceptional symmetric space $F_{4(-20)}/\mathrm{Spin}(1, 8)$ which may (in some sense) be considered as a projective hyperbolic space over the Cayley numbers. For this space problem A was solved earlier by Kosters [18].

Finally, in Section 9, we consider closer those discrete series representations for G/H which are spherical (that is, contain the trivial K-type). We describe the Langlands parameters of this interesting series of unitary spherical representations.

2. Notation

Let $\mathbb{F}$ be one of the classical fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and let $x \to \bar{x}$ be the standard (anti-) involution of $\mathbb{F}$. Let p and q be nonnegative integers, and let $[\ ,\ ]$ be the Hermitian form on $\mathbb{F}^{p+q}$ given by

$$[x, y] = \bar{y}_1 x_1 + \cdots + \bar{y}_p x_p - \bar{y}_{p+1} x_{p+1} - \cdots - \bar{y}_{p+q} x_{p+q} \tag{2.1}$$

for $x, y \in \mathbb{F}$, and let $G = U(p, q; \mathbb{F})$ denote the group of all $(p+q) \times (p+q)$ matrices over $\mathbb{F}$ preserving $[\ ,\ ]$. Here the action of a matrix on $\mathbb{F}^{p+q}$ is given by multiplication on the left, considering elements of $\mathbb{F}^{p+q}$ as $(p+q)$-columns. Thus, $G = O(p, q)$, $U(p, q)$, or $Sp(p, q)$ in standard notation. We put $U(p; \mathbb{F}) = U(p, 0; \mathbb{F})$.

Assume that p and q are positive. Let H be the subgroup of G which is the stabilizer of the line $\mathbb{F}(1, 0, \ldots, 0)$ in $\mathbb{F}^{p+q}$. Then $H \cong U(1; \mathbb{F}) \times U(p-1, q; \mathbb{F})$ and the homogeneous space G/H is identified with the space $X = X(p, q; \mathbb{F})$ which is the projective image of the space

$$Z = Z(p, q; \mathbb{F}) = \{z \in \mathbb{F}^{p+q} \mid [z, z] = 1\}$$

(that is, $X = Z/\sim$ where $\sim$ is the equivalence relation $z \sim zu$ for all $z \in Z$ and $u \in \mathbb{F}$). The space G/H is a reductive symmetric space (an involution of G is obtained as in [25, p. 114 Example e]). We call X a *projective hyperbolic space.*

Let $d = \dim_{\mathbb{R}} \mathbb{F}$, then $\mathbb{F}^{p+q} \cong \mathbb{R}^{dp+dq}$ as a vector space over $\mathbb{R}$, and $Z(p, q; \mathbb{F}) \cong Z(dp, dq; \mathbb{R})$. Hence there is a natural projection

$$X(dp, dq; \mathbb{R}) \to X(p, q; \mathbb{F}), \tag{2.2}$$

and there is a natural action of $U(1; \mathbb{F})$ on $X(dp, dq; \mathbb{R})$, defined via multiplication from the right on $Z(p, q; \mathbb{F})$. The fibers of (2.2) are the $U(1; \mathbb{F})$-orbits in $X(dp, dq; \mathbb{R})$. Moreover, we may consider G naturally as a subgroup of $O(dp, dq)$, and the action of G on $X(p, q; \mathbb{F})$ factorizes through the map (2.2).

Let $\Delta = \Delta(p, q; \mathbb{F})$ be the *Laplace–Beltrami operator* on X. Explicitly, Δ is constructed (and normalized) as follows (cf. [5, p. 377]). Assume first $\mathbb{F} = \mathbb{R}$ and let $\square_{p,q}$ be the generalized wave operator on $\mathbb{R}^{p+q}$

$$\square_{p,q} = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_p^2} + \frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}.$$

For $f \in C^\infty(X)$ let $\tilde{f}(z) = f([z, z]^{-1/2} z)$ for all $z \in \mathbb{R}^{p+q}$ with $[z, z] > 0$, then Δf is the pull back to X of the function $\square_{p,q}\tilde{f}$ on Z. For $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ we can construct Δ similarly, or we can reduce to $\mathbb{F} = \mathbb{R}$ by (2.2). We have $\Delta \in \mathbb{D}(G/H)$, and every element of $\mathbb{D}(G/H)$ is a polynomial in Δ.

For $\lambda \in \mathbb{C}$ the eigenspace $\mathscr{E}_\lambda = \mathscr{E}_\lambda(p, q; \mathbb{F})$ is defined by

$$\mathscr{E}_\lambda = \{ f \in C^\infty(X) \mid \Delta f = (\lambda^2 - \rho^2) f \} \tag{2.3}$$

where $\rho = \frac{1}{2}(dp + dq - 2)$. Let π_λ denote the restriction to $\mathscr{E}_\lambda$ of the regular (Fréchet space-) representation of G on $C^\infty(X)$. Notice that from (2.2) we get a G-homomorphism

$$\mathscr{E}_\lambda(p, q; \mathbb{F}) \to \mathscr{E}_\lambda(dp, dq; \mathbb{R})$$

which is an isomorphism onto the $U(1; \mathbb{F})$ invariant functions in $\mathscr{E}_\lambda(dp, dq; \mathbb{R})$.

Let $K = K(p, q; \mathbb{F}) \subset G$ be the maximal compact subgroup $U(p; \mathbb{F}) \times U(q, \mathbb{F})$, and let $\mathfrak{g} \cong \mathfrak{u}(p, q; \mathbb{F})$ be the Lie algebra of G. For $1 \leqslant i, j \leqslant p + q$ let $E_{i,j}$ denote the $(p+q) \times (p+q)$ matrix with 1 on the (i, j)th entry and zero on all other entries. Assume $q > 0$ and let $Y \in \mathfrak{g}$ denote the element $E_{p+q,1} + E_{1,p+q}$. Let $M = M(p, q; \mathbb{F}) \subset G$ be the centralizer of Y in $K \cap H$, then M consists of the elements in G of the form

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & W & 0 \\ 0 & 0 & 0 & u \end{pmatrix}$$

where $V \in U(p-1; \mathbb{F})$, $W \in U(q-1; \mathbb{F})$, and $u \in U(1, \mathbb{F})$. Notice that M is the stabilizer in K of the line $\mathbb{F}(1, 0, \ldots, 0, 1)$ in $\mathbb{F}^{p+q}$, and that the homogeneous space K/M can be identified with the space $B = B(p, q; \mathbb{F})$ which is the projective image of the product of the unit spheres

$$S = \{ y \in \mathbb{F}^{p+q} \mid |y_1|^2 + \cdots + |y_p|^2 = |y_{p+1}|^2 + \cdots + |y_{p+q}|^2 = 1 \}$$

(that is, $B = S/\sim$ where $y \sim yu$ if $y \in S$ and $u \in U(1; \mathbb{F})$).

Let $\Omega \subset X$ be the image of the set $\{z \in Z \mid (z_{p+1}, ..., z_{p+q}) \neq 0\}$, then Ω is an open dense subset of X. For $t \in \mathbb{R}$ let $a_t = \exp tY$, then the map

$$\begin{aligned} K/M \times]0, \infty[&\to G/H \\ (kM, t) &\to ka_t H \end{aligned} \tag{2.4}$$

is a diffeomorphism onto Ω (cf. [25, Proposition 7.1.3]). Via the identification $K/M \cong B$ this map is the pull back to $B \times]0, \infty[\to X$ of the map $S \times]0, \infty[\to Z$ given by

$$(y, t) \to (y_1 \cosh t, ..., y_p \cosh t, y_{p+1} \sinh t, ..., y_{p+q} \sinh t). \tag{2.5}$$

We will frequently denote by (y, t) the point of X which is the image of (2.5), for $y \in S$ and $t \in \mathbb{R}$. For $(y, t) \in \Omega$, y and t are called the *spherical coordinates* of (y, t).

Let $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

3. The K-Decomposition of $C^\infty(B)$

In this section the irreducible representations of K occurring in $C^\infty(B)$ are determined. We start by recalling some results from [14].

