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# Static Modules and Stable Clifford Theory

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## 1. INTRODUCTION

The object of this paper is to express an important result of Dade [2, Theorem 7.4], on equivalence of categories, from classical stable Clifford theory to one in pure ring theory. In fact, in [2], utilising the properties of fully group-graded rings and modules, Dade described an extended version of Cline's stable Clifford theory [1].

In the sequel we will work in the category of right $\mathscr{A}$-modules, where $\mathscr{A}$ is an arbitrary ring. In our work no group action is involved and the rings and modules are free from any group-grading. Our work is based on a type of module, called a "static module related to a fixed $\mathscr{A}$-module $\mathscr{M}$" or "$\mathscr{M}$-static $\mathscr{A}$-module" (Definition 2.1).

If $\mathscr{A}$ and $\mathscr{B}$ are rings and $\rho: \mathscr{A} \to \mathscr{B}$ is an identity-preserving ring homomorphism, then we will be interested in those $\mathscr{B}$-modules which are $\mathscr{M}$-static as $\mathscr{A}$-modules under the restriction of $\rho$. A source of interest is the following example:

Let $\mathscr{G}$ be a group, $\mathscr{H}$ a normal subgroup of $\mathscr{G}$, and $\mathscr{F}$ any field. Let $\mathscr{M}$ be a $\mathscr{G}$-stable $\mathscr{F}[\mathscr{H}]$-module. Then the $\mathscr{F}[\mathscr{G}]$-modules of interest here are exactly those which are $\mathscr{M}$-static as $\mathscr{F}[\mathscr{H}]$-modules. Our first main theorem, Theorem 4.9, below shows that the category of all such modules is equivalent to a certain category of $\mathscr{E}$-modules, where

$$\mathscr{E} = \operatorname{End}_{\mathscr{F}[\mathscr{G}]}(\mathscr{M} \otimes_{\mathscr{F}[\mathscr{H}]} \mathscr{F}[\mathscr{G}]).$$

In many cases, the latter category is much easier to deal with. Our second main theorem, Theorem 5.5, below shows that the category whose objects are those $\mathscr{F}[\mathscr{G}]$-modules which weakly divide $\mathscr{M}$ as $\mathscr{F}[\mathscr{H}]$-modules is equivalent to the category whose objects are those $\mathscr{E}$-modules which are projective of finite type as $\mathscr{D}$-modules, where

$$\mathscr{D} = \operatorname{End}_{\mathscr{F}[\mathscr{H}]}(\mathscr{M}).$$



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Dade and Cline have proved similar, but less general, equivalences. Dade's work in particular, depends strongly on the fact that $\mathscr{F}[\mathscr{G}]$ and $\mathscr{E}$ are fully $\mathscr{G}/\mathscr{H}$-graded. But this is obviously not necessary, from the work here. Furthermore, the categories of modules dealt within Dade's work appear to be less general than the categories we consider here.

*Notation.* In this paper the term *ring* means *associative ring with identity* and *module* means *unital right module.* For any ring $\mathscr{A}$, MOD-$\mathscr{A}$ means the *category* of all *$\mathscr{A}$-modules and their homomorphisms.* Finally, *fully group-graded* means *strongly group-graded* (see [3] for the change of terminology).

## 2. Static Modules

Let $\mathscr{A}$ be a ring, let $\mathscr{M}$ be a fixed $\mathscr{A}$-module, and let $\mathscr{D} = \mathrm{End}_{\mathscr{A}}(\mathscr{M})$ be the ring of endomorphisms of $\mathscr{M}$.

Definition 2.1. An $\mathscr{A}$-module $\mathscr{V}$ is said to be a *static $\mathscr{A}$-module related to $\mathscr{M}$ or an $\mathscr{M}$-static $\mathscr{A}$-module* if

(2.1.a) $$\mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V}) \otimes_{\mathscr{D}} \mathscr{M} \cong \mathscr{V}$$

via the natural isomorphism

(2.1.b) $$f \otimes m \mapsto f(m)$$

for all $m \in \mathscr{M}$ and $f \in \mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V})$.

The category of all $\mathscr{M}$-static $\mathscr{A}$-modules will be denoted by

(2.2) $$\text{MOD-}\mathscr{A}_{\mathscr{M}}$$

which is a full additive subcategory of MOD-$\mathscr{A}$.

We will also denote by

(2.3) $$\text{MOD-}\mathscr{D}^{\mathscr{M}}$$

the category of all $\mathscr{D}$-modules $\mathscr{W}$ such that

(2.4.a) $$\mathscr{W} \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{W} \otimes_{\mathscr{D}} \mathscr{M})$$

via the natural isomorphism:

(2.4.b) $$w \mapsto f_w,$$

where $f_w$ is defined by

(2.4.c) $$f_w(m) = w \otimes m$$

for all $m \in \mathscr{M}$ and $w \in \mathscr{W}$.