Let r be a positive integer, and let $K_r = U(r, 0; \mathbb{F})$, $M_r = U(r-1, 0; \mathbb{F})$, and $S_r = K_r/M_r \cong Z(r, 0; \mathbb{F})$, the unit sphere in $\mathbb{F}^r$. Let $\Delta_r = \Delta(r, 0; \mathbb{F})$, then Δ_r is (the negative of) the usual Laplacian on S^{dr-1}. The decomposition of $C^\infty(S_r)$ into irreducible representations of K_r is determined from [14, Theorem 3.1]. The K_r-types occurring are parametrized by pairs (a, b) of integers with the following properties (3.1)–(3.5):

$$a \equiv b \bmod 2. \tag{3.1}$$

$$\text{If } r = 1 \text{ then } a = |b|. \tag{3.2}$$

$$\text{If } \mathbb{F} = \mathbb{R} \text{ then } a \geqslant 0 \text{ and } 1 \geqslant b \geqslant 0. \tag{3.3}$$

$$\text{If } \mathbb{F} = \mathbb{C} \text{ then } a \geqslant |b|. \tag{3.4}$$

$$\text{If } \mathbb{F} = \mathbb{H} \text{ then } a \geqslant b \geqslant 0. \tag{3.5}$$

The parametrization (originally due to Kostant [17]) arises as follows (leaving out the trivial case $\mathbb{F} = \mathbb{R}$ with $r = 1$). For $a \geqslant 0$ define $\mathscr{H}^a \subset C^\infty(S_r)$ as the eigenspace

$$\mathscr{H}^a = \{f \in C^\infty(S_r) \mid \Delta_r f = a(a + dr - 2) f\}.$$

Then $\mathscr{H}^a$ is invariant under the action of K_r (from the left) and also under the action by $U(1, \mathbb{F})$ (from the right). For each b as above define a representation δ_b of $U(1; \mathbb{F})$ as follows: If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ let $\delta_b(u) = u^b$ for $u \in U(1; \mathbb{F})$, and if $\mathbb{F} = \mathbb{H}$ let δ_b be the $(b+1)$-dimensional irreducible representation of $Sp(1) \cong SU(2)$. Let χ_b denote the character of δ_b. Now define $\mathscr{H}^{a,b}$ as the space of functions $f \in \mathscr{H}^a$ which transform according to δ_b under the (right) action of $U(1; \mathbb{F})$, that is

$$f(x) = \int_{U(1;\mathbb{F})} f(xu)\, \overline{\chi_b(u)}\, du\, (\dim \delta_b)^{-1} \tag{3.6}$$

for all $x \in S_r$. The space $\mathscr{H}^{a,b}$ is invariant under K_r, and it is irreducible as a representation space for $K_r \times U(1; \mathbb{F})$. Moreover the linear span of all the spaces $\mathscr{H}^{a,b}$, with (a, b) satisfying (3.1)–(3.5), is dense in $C^\infty(B_r)$. If $(a, b) \neq (a', b')$ then $\mathscr{H}^{a,b}$ and $\mathscr{H}^{a',b'}$ are not equivalent as representations of $K_r \times U(1; \mathbb{F})$.

From [14, Theorem 3.1] we get an explicit formula for a specific funtion in $\mathscr{H}^{a,b}$. There is a unique function $h_{a,b}$ in $\mathscr{H}^{a,b}$ with value 1 at the origin, and which is fixed under the subgroup $\{([\begin{smallmatrix} u & 0 \\ 0 & m \end{smallmatrix}], u) \in K_r \times U(1; \mathbb{F}) | m \in M_r\}$ of $K_r \times U(1; \mathbb{F})$. This function is given by the following formula:

$$h_{a,b}(y) = (\dim \delta_b)^{-1}\, \chi_b\left(\frac{y_1}{|y_1|}\right) e(r; d; a, b; \theta) \tag{3.7}$$

for $y = (y_1, \ldots, y_r) \in S_r$ with $y_1 \neq 0$. Here $\theta \in [0, \pi/2[$ is determined by $|y_1| = \cos\theta$, and the function $e(r; d; a, b; \theta)$ is defined by

$$\begin{aligned} &e(r; d; a, b; \theta) \\ &= \cos^a \theta\, F\left(-\tfrac{1}{2}(a+b+d-2),\ -\tfrac{1}{2}(a-b); \frac{d}{2}(r-1);\ -\tan^2\theta\right) \end{aligned} \tag{3.8}$$

where F is the hypergeometric function of type $(2, 1)$. If $r = 1$, (3.8) is replaced by $e(1; d; a, b; 0) = 1$ (only $\theta = 0$ is needed since $|y_1| = 1$).

Notice that if $\mathbb{F} = \mathbb{R}$ then $\mathscr{H}^a = \mathscr{H}^{a,b}$ and $\mathscr{H}^a$ is the space of spherical harmonics of degree a on S_r. Furthermore, $h_{a,b}$ is the zonal spherical harmonic function in $\mathscr{H}^a$.

We now consider $C^\infty(B)$. Let Λ be the set of triples μ of integers (j, k, l) with the following properties (i) and (ii):

(i) Conditions (3.1)–(3.5) hold with $r = p$ and $(a, b) = (j, l)$.

(ii) Conditions (3.1)–(3.5) hold with $r = q$ and $(a, b) = (k, l)$.

For $\mu = (j, k, l) \in \Lambda$ consider the subspace of $C^\infty(S)$ spanned by functions of the form

$$y \to \int_{U(1;\mathbb{F})} h'(y'u)\,\overline{h''(y''u)}\,du \tag{3.9}$$

for $y = (y', y'') \in S = S_p \times S_q$, where $h' \in \mathscr{H}^{j,l} \subset C^\infty(S_p)$ and $h'' \in \mathscr{H}^{k,l} \subset C^\infty(S_q)$. We denote by $\mathscr{H}^\mu$ this space, considered as a space of functions on B. Notice that if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the integrand in (3.9) is constant.

In particular, taking $h' = h_{j,l}$ and $h'' = h_{k,l}$ in (3.9), we obtain a function $h_\mu \in \mathscr{H}^\mu$, which is given by

$$h_\mu(y) = (\dim \delta_l)^{-1}\, \chi_l\left(\frac{y_1 \bar{y}_{p+q}}{|y_1 \bar{y}_{p+q}|}\right) e(p; d; j, l; \xi)\, e(q; d; k, l; \eta) \tag{3.10}$$

for $y = (y_1, ..., y_{p+q}) \in S$, with $y_1 \neq 0$ and $y_{p+q} \neq 0$, where $\xi, \eta \in [0, \pi/2[$ are determined by $|y_1| = \cos \xi$ and $|y_{p+q}| = \cos \eta$. To obtain (3.10) we have used the following formula

$$\int_{U(1,\mathbb{F})} \chi_l(vu)\,\overline{\chi_l(wu)}\,du = \chi_l(v\bar{w}) \dim \delta_l$$

for $v, w \in U(1; \mathbb{F})$. We now have

PROPOSITION 3.1. *The set Λ parametrizes the irreducible representations of K in $C^\infty(B)$. For each $\mu \in \Lambda$ the corresponding K-type has multiplicity one in $C^\infty(B)$ and the space of functions transforming according to this K-type is $\mathscr{H}^\mu$. In $\mathscr{H}^\mu$ there is a unique function with the value* 1 *at the origin and which is fixed under M. This function is h_μ.*

Proof. It follows from the description of K_r-types in $C^\infty(S_r)$ that $C^\infty(B)$ is spanned by functions of the form (3.9) with $h' \in \mathscr{H}^{j,l} \subset C^\infty(S_p)$ and $h'' \in \mathscr{H}^{k,m} \subset C^\infty(S_q)$, where (j, l) (resp. (k, m)) satisfy (3.1)–(3.5) with $r = p$ (resp. $r = q$). However the integral (3.9) vanishes unless $l = m$, because of (3.6). Hence $C^\infty(B)$ is spanned by the spaces $\mathscr{H}^\mu$. The irreducibility of $\mathscr{H}^\mu$ under K follows from the fact that the trivial representation has multiplicity one in $\delta_l \otimes \delta_l^*$ (δ_l^* denotes the representation contragradient to δ_l). Since K-types thus have multiplicity one in $C^\infty(K/M)$, each K-type contains a unique one-dimensional space of M-fixed vectors (Frobenius reciprocity). Therefore, h_μ is unique. ▌

Remark. Proposition 3.1 answers a question posed in [5, p. 399, lines 4–5].

4. The K-Decomposition of the Eigenspaces

In this section the K-types occurring in $\mathscr{E}_\lambda$ are described, for each $\lambda \in \mathbb{C}$. Using the spherical coordinates (2.4) we see that every K-type in $\mathscr{E}_\lambda$ must occur in $C^\infty(B)$ and hence is given by a parameter $\mu \in \Lambda$. For $\mu \in \Lambda$ let

$$\mathscr{E}_{\lambda,\mu} = \{f \in \mathscr{E}_\lambda \mid f \text{ is } K\text{-finite of type } \mu\}. \tag{4.1}$$

The functions in $\mathscr{E}_{\lambda,\mu}$ can be explicitly desribed using special functions. For $j \in \mathbb{Z}$ and $k \geqslant 0$ we define a function $\varphi_{\lambda,j,k} \in C^\infty(\mathbb{R})$ by

$$\varphi_{\lambda,j,k}(t) = \cosh^j t \sinh^k t \,\Gamma(\alpha+1)^{-1} \,\varphi_{i\lambda}^{(\alpha,\beta)}(t) \tag{4.2}$$

where $\alpha = k + dq/2 - 1$, $\beta = j + dp/2 - 1$, and $\varphi_{i\lambda}^{(\alpha,\beta)}$ is the Jacobi function

$$\varphi_{i\lambda}^{(\alpha,\beta)}(t) = F(\tfrac{1}{2}(\alpha+\beta+1+\lambda), \tfrac{1}{2}(\alpha+\beta+1-\lambda); \alpha+1; -\sinh^2 t).$$

(The factor $\Gamma(\alpha+1)^{-1}$ is introduced only to simplify certain formulas). For $\lambda \in \mathbb{C}$, $\mu = (j, k, l) \in \Lambda$, and $h \in \mathscr{H}^\mu$ we define a C^∞-function $\Phi_\lambda(h)$ on Ω by

$$\Phi_\lambda(h)(y, t) = h(y)\, \varphi_{\lambda,j,k}(t), \tag{4.3}$$

for $y \in B$ and $t \in\,]0, \infty[$. Obviously $\Phi_\lambda(h)$ is K-finite of type μ.

Lemma 4.1. *$\Phi_\lambda(h)$ extends to a K-finite C^∞-function on X, also given by* (4.3) (*with* $t \in [0, \infty[$).

Proof. Through (2.2) we may assume $\mathbb{F} = \mathbb{R}$. Since $h \in \mathscr{H}^\mu$ is a sum of products of spherical harmonics, we can extend it to a polynomial $\bar{h}$ on $\mathbb{R}^{p+q}$, homogeneous of degree j on $\mathbb{R}^p$ and homogeneous of degree k on $\mathbb{R}^q$. Then

$$\Phi_\lambda(h)(x) = C\bar{h}(x)\, F(a, b, c; -x_{p+1}^2 - \cdots - x_{p+q}^2)$$

for $x \in X$, for suitable constants C, a, b, and c. From this the lemma follows. ∎

Theorem 4.2. *For each $\lambda \in \mathbb{C}$ and $\mu \in \Lambda$ the map $\Phi_\lambda : \mathscr{H}^\mu \to C^\infty(X)$ is a K-isomorphism onto $\mathscr{E}_{\lambda,\mu}$.*

Proof (*cf.* [5, p. 427] *and for* $p = 1$ [11, Theorem 4.5]). Expressing in spherical coordinates the Laplacian on functions K-finite of type μ, and separating the variables gives an equation of hypergeometric type in t, whose only solutions regular at 0 are proportional to $\varphi_{\lambda,\mu}$. Therefore $\Phi_\lambda(h) \in \mathscr{E}_{\lambda,\mu}$, and Φ_λ is onto. ∎

In particular, each K-type in $\mathscr{E}_\lambda$ has multiplicity one. Also it follows that the space of M-fixed vectors in $\mathscr{E}_{\lambda,\mu}$ is spanned by the function

$$f_{\lambda,\mu} = \Phi_\lambda(h_\mu) \tag{4.4}$$

where h_μ is given by (3.10).

Notice that Eq. (4.2) can be used as the starting point for obtaining Faraut's solution to problem A, using the inversion formula for the Jacobi transform which has been given an elementary proof in [15] (cf. [16, p. 36] and [5, p. 428]). Actually, using (2.2) the solution to problem A can be reduced to the case $\mathbb{F} = \mathbb{R}$, where it was solved in [19].

5. The Action of Y

Let $\mathscr{E}_\lambda^M$ denote the space of M-fixed K-finite functions in $\mathscr{E}_\lambda$. From the previous section we know that $\mathscr{E}^M \cap \mathscr{E}_{\lambda,\mu}$ is spanned by $f_{\lambda,\mu}$, for each $\mu \in \Lambda$. Since M commutes with Y it is clear that $\pi_\lambda(Y)$ leaves $\mathscr{E}_\lambda^M$ invariant. Using the explicit formulas for $f_{\lambda,\mu}$ we can now determine explicitly the action of $\pi_\lambda(Y)$ on $\mathscr{E}_\lambda^M$.

For the statement of the following theorem we define $f_{\lambda,\nu} = 0$ if $\nu \notin \Lambda$.

Theorem 5.1. *Let $\lambda \in \mathbb{C}$ and let $\mu = (j, k, l) \in \Lambda$. Then $\pi_\lambda(Y) f_{\lambda,\mu} = \sum_\nu a_\nu f_{\lambda,\nu}$ with the following coefficients a_ν:*

$$a_{j+1,k+1,l+1} = \left[\frac{j+l+dp-2}{2j+dp-2}\right]\left[\frac{k+l+dq-2}{2k+dq-2}\right]\left[\frac{2l+2d-4}{2l+d-2}\right] \times \frac{1}{4}(j+k+\rho+\lambda)(j+k+\rho-\lambda),$$

$$a_{j+1,k+1,l-1} = \left[\frac{j-l+dp-d}{2j+dp-2}\right]\left[\frac{k-l+dq-d}{2k+dq-2}\right]\left[\frac{2l}{2l+d-2}\right] \times \frac{1}{4}(j+k+\rho+\lambda)(j+k+\rho-\lambda),$$

$$a_{j+1,k-1,l+1} = -\left[\frac{j+l+dp-2}{2j+dp-2}\right]\left[\frac{k-l}{2k+dq-2}\right]\left[\frac{2l+2d-4}{2l+d-2}\right],$$

$$a_{j+1,k-1,l-1} = -\left[\frac{j-l+dp-d}{2j+dp-2}\right]\left[\frac{k+l+d-2}{2k+dq-2}\right]\left[\frac{2l}{2l+d-2}\right],$$

$$a_{j-1,k+1,l+1} = \left[\frac{j-l}{2j+dp-2}\right]\left[\frac{k+l+dq-2}{2k+dq-2}\right]\left[\frac{2l+2d-4}{2l+d-2}\right]$$
$$\times\frac{1}{4}(j-k-dq+\rho+\lambda)(j-k-dq+\rho-\lambda),$$

$$a_{j-1,k+1,l-1} = \left[\frac{j+l+d-2}{2j+dp-2}\right]\left[\frac{k-l+dq-d}{2k+dq-2}\right]\left[\frac{2l}{2l+d-2}\right]$$
$$\times\frac{1}{4}(j-k-dq+\rho+\lambda)(j-k-dq+\rho-\lambda),$$

$$a_{j-1,k-1,l+1} = -\left[\frac{j-l}{2j+dp-2}\right]\left[\frac{k-l}{2k+dq-2}\right]\left[\frac{2l+2d-4}{2l+d-2}\right],$$

$$a_{j-1,k-1,l-1} = -\left[\frac{j+l+d-2}{2j+dp-2}\right]\left[\frac{k+l+d-2}{2k+dq-2}\right]\left[\frac{2l}{2l+d-2}\right],$$

and $a_\nu = 0$ *otherwise, where each term in square brackets is replaced by* 1 *if its denominater is zero.*

Proof. Obviously $\pi_\lambda(Y) f_{\lambda,\mu} = \sum a_\nu f_{\lambda,\nu}$ for some unique constants a_ν. To determine these constants we compute $\pi_\lambda(Y) f_{\lambda,\mu}(x)$ explicitly for $x \in X$. It suffices to consider x in a dense subset of X, so we may assume $x = (y, t) \in \Omega$ with $y_1 \neq 0$ and $y_{p+q} \neq 0$. By definition

$$\pi_\lambda(Y) f_{\lambda,\mu}(x) = \left.\frac{d}{ds}\right|_0 f_{\lambda,\mu}(a_{-s}x) \tag{5.1}$$

We define $y(s) \in B$ and $t(s) > 0$ by

$$a_{-s}x = (y(s), t(s))$$

for s sufficiently small so that $a_{-s}x \in \Omega$. We then have the following set of identities, modulo right multiplication from $U(1; \mathbb{F})$:

$$\begin{aligned}
y_1(s)\cosh t(s) &= y_1 \cosh t \cosh s - y_{p+q} \sinh t \sinh s,\\
y_2(s)\cosh t(s) &= y_2 \cosh t,\\
&\vdots\\
y_p(s)\cosh t(s) &= y_p \cosh t,\\
y_{p+1}(s)\sinh t(s) &= y_{p+1} \sinh t,\\
&\vdots\\
y_{p+q-1}(s)\sinh t(s) &= y_{p+q-1} \sinh t,\\
y_{p+q}(s)\sinh t(s) &= -y_1 \cosh t \sinh s + y_{p+q} \sinh t \cosh s.
\end{aligned}$$

Let ξ, $\eta \in [0, \pi/2[$ be defined by $|y_1| = \cos \xi$ and $|y_{p+q}| = \cos \eta$, and let φ be defined by $\mathrm{Re}(y_1 \bar{y}_{p+q}/|y_1 \bar{y}_{p+q}|) = \cos \varphi$ and $0 \leqslant \varphi \leqslant \pi$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, $y_1 \bar{y}_{p+q}/|y_1 \bar{y}_{p+q}| = e^{i\varphi}$ and $0 \leqslant \varphi < 2\pi$ if $\mathbb{F} = \mathbb{C}$. (Notice that $\varphi \in \{0, \pi\}$ if $\mathbb{F} = \mathbb{R}$).