MOD-$\mathscr{D}^{\mathscr{M}}$ is a full additive subcategory of MOD-$\mathscr{D}$, and we will call it the *companion category of* MOD-$\mathscr{A}_{\mathscr{M}}$.

The restrictions of functors $\mathrm{HOM}_{\mathscr{A}}(\mathscr{M}, \cdot)$ and $\cdot \otimes_{\mathscr{D}} \mathscr{M}$ form an equivalence of the categories MOD-$\mathscr{A}_{\mathscr{M}}$ and MOD-$\mathscr{D}^{\mathscr{M}}$. For, the composition of these functors in one direction,

$$\mathrm{HOM}_{\mathscr{A}}(\mathscr{M}, \cdot) \otimes_{\mathscr{D}} \mathscr{M},$$

is naturally equivalent to the identity functor on MOD-$\mathscr{A}_{\mathscr{M}}$, and in the other direction,

$$\mathrm{HOM}_{\mathscr{A}}(\mathscr{M}, \cdot \otimes_{\mathscr{D}} \mathscr{M}),$$

is naturally equivalent to the identity functor on MOD-$\mathscr{D}^{\mathscr{M}}$. Thus we have the following theorem:

THEOREM 2.5. *The restrictions of the additive functors*

$$\mathrm{HOM}_{\mathscr{A}}(\mathscr{M}, \cdot)\colon \begin{cases} \text{MOD-}\mathscr{A}_{\mathscr{M}} \to \text{MOD-}\mathscr{D}^{\mathscr{M}} \\ \mathscr{V} \mapsto \mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V}) \end{cases}$$

*and*

$$\cdot \otimes_{\mathscr{D}} \mathscr{M}\colon \begin{cases} \text{MOD-}\mathscr{D}^{\mathscr{M}} \to \text{MOD-}\mathscr{A}_{\mathscr{M}} \\ \mathscr{W} \mapsto \mathscr{W} \otimes_{\mathscr{D}} \mathscr{M} \end{cases}$$

*form an equivalence of the categories* MOD-$\mathscr{A}_{\mathscr{M}}$ *and* MOD-$\mathscr{D}^{\mathscr{M}}$.

EXAMPLES 2.6.1. $\mathscr{M}$ itself is an $\mathscr{M}$-static $\mathscr{A}$-module.

2. If an $\mathscr{A}$-module $\mathscr{V}$ weakly divides $\mathscr{M}$ in MOD-$\mathscr{A}$, then $\mathscr{V}$ is $\mathscr{M}$-static (Corollary 3.2(i)).

3. If $\mathscr{M}$ is of finite type, then every $\mathscr{A}$-module $\mathscr{V}$ which divides some direct sum of copies of $\mathscr{M}$ in MOD-$\mathscr{A}$ is $\mathscr{M}$-static (Corollary 3.2.(ii)).

4. If $\mathscr{M}$ is a generator of MOD-$\mathscr{A}$, then every $\mathscr{A}$-module $\mathscr{V}$ is $\mathscr{M}$-static, and in this case MOD-$\mathscr{A}_{\mathscr{M}}$ coincides with MOD-$\mathscr{A}$ [4, p. 289].

5. $\mathscr{D}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$.

6. Every projective $\mathscr{D}$-module of finite type is an object of MOD-$\mathscr{D}^{\mathscr{M}}$ (Corollary 3.4(i)).

7. If $\mathscr{M}$ is of finite type, then every projective $\mathscr{D}$-module is an object of MOD-$\mathscr{D}^{\mathscr{M}}$ (Corollary 3.4(ii)).

## 3. Divisibility and Weak Divisibility

Throughout this section we will continue to assume the notation of Section 2, i.e., $\mathscr{A}$ is a ring, $\mathscr{M}$ is a fixed $\mathscr{A}$-module, and $\mathscr{D} = \mathrm{End}_{\mathscr{A}}(\mathscr{M})$. In this section we will prove a weak version (Theorem 3.7) of our second main result, Theorem 5.5.

Recall that a right $\mathscr{A}$-module $\mathscr{U}$ divides a right $\mathscr{A}$-module $\mathscr{V}$ in MOD-$\mathscr{A}$, if there is a right $\mathscr{A}$-module $\mathscr{U}'$ such that

$$\mathscr{V} \cong \mathscr{U} \oplus \mathscr{U}',$$

and that $\mathscr{U}$ weakly divides $\mathscr{V}$ in MOD-$\mathscr{A}$ if $\mathscr{U}$ divides a finite direct sum of copies of $\mathscr{V}$ in MOD-$\mathscr{A}$.