Define $\xi(s)$, $\eta(s)$, and $\varphi(s)$ similarly. For x in a dense subset of X these functions depend smoothly on s in a neighborhood of 0. By straightforward calculations it follows from the identities above that

$$\left.\frac{dt}{ds}\right|_0 = -\cos \xi \cos \eta \cos \varphi, \tag{5.2}$$

$$\left.\frac{d\xi}{ds}\right|_0 = \sin \xi \cos \eta \cos \varphi \tanh t, \tag{5.3}$$

$$\left.\frac{d\eta}{ds}\right|_0 = \cos \xi \sin \eta \cos \varphi \coth t, \tag{5.4}$$

$$\left.\frac{d\varphi}{ds}\right|_0 = \left(\frac{\cos \eta}{\cos \xi} \tanh t + \frac{\cos \xi}{\cos \eta} \coth t\right) \sin \varphi. \tag{5.5}$$

Let $\chi(d; l; \varphi)$ denote the following function of φ:

$$\chi(d; l; \varphi) = \begin{cases} e^{il\varphi} & \text{if } \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}, \\ (l+1)^{-1} \dfrac{\sin(l+1)\varphi}{\sin \varphi} & \text{if } \mathbb{F} = \mathbb{H}, \end{cases} \tag{5.6}$$

then it follows from (3.10) and (4.4) that $f_{\lambda,\mu}$ is given by the equation

$$f_{\lambda,\mu}(x) = \chi(d; l; \varphi)\, e(p; d; j, l; \xi)\, e(q; d; k, l; \eta)\, \varphi_{\lambda,j,k}(t) \tag{5.7}$$

since

$$(\dim \delta_l)^{-1} \chi_l\left(\frac{y_1 \bar{y}_{p+q}}{|y_1 \bar{y}_{p+q}|}\right) = \chi(d; l; \varphi)$$

(for $\mathbb{F} = \mathbb{H}$, cf. [28, p. 49]).

From (5.1)–(5.5) and (5.7) it follows that

$$\pi_\lambda(Y) f_{\lambda,\mu}(x)$$

$$= \Big\{ \sin \xi \frac{d}{d\xi} e(p; d; j, l; \xi) \cos \eta \; e(q; d; k, l; \eta) \tanh t \; \varphi_{\lambda,j,k}(t)$$

$$
\begin{aligned}
&+\cos\xi\, e(p; d; j, l; \xi) \sin\eta \frac{d}{d\eta} e(q; d; k, l; \eta) \coth t\, \varphi_{\lambda,j,k}(t) \\
&\left.-\cos\xi\, e(p; d; j, l; \xi) \cos\eta\, e(q; d; k, l; \eta) \frac{d}{dt} \varphi_{\lambda,j,k}(t)\right\} \cos\varphi\, \chi(d; l; \varphi) \\
&+ e(p; d; j, l; \xi)\, e(q; d; k, l; \eta)\, \varphi_{\lambda,j,k}(t) \\
&\times \left(\frac{\cos\eta}{\cos\xi} \tanh t + \frac{\cos\xi}{\cos\eta} \coth t\right) \sin\varphi \frac{d}{d\varphi} \chi(d; l; \varphi).
\end{aligned} \tag{5.8}
$$

The terms involving φ in (5.8) are determined from the following lemma, the proof of which is elementary.

LEMMA 5.2. *We have*

(i)
$$
\begin{aligned}
\cos\varphi\, \chi(d; l; \varphi) = {} & \frac{1}{2}\left[\frac{2l+2d-4}{2l+d-2}\right] \chi(d; l+1; \varphi) \\
&+ \frac{1}{2}\left[\frac{2l}{2l+d-2}\right] \chi(d; l-1; \varphi),
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\sin\varphi \frac{d}{d\varphi} \chi(d; l; \varphi) = {} & \frac{l}{2}\left[\frac{2l+2d-4}{2l+d-2}\right] \chi(d; l+1; \varphi) \\
&- \frac{l+d-2}{2}\left[\frac{2l}{2l+d-2}\right] \chi(d; l-1; \varphi),
\end{aligned}
$$

where each term in square brackets is replaced by 1 *if its denominator is zero.*

Insertion into (5.8) gives

$$
\begin{aligned}
\pi_\lambda(Y) f_{\lambda,\mu} = {} & \frac{1}{2}\left[\frac{2l+2d-4}{2l+d-2}\right] A_1 \chi(d; l+1; \varphi) \\
&+ \frac{1}{2}\left[\frac{2l}{2l+d-2}\right] A_2 \chi(d; l-1; \varphi)
\end{aligned} \tag{5.9}
$$

where

$$
\begin{aligned}
A_1 = {} & \left(\frac{l}{\cos\xi} + \sin\xi \frac{d}{d\xi}\right) e(p; d; j, l; \xi) \\
& \times \cos\eta\, e(q; d; k, l; \eta) \tanh t\, \varphi_{\lambda,j,k}(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\xi\, e(p; d; j, l; \xi) \\
&\times\left(\frac{l}{\cos\eta} + \sin\eta\,\frac{d}{d\eta}\right) e(q; d; k, l; \eta) \coth t\, \varphi_{\lambda,j,k}(t) \\
&-\cos\xi\, e(p; d; j, l; \xi) \cos\eta\, e(q; d; k, l; \eta) \frac{d}{dt}\varphi_{\lambda,j,k}(t)
\end{aligned}
$$

and A_2 is obtained from A_1 by formally replacing l with $-l-d+2$.

The terms involving ξ and η are now determined from the next lemma, the proof of which follows from standard formulas for hypergeometric functions (e.g. [14, Lemma 4.1]). For the statement we define $e(r; d; a, b; \theta) = 0$ whenever a and b do not satisfy (3.1)–(3.5).

LEMMA 5.3. *Let* $e(\theta) = e(r; l; a, b; \theta)$ *be defined by* (3.8), *and let* $\theta \in [0, \pi/2[$ *if* $r > 1$, *or* $\theta = 0$ *if* $r = 1$. *Then*

$$
\text{(i)} \qquad \cos\theta\, e(\theta) = \left[\frac{a+b+dr-2}{2a+dr-2}\right] e(r; d; a+1, b+1; \theta) + \left[\frac{a-b}{2a+dr-2}\right] e(r; d; a-1, b+1; \theta),
$$

$$
\begin{aligned}
\text{(ii)} \qquad &\left(\frac{b}{\cos\theta} + \sin\theta\,\frac{d}{d\theta}\right) e(\theta) \\
&= a\left[\frac{a+b+dr-2}{2a+dr-2}\right] e(r; d; a+1, b+1; \theta) \\
&\quad -(a+dr-2)\left[\frac{a-b}{2a+dr-2}\right] e(r; d; a-1, b+1; \theta),
\end{aligned}
$$

where each term in square brackets is replaced by 1 *if its denominator is zero.*

When these formulas are inserted into A_1 in (5.9) the result is the following, after rearrangement of the terms

$$
\begin{aligned}
A_1 &= \left[\frac{j+l+dp-2}{2j+dp-2}\right]\left[\frac{k+l+dq-2}{2k+dq-2}\right] \\
&\quad \times e(p; d; j+1, l+1; \xi)\, e(q; d; k+1, l+1; \eta)\, B_1 \\
&\quad + \left[\frac{j+l+dp-2}{2j+dp-2}\right]\left[\frac{k-l}{2k+dq-2}\right] \\
&\quad \times e(p; d; j+1, l+1; \xi)\, e(q; d; k-1, l+1; \eta)\, B_2
\end{aligned}
$$

$$+\left[\frac{j-l}{2j+dp-2}\right]\left[\frac{k+l+dq-2}{2k+dq-2}\right]$$
$$\times e(p; d; j-1, l+1; \xi)\, e(q; d; k+1, l+1; \eta)\, B_3$$
$$+\left[\frac{j-l}{2j+dp-2}\right]\left[\frac{k-l}{2k+dq-2}\right]$$
$$\times e(p; d; j-1; l+1; \xi)\, e(q; d; k-1, l+1; \eta)\, B_4$$

where

$$B_1 = (j \tanh t + k \coth t)\, \varphi_{\lambda,j,k}(t) - \frac{d}{dt}\varphi_{\lambda,j,k}(t),$$

$$B_2 = (j \tanh t - (k+dq-2) \coth t)\, \varphi_{\lambda,j,k}(t) - \frac{d}{dt}\varphi_{\lambda,j,k}(t),$$

$$B_3 = (-(j+dp-2) \tanh t + k \coth t)\, \varphi_{\lambda,j,k}(t) - \frac{d}{dt}\varphi_{\lambda,j,k}(t),$$

$$B_4 = (-(j+dp-2) \tanh t - (k+dq-2) \coth t)\, \varphi_{\lambda,j,k}(t) - \frac{d}{dt}\varphi_{\lambda,j,k}(t).$$

The terms B_1, B_2 etc. are determined from the following lemma, the proof of which easily follows from standard relations for hypergeometric functions. Notice that B_3 and B_4 are obtained from B_1 and B_2 by a formal replacement of j with $-j-dp+2$.

LEMMA 5.4. *Let $\varphi_{\lambda,j,k}(t)$ be defined by* (4.2). *Then*

(i) $$(j \tanh t + k \coth t)\, \varphi_{\lambda,j,k} - \frac{d}{dt}\varphi_{\lambda,j,k}$$
$$= \tfrac{1}{2}(j+k+\rho+\lambda)(j+k+\rho-\lambda)\, \varphi_{\lambda,j+1,k+1}, \quad \textit{and if } k>0$$

(ii) $$(j \tanh t - (k+dq-2) \coth t)\, \varphi_{\lambda,j,k} - \frac{d}{dt}\varphi_{\lambda,j,k}$$
$$= -2\varphi_{\lambda,j+1,k-1}.$$

Inserting these relations into the expression for A_1 we get

$$\frac{1}{2}\left[\frac{2l+2d-4}{2l+d-2}\right] A_1 \chi(d; l+1; \varphi) = \sum a_{j\pm 1,k\pm 1,l+1} f_{\lambda,j\pm 1,k\pm 1,l+1}$$

where the coefficients are as stated in the theorem. Replacing l with $-l-d+2$ we get a similar expression for the second term in (5.9), and the theorem is proved. ∎

6. The Composition Series of π_λ

The composition series of π_λ is completely described if all closed invariant subspaces of $\mathscr{E}_\lambda$ are determined. With this purpose we define some subspaces of $\mathscr{E}_\lambda$:

If $\lambda \in \rho + 2\mathbb{Z}_+$ let

$$T_\lambda = \{\mu = (j, k, l) \in \Lambda \mid j + k \leqslant \lambda - \rho\}$$

and let $\mathscr{T}_\lambda$ be the finite dimensional subspace $\sum_{\mu \in T_\lambda} \mathscr{E}_{\lambda,\mu}$ of $\mathscr{E}_\lambda$.

If $q > 1$ and $\lambda \in \rho + dp + 2\mathbb{Z}$, let

$$V_\lambda = \{\mu = (j, k, l) \in \Lambda \mid k - j \leqslant \lambda + \rho - dq\}$$

and let $\mathscr{V}_\lambda$ be the closure in $\mathscr{E}_\lambda$ of $\Sigma_{\mu \in V_\lambda} \mathscr{E}_{\lambda,\mu}$. Notice that $\mathscr{V}_\lambda = \{0\}$ if $p = 1$ and $\lambda \leqslant \rho - d$.

If $q > p = 1$, $\mathbb{F} = \mathbb{C}$, and $\lambda \in \rho + 2\mathbb{Z}_+$, let V_λ^+ and V_λ^- be defined by

$$V_\lambda^\pm = \{\mu = (j, k, l) \in \Lambda \mid k \pm l \leqslant \lambda - \rho\}.$$

and let $\mathscr{V}_\lambda^\pm$ be the closure of $\Sigma_{\mu \in V_\lambda^\pm} \mathscr{E}_{\lambda,\mu}$. Notice that $\mathscr{T}_\lambda = \mathscr{V}_\lambda^+ \cap \mathscr{V}_\lambda^-$ and $\mathscr{V}_\lambda = \mathscr{V}_\lambda^+ + \mathscr{V}_\lambda^-$.

If $p > 1$ and $\lambda \in \rho + dq + 2\mathbb{Z}$, let

$$U_\lambda = \{\mu = (j, k, l) \in \Lambda \mid j - k \geqslant \lambda - \rho + dq\},$$

and let $\mathscr{U}_\lambda$ be the closure of $\Sigma_{\mu \in U_\lambda} \mathscr{E}_{\lambda,\mu}$. If moreover dq is even and $\lambda \geqslant \rho$, let $\mathscr{W}_\lambda = \mathscr{T}_\lambda + \mathscr{U}_\lambda$. Notice that if $q > 1$ then $\mathscr{U}_\lambda = \mathscr{V}_{-\lambda}$, and if $q = 1$ and $\lambda \leqslant \rho - d$ then $\mathscr{U}_\lambda = \mathscr{E}_\lambda$.

If $p > q = 1$, $\mathbb{F} = \mathbb{C}$, and $\lambda \in \rho + 2\mathbb{Z}_+$, let W_λ^+ and W_λ^- be defined by

$$W_\lambda^\pm = \{\mu = (j, k, l) \in \Lambda \mid j \pm l \geqslant \lambda - \rho + 2 \text{ or } j \mp l \leqslant \lambda - \rho\}$$

and let $\mathscr{W}_\lambda^\pm$ be the closure of $\Sigma_{\mu \in W_\lambda^\pm} \mathscr{E}_{\lambda,\mu}$. Notice that $\mathscr{W}_\lambda = \mathscr{W}_\lambda^+ \cap \mathscr{W}_\lambda^-$ and $\mathscr{E}_\lambda = \mathscr{W}_\lambda^+ + \mathscr{W}_\lambda^-$.

Notice that since $\pi_{-\lambda} = \pi_\lambda$ we only have to consider λ with $\operatorname{Re} \lambda \geqslant 0$.

THEOREM 6.1. *Let $\lambda \in \mathbb{C}$ and assume* $\operatorname{Re} \lambda \geqslant 0$. *The closed G-invariant subspaces of $\mathscr{E}_\lambda$ are those spaces among the following, which are defined for the value of d, p, q, and λ in question*: $\{0\}$, $\mathscr{T}_\lambda$, $\mathscr{U}_\lambda$, $\mathscr{V}_\lambda$, $\mathscr{W}_\lambda$, $\mathscr{V}_\lambda^\pm$, $\mathscr{W}_\lambda^\pm$, $\mathscr{E}_\lambda$.

COROLLARY 6.2. *Let $\lambda \in \mathbb{C}$ and* $\operatorname{Re} \lambda \geqslant 0$. *$\pi_\lambda$ is irreducible if and only if one of the following conditions hold*

(i) $\lambda - \rho \notin \mathbb{Z}$.

(ii) *dp and dq are even and* $\lambda - \rho \in 2\mathbb{Z} + 1$.

(iii) *p and q are odd, $\mathbb{F}=\mathbb{R}$, $\lambda-\rho\in 2\mathbb{Z}$ and $\lambda<\rho$.*

(iv) *$p=1$ or $q=1$ and $\lambda\leqslant\rho-d$.*

(v) *$p=q=1$, $\mathbb{F}=\mathbb{H}$ and $\lambda=1$.*

(vi) *$p=q=1$, $\mathbb{F}=\mathbb{R}$ and $\lambda\neq 0$.*

COROLLARY 6.3. *Let $\lambda\in\mathbb{C}$ and* $\operatorname{Re}\lambda\geqslant 0$. *$\pi_\lambda$ has a unique irreducible non-zero subrepresentation except when $p>1$, dq is even, and $\lambda\in\rho+2\mathbb{Z}_+$, where both $\mathcal{T}_\lambda$ and $\mathcal{U}_\lambda$ are closed invariant subspaces.*

Proof of Theorem 6.1. From general principles (cf. [3, Theorem 3.17]) it follows that the G-invariant closed subspaces of $\mathcal{E}_\lambda$ are precisely the closures of the $(\mathfrak{g}, K)$-invariant subspaces of the space of K-finite vectors in $\mathcal{E}_\lambda$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Since the K-orbit of Y spans $\mathfrak{p}$ linearly, a subspace of K-finite vectors in $\mathcal{E}_\lambda$ is $(\mathfrak{g}, K)$-stable if and only if it is stable under $\pi_\lambda(Y)$ and K. From the expression for $\pi_\lambda(Y)f_{\lambda,\mu}$ in Theorem 5.1 we can determine the $(\pi_\lambda(Y), K)$-stable subspaces just by determining which coefficients vanish. From this the theorem is easily obtained. ∎

The justification of the corollaries is straightforward from Theorem 6.1.

Remarks. For $\mathbb{F}=\mathbb{R}$ the irreducibility of $\mathcal{U}_\lambda$ is proved in [21] by a similar method, and the irreducitibility of $\mathcal{E}_\lambda$ under condition (i) is proved in [22]. Corollary 6.3 is closely related to [5, Theorem 3.2].

The subspaces $\mathcal{U}_\lambda$ are of particular interest because they are related to the discrete series for X. Let dx be the measure on X given by

$$\int_X f(x)\,dx=\int_B\int_0^\infty f(y,t)\,A(t)\,dt\,dy$$

where $A(t)=(2\cosh t)^{dp-1}(2\sinh t)^{dq-1}$, and dy is invariant measure on $B=K/M$. Up to normalization dx is the unique G-invariant measure on X (cf. [25, Theorem 8.1.1]). Let $D\subset\mathbb{C}$ be defined by

$$D=\{\lambda\in\mathbb{C}\mid\lambda>0,\ \lambda-\rho\in dq+2\mathbb{Z}\}.$$

THEOREM 6.4. *Let $\lambda\in\mathbb{C}$ and* $\operatorname{Re}\lambda\geqslant 0$, *and let $\mu\in\Lambda$. Then $\mathcal{E}_{\lambda,\mu}\subset L^2(X)$ if and only if $p>1$, $\lambda\in D$ and $\mu\in U_\lambda$.*

Proof. Follows immediately from Theorem 4.2 and [6, Lemma A.3]. ∎

From Theorem 6.4 one concludes that the discrete series for X consists precisely of the closures of $\mathcal{U}_\lambda$ in $L^2(X)$ for $\lambda\in D$ (using, e.g. [1, Theorem 1.5]). Since we know from Theorem 6.2 that each of these spaces is irreducible we thus have a complete description of the discrete series (the parametrization by D is due to Faraut [5, p. 432]). In particular, we have explicitly described the K-type structure of each discrete series representation.

Remark. It is striking that for $q=1$ all K-finite functions in $\mathscr{E}_\lambda$ are square integrable if $0<\lambda\leqslant\rho-d$ and $\lambda\equiv\rho+dq \bmod 2$.