Since the functor $\cdot \otimes_{\mathscr{D}} \mathscr{M}$ preserves arbitrary direct sums while the functor $\mathrm{HOM}_{\mathscr{A}}(\mathscr{M}, \cdot)$ preserves finite direct sums, and even arbitrary direct sums if $\mathscr{M}$ has finite type (or "small," in the sense of Proposition 4.25C of [4]), it can immediately be observed that a finite direct sum of $\mathscr{M}$-static $\mathscr{A}$-modules is an $\mathscr{M}$-static $\mathscr{A}$-module, and in case $\mathscr{M}$ has finite type (or small), an arbitrary direct sum of $\mathscr{M}$-static $\mathscr{A}$-modules is an $\mathscr{M}$-static $\mathscr{A}$-module. The converse of this statement also holds, i.e.,

Proposition 3.1. *If $\mathscr{V}$ is an $\mathscr{M}$-static $\mathscr{A}$-module, then every direct summand of $\mathscr{V}$ is an $\mathscr{M}$-static $\mathscr{A}$-module.*

An immediate consequence of the above proposition and the definitions of divisibility and weak divisibility is the following:

Corollary 3.2. (i) *If an $\mathscr{A}$-module $\mathscr{V}$ weakly divides $\mathscr{M}$ in* MOD-$\mathscr{A}$, *then $\mathscr{V}$ is an $\mathscr{M}$-static $\mathscr{A}$-module.*

(ii) *If $\mathscr{M}$ is of finite type (or small), and if $\mathscr{V}$ divides some direct sum of copies of $\mathscr{M}$ in* MOD-$\mathscr{A}$, *then $\mathscr{V}$ is an $\mathscr{M}$-static $\mathscr{A}$-module.*

Similar to that of the objects of MOD-$\mathscr{A}_{\mathscr{M}}$, it is straightforward to verify that a finite direct sum of objects of MOD-$\mathscr{D}^{\mathscr{M}}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$. If $\mathscr{M}$ has finite type (or small) then an arbitrary direct sum of objects of MOD-$\mathscr{D}^{\mathscr{M}}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$. Moreover, the converse of the above also holds, i.e.,

Proposition 3.3. *Every direct summand of an object of* MOD-$\mathscr{D}^{\mathscr{M}}$ *is an object of* MOD-$\mathscr{D}^{\mathscr{M}}$.

The following corollary gives examples of the companion category of MOD-$\mathscr{A}_{\mathscr{M}}$.

COROLLARY 3.4. (i) *If* $\mathscr{W}$ *is a projective* $\mathscr{D}$*-module of finite type, then* $\mathscr{W}$ *is an object of* MOD-$\mathscr{D}^{\mathscr{M}}$.

(ii) *If* $\mathscr{M}$ *is of finite type* (*or small*) *and* $\mathscr{W}$ *is a projective* $\mathscr{D}$*-module, then* $\mathscr{W}$ *is an object of* MOD-$\mathscr{D}^{\mathscr{M}}$.

*Proof.* (i) Let $\mathscr{W}$ be projective of finite type. Then there is a projective complement $\mathscr{W}'$ of $\mathscr{W}$ of finite type such that their direct sum

$$\mathscr{W} \oplus \mathscr{W}'$$

is a free $\mathscr{D}$-module of finite type. But a free $\mathscr{D}$-module of finite type is a direct sum of a finite number of copies of $\mathscr{D}$, so we can find an integer $k$ such that

$$\mathscr{W} \oplus \mathscr{W}' \cong k\mathscr{D},$$

where $k\mathscr{D}$ is the direct sum of $k$ copies of $\mathscr{D}$. Clearly $k\mathscr{D}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$. Hence, from Proposition 3.3, $\mathscr{W}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$.

(ii) Let $\mathscr{M}$ and $\mathscr{W}$ be as in the hypothesis. Then there is a projective complement $\mathscr{W}'$ of $\mathscr{W}$ such that

$$\mathscr{W} \oplus \mathscr{W}'$$

is a free $\mathscr{D}$-module, which is a direct sum of copies of $\mathscr{D}$ and so is an object of MOD-$\mathscr{D}^{\mathscr{M}}$. Hence, from Proposition 3.3, $\mathscr{W}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$. ∎

We define the subcategories MOD($\mathscr{A} \mid \mathscr{M}$) and MOD($\mathscr{A} \mid$weak $\mathscr{M}$) of MOD-$\mathscr{A}$ as follows:

DEFINITION 3.5. (a) MOD($\mathscr{A} \mid \mathscr{M}$) is the full additive subcategory of MOD-$\mathscr{A}$ having as objects all $\mathscr{A}$-modules $\mathscr{V}$ such that $\mathscr{V}$ divides some direct sum of copies of $\mathscr{M}$ in MOD-$\mathscr{A}$.

(b) MOD($\mathscr{A} \mid$weak $\mathscr{M}$) is the full additive subcategory of MOD-$\mathscr{A}$ having as objects all $\mathscr{A}$-modules $\mathscr{V}$ such that $\mathscr{V}$ weakly divides $\mathscr{M}$ in MOD-$\mathscr{A}$.

Similarly we define full additive subcategories MOD($\mathscr{D} \mid \mathscr{D}$) and MOD($\mathscr{D} \mid$weak $\mathscr{D}$) as follows:

DEFINITION 3.6. (a) MOD($\mathscr{D} \mid \mathscr{D}$) is the full additive subcategory of MOD-$\mathscr{D}$ of all projective $\mathscr{D}$-modules.

(b) MOD($\mathscr{D} \mid$weak $\mathscr{D}$) is the full additive subcategory of MOD-$\mathscr{D}$ of all projective $\mathscr{D}$-modules of finite type.

Let $\mathscr{V}$ be an object of $\mathrm{MOD}(\mathscr{A} \mid \text{weak } \mathscr{M})$ (resp. of $\mathrm{MOD}(\mathscr{A} \mid \mathscr{M})$ if $\mathscr{M}$ is of finite type (or small)). In other words $\mathscr{V}$ weakly divides (resp. divides some direct sum of copies of) $\mathscr{M}$ in MOD-$\mathscr{A}$. Then $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V})$, which now becomes a direct summand of a free $\mathscr{D}$-module of finite type (resp. a direct summand of a free $\mathscr{D}$-module), is a projective $\mathscr{D}$-module of finite type (resp. a projective $\mathscr{D}$-module), and so is an object of $\mathrm{MOD}(\mathscr{D} \mid \text{weak } \mathscr{D})$ (resp. of $\mathrm{MOD}(\mathscr{D} \mid \mathscr{D})$).

Conversely, let $\mathscr{W}$ be an object of $\mathrm{MOD}(\mathscr{D} \mid \text{weak } \mathscr{D})$ (resp. of $\mathrm{MOD}(\mathscr{D} \mid \mathscr{D})$ if $\mathscr{M}$ is of finite type (or small)). In other words $\mathscr{W}$ is a projective $\mathscr{D}$-module of finite type (resp. a projective $\mathscr{D}$-module). Then $\mathscr{W} \otimes_{\mathscr{D}} \mathscr{M}$ weakly divides (resp. divides some direct sum of copies of) $\mathscr{M}$. So $\mathscr{W} \otimes_{\mathscr{D}} \mathscr{M}$ is an object of $\mathrm{MOD}(\mathscr{A} \mid \text{weak } \mathscr{M})$ (resp. of $\mathrm{MOD}(\mathscr{A} \mid \mathscr{M})$).

Hence we conclude that:

THEOREM 3.7. *The restrictions of the additive functors*

$$\mathrm{HOM}_{\mathscr{A}}(\mathscr{M}, \cdot)\colon \mathrm{MOD}(\mathscr{A} \mid \text{weak } \mathscr{M}) \to \mathrm{MOD}(\mathscr{D} \mid \text{weak } \mathscr{D})$$

*and*

$$\cdot \otimes_{\mathscr{D}} \mathscr{M}\colon \mathrm{MOD}(\mathscr{D} \mid \text{weak } \mathscr{D}) \to \mathrm{MOD}(\mathscr{A} \mid \text{weak } \mathscr{M})$$

*form an equivalence of the categories* $\mathrm{MOD}(\mathscr{A} \mid \textit{weak } \mathscr{M})$ *and* $\mathrm{MOD}(\mathscr{D} \mid \textit{weak } \mathscr{D})$. *If* $\mathscr{M}$ *is of finite type* (*or small*) *then the restrictions of those functors form an equivalence of* $\mathrm{MOD}(\mathscr{A} \mid \mathscr{M})$ *and* $\mathrm{MOD}(\mathscr{D} \mid \mathscr{D})$.

## 4. INDUCTION AND RESTRICTION

In the remaining part of this paper we will adopt the following axioms and notation:

(4.1.a) $\mathscr{A}$ and $\mathscr{B}$ are rings with identities $1_{\mathscr{A}}$ and $1_{\mathscr{B}}$, respectively.

(4.1.b) $\rho\colon \mathscr{A} \to \mathscr{B}$ is an identity-preserving ring homomorphism: $\rho(1_{\mathscr{A}}) = 1_{\mathscr{B}}$, which we use to define both the *induction functor*,

$$\mathscr{U} \mapsto \mathscr{U} \otimes_{\mathscr{A}} \mathscr{B},$$

from MOD-$\mathscr{A}$ to MOD-$\mathscr{B}$ and the *restriction functor*,

$$\mathscr{V} \mapsto \mathscr{V}|_{\mathscr{A}},$$

from MOD-$\mathscr{B}$ to MOD-$\mathscr{A}$.