7. Poisson Transformations

Consider the *principal series* representations of G associated to X, defined as follows. Let Γ be the G-invariant cone

$$\Gamma=\{z\in\mathbb{F}^{p+q}\mid[z,z]=0, z\neq 0\}$$

and let $\Xi=\Gamma/\sim$ where $\sim$ is the equivalence relation $z\sim zu$ for all $z\in\Gamma$, $u\in U(1;\mathbb{F})$. For $\lambda\in\mathbb{C}$ let $C_\lambda^\infty(\Xi)$ denote the space of C^∞-functions φ on Γ satisfying

$$\varphi(zv)=|v|^{\lambda-\rho}\,\varphi(z)$$

for all $z\in\Gamma$ and $v\in\mathbb{F}$, $v\neq 0$ (in particular, we may consider φ a function on Ξ). Notice that $S\subset\Gamma$, and by restriction we can identify $C_\lambda^\infty(\Xi)$ with $C^\infty(B)$ for each $\lambda\in\mathbb{C}$. Denote by σ_λ the regular representation of G on $C_\lambda^\infty(\Xi)$. Notice that σ_λ is unitarisable in $L^2(B)$ when λ is purely imaginary ([5, Proposition 5.1]).

It is of interest to compare the representations σ_λ and π_λ of G. When $p=1$, X is Riemannian and comparison of Theorem 6.1 with [14, Theorem 5.1] shows that the composition series of σ_λ and π_λ resemble each other if $\operatorname{Re}\lambda\geqslant 0$. In this case it actually follows from [10, Theorem IV 1.4] that the $(\mathfrak{g}, K)$-modules of σ_λ and π_λ are isomorphic. Moreover the isomorphism is given by the Poisson transformation defined by

$$\mathscr{P}_\lambda\varphi(x)=\int_B|[x,b]|^{-\lambda-\rho}\,\varphi(b)\,db \tag{7.1}$$

for $\varphi\in C^\infty(B)$ and $x\in X$. Here, by definition, $|[x,b]|=|[z,y]|$ where $z\in Z$ and $y\in S$ are chosen in the class of x and b, respectively.

We consider now $p>1$. In analogy with (7.1) we define

$$\mathscr{P}_\lambda\varphi(x)=\Gamma(-\tfrac{1}{2}(\lambda+\rho-d))^{-1}\int_B|[x,b]|^{-\lambda-\rho}\,\varphi(b)\,db \tag{7.2}$$

for $\varphi\in C^\infty(B)$ and $x\in X$. From [5, Proposition 5.3] we have that $\mathscr{P}_\lambda\varphi$ is well defined by this integral if $\operatorname{Re}\lambda$ is sufficiently small, and by analytic continuation for all $\lambda \in \mathbb{C}$ (the factor in front cancels the poles in the meromorphic continuation of the integral). Furthermore, $\mathscr{P}_\lambda \varphi$ is in $\mathscr{E}_\lambda$ and

$$\mathscr{P}_\lambda : C_\lambda^\infty(\Xi) \to \mathscr{E}_\lambda$$

is G-equivariant. $\mathscr{P}_\lambda$ is called the *Poisson transformation* for X (with parameter λ).

Let $\mathscr{I}_\lambda \subset \mathscr{E}_\lambda$ denote the closure of the image of $\mathscr{P}_\lambda$. We now determine $\mathscr{I}_\lambda$, which due to the equivariance of $\mathscr{P}_\lambda$ must be one of the spaces mentioned in Theorem 6.1.

THEOREM 7.1. *Assume* $p > 1$ *and* $\operatorname{Re}\lambda \geq 0$. *If* $\lambda - \rho \in dq + 2\mathbb{Z}$ *then* $\mathscr{I}_\lambda = \mathscr{U}_\lambda$, *otherwise* $\mathscr{I}_\lambda = \mathscr{E}_\lambda$.

Proof. Using the expansion of the functions in $C^\infty(B)$ into their K-type components, it suffices to determine those K-types $\mu \in \Lambda$ for which $\mathscr{P}_\lambda(\mathscr{H}^\mu) \neq \{0\}$. This is done using the following lemma, after which we will resume the proof of Theorem 7.1. Let the measure on B be normalized such that B has volume 1.

LEMMA 7.2. *Let* $\mu = (j, k, l) \in \Lambda$ *and* $h \in \mathscr{H}^\mu$. *Then*

$$\mathscr{P}_\lambda h = \beta(\lambda, \mu)\, \Phi_\lambda(h) \tag{7.3}$$

where $\Phi_\lambda(h)$ *is given by Lemma* 4.1, *and* $\beta(\lambda, \mu)$ *is the constant*

$$\beta(\lambda, \mu) = \frac{\Gamma\left(\frac{dp}{2}\right)\Gamma\left(\frac{dq}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} (-1)^{(j-k)/2} \frac{\Gamma(\frac{1}{2}(\lambda + \rho + j + k))}{\Gamma(\frac{1}{2}(\lambda + \rho))\,\Gamma(-\frac{1}{2}(\lambda + \rho - dp - j + k))}. \tag{7.4}$$

Proof. The existence of some constant $\beta(\lambda, \mu)$ such that (7.3) holds follows from the equivariance of $\mathscr{P}_\lambda$ and Theorem 4.2. Taking $h = h_\mu$ (given by (3.10)) the value of $\beta(\lambda, \mu)$ can be determined. The computation is nearly identical with Appendix B of [5]. ∎

From the lemma we see that $\mathscr{P}_\lambda(\mathscr{H}^\mu) = \{0\}$ if and only if

(i) $j - k \in \lambda + \rho - dp - 2\mathbb{Z}_+$ or

(ii) $\lambda + \rho \in -2\mathbb{Z}_+$ and $j + k > -\lambda - \rho$.

Now (ii) is excluded since $\operatorname{Re}\lambda \geq 0$, and Theorem 7.1 follows. ∎

Remark. For $p=1$, where $\mathscr{P}_\lambda$ is defined by (7.1) we also have Lemma 7.2, but with (7.4) replaced by

$$\beta(\lambda,\mu)=\Gamma\left(\frac{dq}{2}\right)(-1)^{(j-k)/2}\frac{\Gamma(\frac{1}{2}(\lambda+\rho+j+k))\,\Gamma(-\frac{1}{2}(\lambda+\rho-d))}{\Gamma(\frac{1}{2}(\lambda+\rho))\,\Gamma(-\frac{1}{2}(\lambda+\rho-d-j+k))} \tag{7.5}$$

(cf., also [11, Theorem 4.5]). In this case it is well known (and follows from (7.5)) that $\mathscr{I}_\lambda=\mathscr{E}_\lambda$ for $\operatorname{Re}\lambda\geqslant 0$ (except in the trivial case $\mathbb{F}=\mathbb{R}$, $p=q=1$).

From the proof of Theorem 7.1 we obtain the following corollary:

COROLLARY 7.3. *Let* $\lambda\in\mathbb{C}$ *and* $\operatorname{Re}\lambda\geqslant 0$.

(a) *If* $\lambda-\rho\in 2\mathbb{Z}$ *and* $\lambda\geqslant\rho$ *then* $\mathscr{I}_{-\lambda}=\mathscr{T}_\lambda$.

(b) *If* $q>1$ *and* (1): $\lambda-\rho\in 2\mathbb{Z}$, $\lambda<\rho$, *and* dp *is even, or* (2): $\lambda-\rho\in 1+2\mathbb{Z}$ *and* dp *is odd, then* $\mathscr{I}_{-\lambda}=\mathscr{V}_\lambda$.

(c) *In all other cases than* (a) *and* (b), $\mathscr{I}_{-\lambda}=\mathscr{E}_\lambda$.

In particular, for every G-invariant irreducible closed subspace $\mathscr{S}$ *of* $\mathscr{E}_\lambda$ *we have* $\mathscr{S}=\mathscr{I}_\lambda$ *or* $\mathscr{S}=\mathscr{I}_{-\lambda}$.

Proof. Statements (a), (b), and (c) follow by the same reasoning as Theorem 7.1. The last statement follows from Theorem 6.1 by inspection of each case. ∎

From Corollary 7.3 we obtain that every irreducible subrepresentation of π_λ is equivalent to a quotient (and hence by duality a subrepresentation) of σ_λ or $\sigma_{-\lambda}$ (This is an analog of Casselman's theorem [4]).

For $\varphi\in C^\infty(B)$ we now define a function $\mathscr{P}'_\lambda\varphi$ on X by

$$\mathscr{P}'_\lambda\varphi(x)=\frac{d}{dv}\bigg|_{v=\lambda}\mathscr{P}_v\varphi(x). \tag{7.6}$$

We have

$$\begin{aligned}\Delta\mathscr{P}'_\lambda\varphi&=\frac{d}{dv}\bigg|_{v=\lambda}\Delta\mathscr{P}_v\varphi=\frac{d}{dv}\bigg|_{v=\lambda}(v^2-\rho^2)\,\mathscr{P}_v\varphi\\&=(\lambda^2-\rho^2)\,\mathscr{P}'_v\varphi+2\lambda\mathscr{P}_\lambda\varphi\end{aligned}$$

and thus, if $\mathscr{P}_\lambda\varphi=0$, then $\mathscr{P}'_\lambda\varphi\in\mathscr{E}_\lambda$. Let

$$\tilde{\mathscr{P}}'_\lambda\colon\ker\mathscr{P}_\lambda\to\mathscr{E}_\lambda/\mathscr{I}_\lambda$$

be the map obtained by composing $\mathscr{P}'_\lambda$ with the projection onto $\mathscr{E}_\lambda/\mathscr{I}_\lambda$.