(4.1.c) $\mathcal{M}$ is a fixed $\mathcal{A}$-module such that

$$(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{B})|_{\mathcal{A}}$$

is an object of MOD-$\mathcal{A}_{\mathcal{M}}$.

(4.1.a) $\mathcal{D}$ and $\mathcal{E}$ are endomorphism rings $\mathrm{End}_{\mathcal{A}}(\mathcal{M})$ and $\mathrm{End}_{\mathcal{B}}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{B})$, respectively, while

$$\sigma\colon \mathcal{D}\mapsto\mathcal{E}$$

is the natural identity-preserving ring homomorphism determined by the formula

$$[\sigma(d)](m\otimes b)=d(m)\otimes b$$

for all $d\in\mathcal{D}$, $m\in\mathcal{M}$, and $b\in\mathcal{B}$.

We define *restricted categories* MOD($\mathcal{B}$ rest. $\mathcal{A}_{\mathcal{M}}$) and MOD($\mathcal{E}$ rest. $\mathcal{D}^{\cdot\mathcal{M}}$), which are full additive subcategories of MOD-$\mathcal{B}$ and MOD-$\mathcal{E}$ respectively, as follows:

DEFINITION 4.2. MOD($\mathcal{B}$ rest. $\mathcal{A}_{\mathcal{M}}$) is the category of all those $\mathcal{B}$-modules $\mathcal{V}$ for which $\mathcal{V}|_{\mathcal{A}}$ is an object of MOD-$\mathcal{A}_{\mathcal{M}}$.

DEFINITION 4.3. MOD($\mathcal{E}$ rest. $\mathcal{D}^{\cdot\mathcal{M}}$) is the category of all those $\mathcal{E}$ modules $\mathcal{W}$ for which $\mathcal{W}|_{\mathcal{D}}$ is an object of MOD-$\mathcal{D}^{\cdot\mathcal{M}}$.

In this section our main object is to show that the categories defined in 4.2 and 4.3 are equivalent categories. This will be proved in our first main theorem, Theorem 4.9. But before this we will determine what these categories are. We answer this question in Propositions 4.6 and 4.8.

By using the *adjoint associativity theorem* [4, p. 424] we can get the following isomorphism of $\mathcal{D}$-modules,

$$\text{(4.4.a)}\quad \mathcal{E}=\mathrm{Hom}_{\mathcal{B}}(\mathcal{M}\otimes_{\mathcal{A}}\mathcal{B},\mathcal{M}\otimes_{\mathcal{A}}\mathcal{B})\underset{\varphi}{\overset{\eta}{\rightleftarrows}}\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{M}\otimes_{\mathcal{A}}\mathcal{B})$$

defined by the formula

$$\text{(4.4.b)}\qquad [\eta(f)](m)=f(m\otimes 1_{\mathcal{A}})$$

for all $m\in\mathcal{M}$ and $f\in\mathcal{E}$. The inverse isomorphism is defined by

$$\text{(4.4.c)}\qquad \varphi(g)(m\otimes b)=g(m)b,$$

for all $b\in\mathcal{B}$, $m\in\mathcal{M}$, and $g\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{M}\otimes_{\mathcal{A}}\mathcal{B})$.

Tensoring both sides of (4.4.a) by $\mathcal{M}$ we will get the $\mathcal{A}$-module isomorphism

$$\text{(4.4.d)} \qquad \mathcal{E} \otimes_{\mathcal{D}} \mathcal{M} \underset{\varphi \otimes l_{\mathcal{M}}}{\overset{\eta \otimes l_{\mathcal{M}}}{\rightleftarrows}} \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}) \otimes_{\mathcal{D}} \mathcal{M},$$

where $\eta$ and $\varphi$ are as defined above in (4.4.b) and (4.4.c) and $l_{\mathcal{M}}$ is the identity map on $\mathcal{M}$.

According to our assumption (4.1.c) and by Definition 4.2, $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$ is an object of MOD($\mathcal{B}$ rest. $\mathcal{A}_{\mathcal{M}}$). Hence, we have the following natural isomorphism of $\mathcal{A}$-modules,

$$\text{(4.4.e)} \qquad \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}) \otimes_{\mathcal{D}} \mathcal{M} \underset{\lambda^{-1}}{\overset{\lambda}{\rightleftarrows}} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B},$$

where

$$\lambda(f \otimes m) = f(m),$$

for all $m \in \mathcal{M}$ and $f \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M} \otimes_A \mathcal{B})$ and $\lambda^{-1}$ is the inverse of the isomorphism $\lambda$.