THEOREM 7.4. *Let $\lambda \in \mathbb{C}$. The map $\tilde{\mathscr{P}}'_\lambda$ is a G-invariance. If* $\operatorname{Re} \lambda \geqslant 0$ *it is an isomorphism onto a dense subset of $\mathscr{E}_\lambda / \mathscr{I}_\lambda$.*

Proof. Let $\varphi \in \ker \mathscr{P}_\lambda \subset C^\infty(B)$. Using (7.2) and analytic continuation we get

$$\begin{aligned}\pi_\lambda(g)\,\mathscr{P}'_\lambda \varphi &= \frac{d}{dv}\bigg|_{v=\lambda} (\pi_v(g)\,\mathscr{P}_v \varphi)\\ &= \frac{d}{dv}\bigg|_{v=\lambda} \mathscr{P}_v(\sigma_v(g)\,\varphi)\\ &= \mathscr{P}'_\lambda(\sigma_\lambda(g)\,\varphi) + \mathscr{P}_\lambda\left(\frac{d}{dv}\bigg|_{v=\lambda} \sigma_v(g)\,\varphi\right) \in \mathscr{P}_\lambda(\sigma_\lambda(g)\,\varphi) + I_\lambda.\end{aligned}$$

from which the equivariance property follows.

Assume $\mu \in \Lambda$ and $\mathscr{H}^\mu \subset \ker \mathscr{P}_\lambda$, then $\beta(\lambda, \mu) = 0$ (cf. (7.3)). From (7.6) and (7.3) we obtain

$$\mathscr{P}'_\lambda h = \beta'(\lambda, \mu)\,\Phi_\lambda(h) \tag{7.7}$$

for $h \in \mathscr{H}^\mu$, where $\beta'(\lambda, \mu) = d/dv\,|_{v=\lambda}\, \beta(v, \mu)$. Thus $\mathscr{P}'_\lambda(\mathscr{H}^\mu) = 0$ if and only if $\beta'(\lambda, \mu) = 0$. However it follows from Lemma 7.2 that the zeros of $\beta(\lambda, \mu)$ are of order one in λ when $\operatorname{Re} \lambda \geqslant 0$. Therefore $\beta'(\lambda, \mu) \neq 0$ and the theorem follows. ∎

Remark. Let $\mathscr{I}'_\lambda \subset \mathscr{E}_\lambda$ be the sum of $\mathscr{I}_\lambda$ and the closure of the image $\mathscr{P}'_\lambda$ $(\ker \mathscr{P}_\lambda)$. From the theorem it follows that $\mathscr{I}'_\lambda = \mathscr{E}_\lambda$ if $\operatorname{Re} \lambda \geqslant 0$. Actually it follows from the proof and Lemma 7.2 that $\mathscr{I}'_\lambda = \mathscr{E}_\lambda$ for all $\lambda \in \mathbb{C}$, except when dp is even, $q > 1$, and $\lambda \in -\rho - 2\mathbb{Z}_+$. In this case $\mathscr{I}'_\lambda = \mathscr{V}_{-\lambda}$ and differentiating once more with respect to λ, we get an isomorphism $\tilde{\mathscr{P}}''_\lambda$ of $\ker \mathscr{P}'_\lambda$ onto a subset of $\mathscr{E}_\lambda / \mathscr{I}'_\lambda$.

From Theorem 7.4 we obtain

COROLLARY 7.5. *Let $\lambda \in \mathbb{C}$. Every irreducible subquotient of π_λ is infinitesimally equivalent to an irreducible subquotient of σ_λ, and vice versa.*

8. THE EXCEPTIONAL SPACES

Let $G = F_4^1$ be the simply connected Lie group whose Lie algebra $\mathfrak{g}$ is a normal real from of the exceptional simple Lie algebra $\mathfrak{f}_4$ (i.e. $\mathfrak{g} = \mathfrak{f}_{4(-20)}$), and let $K = \mathrm{Spin}(9)$ be maximally compact in G. G contains the group $H = \mathrm{Spin}(1, 8)$ as a symmetric subgroup (cf. [2]). In this section we consider the eigenspace representations for G/K and G/H, and sketch results similar to those proved in the previous sections for the projective hyperbolic spaces. The symmetric space G/K has previously considered been in [13, 29], and G/H in [18] (where problem A is solved). To emphasize the analogy with the hyperbolic spaces we denote $G/K = X(1, 2; \mathbb{O})$ and $G/H = X(2, 1; \mathbb{O})$, where $\mathbb{O}$ is the algebra of Cayley numbers. Let $d = \dim_{\mathbb{R}} \mathbb{O} = 8$.

Choose an Iwasawa decomposition $G = KAN$ such that the Lie algebra $\mathfrak{a}$ of A is also a Cartan subspace for G/H, and let $Y \in \mathfrak{a}$ be such that the positive restricted roots are α and 2α, where $\alpha(Y) = 1$. Let M be the centralizer of $\mathfrak{a}$ in K, then $M \cong \mathrm{Spin}(7)$ and $K/M \cong S^{15}$ (cf. [13]). Since M is connected and $\mathfrak{a}$ is a Cartan subspace also for G/H we have that $M \subset H$. We then have spherical coordinates on $X = X(p, q; \mathbb{O})$ (cf. [25, Proposition 7.1.7]):

$$\begin{aligned} K/M \times]0, \infty[&\simeq \Omega \subset X \\ (kM, t) &\to ka_t \cdot 0 \end{aligned} \tag{8.1}$$

where $a_t = \exp tY$, $0 \in X$ is the origin, and Ω is an open dense subset of X. Let ω be the Casimir element for $\mathfrak{g}$, and let π denote the regular representation of G on $C^\infty(X)$. We define the Laplace operator Δ by

$$\Delta = 72\pi(\omega)$$

(72 is the square of the Killing norm of Y), and for $\lambda \in \mathbb{C}$ we define the eigenspace $\mathscr{E}_\lambda$ by (2.3) where $\rho = 11$.

The K-decomposition of $C^\infty(K/M)$ follows from [13, Theorem 3.1]: The K-types occurring are parametrized by pairs (a, b) of integers satisfying $a \geqslant b \geqslant 0$ and $a \equiv b \bmod 2$, each occurring with multiplicity one. Moreover the corresponding subspace $\mathscr{H}^{a,b} \subset C^\infty(K/M)$ contains a unique M-fixed function $h_{a,b}$ with value 1 at the origin. The function $h_{a,b}$ is given by the formula

$$h_{a,b}(y) = \chi(8; b; \varphi)\, e(2; 8; a, b; \theta) \tag{8.2}$$

for $y \in K/M$, where $\theta \in [0, \pi/2]$ and $\varphi \in [0, \pi]$ are determined by $y = (y_1, ..., y_{16}) \in S^{15}$, $\cos^2\theta = y_1^2 + \cdots + y_8^2$, and $y_1 = \cos\theta\cos\varphi$, and where $\chi(8; b; \varphi) = \cos^b \varphi F(-b/2,\ (1-b)/2;\ 7/2;\ -\tan^2\varphi)$ and $e(2; 8; a, b; \theta)$ is defined by (3.8) (for $\varphi \neq \pi/2$ and $\theta \neq \pi/2$).

In order to keep the analogy with Section 3 we define Λ as the set of triples $\mu = (j, k, l)$ of integers satisfying respectively $k \geqslant j = l \geqslant 0$ if $(p, q) = (1, 2)$, and $j \geqslant k = l \geqslant 0$ if $(p, q) = (2, 1)$. For $\mu \in \Lambda$ let $\mathscr{H}^\mu = \mathscr{H}^{a,b}$ and $h_\mu = h_{a,b}$ with $(a, b) = (k, l)$, respectively $(a, b) = (j, l)$. Let $\mathscr{E}_{\lambda,\mu}$ be given by (4.1), define $\varphi_{\lambda,j,k}$ by (4.2), and $\Phi_\lambda(h) \in C^\infty(\Omega)$ by (4.3) for $h \in \mathscr{H}^\mu$. In particular let $f_{\lambda,\mu} = \Phi_\lambda(h_\mu)$.