The composition of the isomorphisms (4.4.d) and (4.4.e) gives us the following isomorphism:

LEMMA 4.5. $\mathcal{E} \otimes_{\mathcal{D}} \mathcal{M} \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$ *(as $\mathcal{A}$-modules), where*

$$\alpha = \lambda \circ (\eta \otimes l_{\mathcal{M}})$$

*and*

$$\alpha^{-1} = (\varphi \otimes l_{\mathcal{M}}) \circ \lambda^{-1},$$

*where $\eta$, $\varphi$, $\lambda$, and $\lambda^{-1}$ are as defined above.*

In the following proposition we will demonstrate that MOD($\mathcal{B}$ rest. $\mathcal{A}_{\mathcal{M}}$) is precisely the category of all $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$-static $\mathcal{B}$-modules.

PROPOSITION 4.6. *$\mathcal{V}$ is an object of* MOD($\mathcal{B}$ *rest.* $\mathcal{A}_{\mathcal{M}}$) *if and only if $\mathcal{V}$ is an $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$-static $\mathcal{B}$-module, i.e.,*

$$\text{(4.6.a)} \quad \operatorname{Hom}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{V}) \otimes_{\mathcal{E}} (\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}) \cong \mathcal{V} \qquad (\textit{as } \mathcal{B}\textit{-modules})$$

*via the natural isomorphism*

$$\text{(4.6.b)} \qquad f \otimes x \mapsto f(x)$$

*for all $x \in \mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}$ and $f \in \operatorname{Hom}_{\mathcal{B}}(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{B}, \mathcal{V})$.*

*Proof.* Let $\mathscr{V}$ be an object of MOD-$\mathscr{B}$. Then, from (4.1.b), considering $\mathscr{V}$ an object of MOD-$\mathscr{A}$, we have the following sequence of $\mathscr{A}$-module isomorphisms,

$$\begin{aligned}(4.6.c) \qquad \operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V}) \otimes_{\mathscr{E}} (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}) &\cong \operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V}) \otimes_{\mathscr{E}} (\mathscr{E} \otimes_{\mathscr{S}} \mathscr{M}) \\ &\cong \operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V}) \otimes_{\mathscr{S}} \mathscr{M} \\ &\cong \operatorname{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V}) \otimes_{\mathscr{S}} \mathscr{M},\end{aligned}$$

where the first isomorphism is by Lemma 4.5, the second isomorphism is by associativity and by the canonical isomorphism

$$\operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V}) \otimes_{\mathscr{E}} \mathscr{E} \cong \operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V}),$$

and the third isomorphism is by the adjoint associativity theorem [4, p. 424].

If $\mathscr{V}$ is an object of MOD($\mathscr{B}$ rest. $\mathscr{A}_{\mathscr{M}}$), then by Definition 4.2, $\mathscr{V}|_{\mathscr{A}}$ is an object of MOD-$\mathscr{A}_{\mathscr{M}}$, and so from (2.1.a) and (4.6.c),

$$\operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V}) \otimes_{\mathscr{E}} (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}) \cong \mathscr{V}$$

as $\mathscr{A}$-modules. Indeed, this is a $\mathscr{B}$-module isomorphism, for

$$(f \otimes x)\, b = f \otimes (xb) \mapsto f(xb) = f(x)\, b,$$

for all $b \in \mathscr{B}$, $x \in \mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}$, and $f \in \operatorname{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V})$. Hence $\mathscr{V}$ is an $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}$-static $\mathscr{B}$-module.

Conversely, assume that $\mathscr{V}$ is an $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}$-static $\mathscr{B}$-module, i.e., (4.6.a) and (4.6.b) hold. Then, from (4.6.c), $\mathscr{V}|_{\mathscr{A}}$ is an $\mathscr{M}$-static $\mathscr{A}$-module. Hence, from Definition 4.2, $\mathscr{V}$ is an object of MOD($\mathscr{B}$ rest. $\mathscr{A}_{\mathscr{M}}$). ∎

LEMMA 4.7. *For any $\mathscr{E}$-module $\mathscr{W}$,*

$$(4.7.a) \qquad \mathscr{W} \otimes_{\mathscr{S}} \mathscr{M} \cong \mathscr{W} \otimes_{\mathscr{E}} (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}) \qquad (\textit{as } \mathscr{A}\textit{-modules})$$

*via the natural map*

$$(4.7.b) \qquad \sum_{\text{finite}} w_i \otimes m_i \mapsto \sum_{\text{finite}} w_i \otimes m_i \otimes 1_{\mathscr{B}},$$