THEOREM 8.1. *Let $(p, q) = (1, 2)$ or $(2, 1)$, and let $X = X(p, q; \mathbb{O})$. Let $\lambda \in \mathbb{C}$.*

(i) *The K-types in $\mathscr{E}_\lambda(X)$ are parametrized by Λ, each occurring with multiplicity one. For each $\mu \in \Lambda$ and $h \in \mathscr{H}^\mu$ the function $\Phi_\lambda(h)$ on Ω extends to a function in $\mathscr{E}_{\lambda,\mu}$ and Φ_λ is a K-isomorphism of $\mathscr{H}^\mu$ onto $\mathscr{E}_{\lambda,\mu}$.*

(ii) *The statement of Theorem* 5.1 *holds word for word for X.*

Proof (sketch). We use the notation from [29]. By [29, p. 538] (resp. [18, p. 53]) $X(1, 2; \mathbb{O})$ (resp. $X(2, 1; \mathbb{O})$) can be realized as the G-orbit in $\mathscr{J}_{1,2}$ through E_1 (resp. E_3). From [29, p. 540(12) and p. 531(2)] it follows that if the point $X(\zeta, w) \in \mathscr{J}_{1,2}$ ($\zeta \in \mathbb{R}^3$, $w \in \mathbb{O}^3$, notation [29, p. 523]) equals ka_tE_1 (resp. ka_tE_3) then

$$\zeta_1 = \cosh^2 t \qquad (\text{resp. } \zeta_1 = \sinh^2 t) \tag{8.3}$$

$$\zeta_3 = -\sinh^2 t\, |u|^2 \qquad (\text{resp. } \zeta_3 = \cosh^2 t\, |u|^2) \tag{8.4}$$

$$w_2 = -\sinh t \cosh t\, u \qquad (\text{resp. same relation}) \tag{8.5}$$

$$w_3 = -\sinh t \cosh t\, v \qquad (\text{resp. same relation}) \tag{8.6}$$

where u, $v \in \mathbb{O}$ are determined from $kM \in K/M$ via $K/M \cong \{(u, v) \in \mathbb{O} \mid |u|^2 + |v|^2 = 1\}$ (expressions for ζ_2 and w_1 are not needed in the following).

Using these relations the extension of $\Phi_\lambda(h)$ to $C^\infty(X)$ is proved in a similar manner as Lemma 4.1. Expressing the Laplacian in spherical coordinates (cf. [18, p. 32]) (i) is then obtained.

Let ξ, $\eta \in [0, \pi/2]$ and $\varphi \in [0, \pi]$ be defined from $kM \in K/M$ as follows: Let $\xi = 0$, $|u| = \cos \eta$, and $\operatorname{Re} u = \cos \eta \cos \varphi$ if $p = 1$ (resp. $|u| = \cos \xi$, $\eta = 0$, and $\operatorname{Re} u = \cos \xi \cos \varphi$ if $p = 2$). Consider the action of the one-parameter group $(a_s)_{s \in \mathbb{R}}$ on $ka_t \cdot 0$, and define $k(s) M$ and $t(s)$ by $a_{-s}ka_t \cdot 0 = k(s)\, a_{t(s)} \cdot 0$. Define $\xi(s)$, $\eta(s)$, and $\varphi(s)$ from $k(s)\, M$ similarly as above. Using [29, p. 540(12)] and (8.3)–(8.6) the identities (5.2)–(5.5) are proved, and the proof of Theorem 5.1 can then be repeated. ∎

We now define subspaces $\mathscr{T}_\lambda$, $\mathscr{U}_\lambda$, $\mathscr{V}_\lambda$, and $\mathscr{W}_\lambda$ of $\mathscr{E}_\lambda$ as in Section 6 and obtain

THEOREM 8.2. *Let $(p, q) = (2, 1)$ or $(1, 2)$, $X = X(p, q; \mathbb{O})$, $\lambda \in \mathbb{C}$ and $\operatorname{Re} \lambda \geq 0$.*

(i) *π_λ is irreducible if and only if $\lambda \notin 5 + 2\mathbb{Z}_+$.*

(ii) *If $\lambda \in 5 + 2\mathbb{Z}_+$ the only nontrivial closed invariant subspaces of $\mathscr{E}_\lambda$ are $\mathscr{T}_\lambda$ (if $\lambda \geq 11$) and $\mathscr{V}_\lambda$ (if $p = 1$), respectively $\mathscr{U}_\lambda$ (if $p = 2$).*

(iii) *Let $\mu \in \Lambda$. Then $\mathscr{E}_{\lambda,\mu} \subset L^2(X)$ if and only if $p = 2$, $\lambda \in 1 + 2\mathbb{Z}_+$ and $\mu \in \mathscr{U}_\lambda$. Thus $\lambda \in 1 + 2\mathbb{Z}_+$ parametrizes the discrete series for G/H.*

The parametrization of the discrete series for $F_4^1/\mathrm{Spin}(1,8)$ was first obtained by Kosters [18].

Finally, for $\varphi \in C^\infty(K/M)$ the Poisson transforms are defined as follows: Let $\xi^0 = E_1 - E_3 + F_2^1 \in \mathscr{J}_{1,2}$ (notation of [29]) and let

$$\mathscr{P}_\lambda \varphi(gK) = \int_{K/M} |(gE_1 \,|\, k\xi^0)|^{-1/2(\lambda+\rho)} \varphi(k)\, dk \tag{8.7}$$

and

$$\mathscr{P}_\lambda \varphi(gH) = \Gamma(\tfrac{1}{2}(-\lambda-\rho+8))^{-1} \int_{K/M} |(gE_3 \,|\, k\xi^0)|^{-1/2(\lambda+\rho)} \varphi(k)\, dk \tag{8.8}$$

(cf. [29, p. 545] and [18, p. 58]. Notice that (8.8) involves analytic continuation). For these transformations (7.3) holds again, with β given by (7.5) if $p=1$ (cf. [11, Theorem 4.5]) and by (7.4) if $p=2$ (cf. [18, p. 70]). Therefore the results of Section 7 also hold for the exceptional spaces.

9. Remarks on Spherical Discrete Series

Let $p>1$. We have seen in Section 6 and 8 that the discrete series for $X(p,q;\mathbb{F})$ is parametrized by the set D, and that U_λ is the set of K-types (each occurring with multiplicity one) in the discrete series representation with parameter λ. In particular we see that this discrete series representation is spherical (i.e., contains the trivial K-type) if and only if $\lambda \leqslant \rho - dq$ (cf. also [7, Section 8]). In this case it is of particular interest to determine the Langlands parameters of this spherical unitary representation. Actually, for the groups in question (the identity component of G) an irreducible spherical representation is uniquely determined by its infinitesimal character (this follows from [9, Theorem 7.5]). Since we know that the Laplacian Δ acts on $\mathscr{U}_\lambda$ with the eigenvalue $\lambda^2 - \rho^2$, the infinitesimal character of the discrete series representation can be determined, and the Langlands parameters can be deduced.

From the condition $0 < \lambda \leqslant \rho - dq$ it follows that $p>q$, so $\mathfrak{g}$ contains a q-dimensional split abelian subspace $\mathfrak{a}_{\mathfrak{p}}$. Let $f_1,\dots,f_q$ be coordinates on $\mathfrak{a}_{\mathfrak{p}}$ such that a positive system of roots of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{g}$ is given by

$$\{f_i \pm f_j,\, f_l,\, 2f_l \mid 1 \leqslant i < j \leqslant q,\ 1 \leqslant l \leqslant q\}$$

with the multiplicities d, $(p-q)\,d$, and $d-1$ respectively. The corresponding ρ is

$$\rho(\mathfrak{a}_{\mathfrak{p}}) = \left(\frac{d}{2}(p+q)-1,\ \frac{d}{2}(p+q-2)-1,\dots,\ \frac{d}{2}(p-q+2)-1\right).$$

Reasoning as indicated above we obtain:

THEOREM 9.1. *Let $p > 1$ and let the projective hyperboloid $X = X(p, q; \mathbb{F})$ be realized as a semisimple symmetric space G/H where $G = SO_0(p, q)$, $SU(p, q)$, $Sp(p, q)$, or F_4^1. The discrete series for X contains spherical representations of G if and only if $d/2(p-q) > 1$. Under this condition it contains precisely the spherical represenations π_n determined by the Langlands parameters $\nu = (\nu_1, ..., \nu_q)$ on $\mathfrak{a}$ as follows (with respect to the coordinates on $\mathfrak{a}$ as above):*

$$\nu_j = \frac{d}{2}(p + q - 2j) - 1 \qquad (j = 1, ..., q - 1)$$

$$\nu_q = \frac{d}{2}(p - q) - 1 - 2n$$

with $n = 0, 1, ...$ and $2n < d/2(p-q) - 1$. In particular these spherical representations of G are unitary.

For $Sp(p, 2)$ the unitarity of π_n was suggested and given a different proof by Baldoni Silva and Knapp. Their result and technique of proof was announced in [31]. The $(\mathfrak{g}, K)$-modules of the representations mentioned in Theorem 9.1 can also be realized via derived functor modules (which are unitarizable by [30]).

EXAMPLE. Let $G = \mathrm{Sp}(5, 2)$. The spherical representations in $L^2(G/H)$ for $H = \mathrm{Sp}(1) \times \mathrm{Sp}(4, 2)$ are indicated by $*$'s in the following diagram. Other unitary spherical representations related to semisimple symmetric spaces (but not hyperboloids) are indicated by x's (cf. [24, Corollary 7.10]). (See Fig. 1.)

Remark. The projective hyperboloids are isotropic symmetric spaces of rank one. The non-isotropic symmetric spaces of rank one (cf. [8, Table 1]) contain no spherical representations in their discrete series (this follows from [20] and [24, Theorem 5.4]).

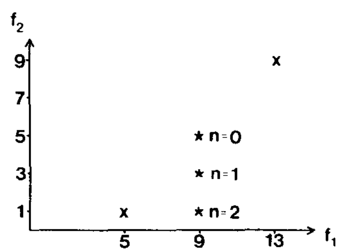

FIGURE 1

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