*for all $m_i \in \mathscr{M}$ and $w_i \in \mathscr{W}$.*

*Proof.*

$$\begin{aligned}\mathscr{W}\otimes_{\mathscr{D}}\mathscr{M} &\cong (\mathscr{W}\otimes_{\mathscr{E}}\mathscr{E})\otimes_{\mathscr{D}}\mathscr{M}\\ &\cong \mathscr{W}\otimes_{\mathscr{E}}(\mathscr{E}\otimes_{\mathscr{D}}\mathscr{M})\\ &\cong \mathscr{W}\otimes_{\mathscr{E}}(\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}) \qquad (\text{as } \mathscr{A}\text{-module}),\end{aligned}$$

where the first two isomorphism are the obvious ones and the last isomorphism is by Lemma 4.5. ▌

In the next proposition we will show that MOD($\mathscr{E}$ rest. $\mathscr{D}^{\mathscr{M}}$) is precisely the companion category, MOD-$\mathscr{E}^{\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}}$, of the category of all $\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}$-static $\mathscr{B}$-modules.

PROPOSITION 4.8. *$\mathscr{W}$ is an object of* MOD($\mathscr{E}$ *rest.* $\mathscr{D}^{\mathscr{M}}$) *if and only if*

(4.8.a) $\mathscr{W}\cong \mathrm{Hom}_{\mathscr{B}}(\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}, \mathscr{W}\otimes_{\mathscr{E}}\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B})$ (*as $\mathscr{E}$-modules*)

*via the map*

(4.8.b) $$w \overset{\mu}{\mapsto} \mu_w,$$

*such that*

(4.8.c) $$\mu_w(x) = w\otimes x,$$

*for all $w\in\mathscr{W}$ and $x\in\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}$.*

*Proof.* Let $\mathscr{W}$ be an object of MOD($\mathscr{E}$ rest. $\mathscr{D}^{\mathscr{M}}$); then we have the following $\mathscr{D}$-module isomorphisms:

$$\begin{aligned}\mathscr{W} &\cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{W}\otimes_{\mathscr{D}}\mathscr{M})\\ &\cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{W}\otimes_{\mathscr{E}}(\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}))\\ &\cong \mathrm{Hom}_{\mathscr{B}}(\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B}, \mathscr{W}\otimes_{\mathscr{E}}(\mathscr{M}\otimes_{\mathscr{A}}\mathscr{B})).\end{aligned}$$

The first isomorphism is by Definition 4.3, the second is by Lemma 4.7, and the last one is due to the adjoint associativity theorem [4, p. 424]. Composition of these isomorphisms is an isomorphism which is the required $\mathscr{E}$-module isomorphism.

The converse is an immediate consequence of Lemma 4.7. ▌

From Propositions 4.6 and 4.8 and Theorem 2.5 we conclude the following interesting fact:

THEOREM 4.9. THE FIRST MAIN THEOREM. *The restrictions of additive functors*

$$\mathrm{HOM}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \cdot)\colon \mathrm{MOD}(\mathscr{B} \textit{ rest. } \mathscr{A}_{\mathscr{M}}) \to \mathrm{MOD}(\mathscr{E} \textit{ rest. } \mathscr{D}^{\mathscr{M}})$$

*and*

$$\cdot \otimes_{\mathscr{E}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B})\colon \mathrm{MOD}(\mathscr{E} \textit{ rest. } \mathscr{D}^{\mathscr{M}}) \to \mathrm{MOD}(\mathscr{B} \textit{ rest. } \mathscr{A}_{\mathscr{M}})$$

*form an equivalence of the full additive subcategories*

$\mathrm{MOD}(\mathscr{B} \textit{ rest. } \mathscr{A}_{\mathscr{M}})$ *and* $\mathrm{MOD}(\mathscr{E} \textit{ rest. } \mathscr{D}^{\mathscr{M}})$ *of* MOD-$\mathscr{B}$ *and* MOD-$\mathscr{E}$,

*respectively.* ∎

## 5. THE SECOND MAIN THEOREM

Finally, we prove our second main theorem which, indeed, is a worthy extension of Cline's [1] and Dade's [2] work. Throughout this section we will continue to assume the hypotheses of Section 4.1.

Define full additive subcategories $\mathrm{MOD}(\mathscr{B} \mid \mathscr{M})$ and $\mathrm{MOD}(\mathscr{B} \mid \text{weak } \mathscr{M})$ of MOD-$\mathscr{B}$ as follows:

DEFINITION 5.1. (a) $\mathrm{MOD}(\mathscr{B} \mid \mathscr{M})$ is the full additive subcategory of MOD-$\mathscr{B}$ having as objects all $\mathscr{B}$-modules $\mathscr{V}$ such that $\mathscr{V}|_{\mathscr{A}}$ divides some direct sum of copies of $\mathscr{M}$ in MOD-$\mathscr{A}$.

(b) $\mathrm{MOD}(\mathscr{B} \mid \text{weak } \mathscr{M})$ is the full additive subcategory of MOD-$\mathscr{B}$ having as objects all $\mathscr{B}$-modules $\mathscr{V}$ such that $\mathscr{V}|_{\mathscr{A}}$ weakly divides $\mathscr{M}$ in MOD-$\mathscr{A}$.

Similarly, we define full additive subcategories $\mathrm{MOD}(\mathscr{E} \mid \mathscr{D})$ and $\mathrm{MOD}(\mathscr{E} \mid \text{weak } \mathscr{D})$ of MOD-$\mathscr{E}$ as follows:

DEFINITION 5.2. (a) $\mathrm{MOD}(\mathscr{E} \mid \mathscr{D})$ is the full additive subcategory of MOD-$\mathscr{E}$ having as objects all $\mathscr{E}$-modules $\mathscr{W}$ such that $\mathscr{W}|_{\mathscr{D}}$ is projective.

(b) $\mathrm{MOD}(\mathscr{E} \mid \text{weak } \mathscr{D})$ is the full additive subcategory of MOD-$\mathscr{E}$ having as objects all $\mathscr{E}$-modules $\mathscr{W}$ such that $\mathscr{W}|_{\mathscr{D}}$ is projective of finite type.

(These subcategories can be compared with the subcategories as defined in [2, Sect. 7].)

(5.3.a) Let $\mathscr{V}$ be an object of MOD($\mathscr{B}$ | weak $\mathscr{M}$) (resp. of MOD($\mathscr{B}$|$\mathscr{M}$) if $\mathscr{M}$ is of finite type (or small)). Then by Definition 5.1(a) and 5.1(b) and by Corollary 3.2, $\mathscr{V}|_{\mathscr{A}}$ is an object of MOD-$\mathscr{A}_{\mathscr{M}}$, and by Definition 4.2 MOD($\mathscr{B}$ | weak $\mathscr{M}$) (resp. MOD($\mathscr{B}$ | $\mathscr{M}$)) becomes a full additive subcategory of MOD($\mathscr{B}$ rest. $\mathscr{A}_{\mathscr{M}}$).

(5.3.b) Similarly, if $\mathscr{W}$ is an object of MOD($\mathscr{E}$ | weak $\mathscr{D}$) (resp. of MOD($\mathscr{E}$ | $\mathscr{D}$) if $\mathscr{M}$ is of finite type (or small)), then by Definitions 5.2(a) and 5.2(b) and by Corollary 3.4, $\mathscr{W}|_{\mathscr{D}}$ is an object of MOD-$\mathscr{D}^{\mathscr{M}}$. From Definition 4.3 MOD($\mathscr{E}$ | weak $\mathscr{D}$) becomes a full additive subcategory of MOD($\mathscr{E}$ rest. $\mathscr{D}^{\mathscr{M}}$).

(5.4.a) An object $\mathscr{V}$ of MOD($\mathscr{B}$ | weak $\mathscr{M}$) (resp. MOD($\mathscr{B}$ | $\mathscr{M}$) if $\mathscr{M}$ is of finite type (or small)) weakly divides (resp. divides some direct sum of copies of) $\mathscr{M}$ in MOD-$\mathscr{A}$, and so $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V}|_{\mathscr{A}})$ is a projective $\mathscr{D}$-module of finite type (resp. a projective $\mathscr{D}$-module). By the adjoint associativity theorem [4, p. 424],

$$\mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{V}|_{\mathscr{A}}) \cong \mathrm{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V})$$

as $\mathscr{D}$-modules. Hence we deduce that if $\mathscr{V}$ is an object of MOD($\mathscr{B}$|weak $\mathscr{M}$) (resp. of MOD($\mathscr{B}$ | $\mathscr{M}$) if $\mathscr{M}$ is of finite type (or small)) then $\mathrm{Hom}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \mathscr{V})$ is an object of MOD($\mathscr{E}$ | weak $\mathscr{D}$) (resp. of MOD($\mathscr{E}$ | $\mathscr{D}$)).

(5.4.b) An object $\mathscr{W}$ of MOD($\mathscr{E}$ | weak $\mathscr{D}$) (resp. of MOD($\mathscr{E}$ | $\mathscr{D}$) if $\mathscr{M}$ is of finite type (or small)) is projective as a $\mathscr{D}$-module of finite type (resp. projective as a $\mathscr{D}$-module). So $\mathscr{W} \otimes_{\mathscr{D}} \mathscr{M}$ weakly divides (resp. divides some direct sum of copies of) $\mathscr{M}$ in MOD-$\mathscr{A}$. But from Lemma 4.7

$$\mathscr{W} \otimes_{\mathscr{D}} \mathscr{M} \cong \mathscr{W} \otimes_{\mathscr{E}} (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B})$$

as $\mathscr{D}$-modules. Hence we deduce that if $\mathscr{W}$ is an object of MOD($\mathscr{E}$ | weak $\mathscr{D}$) (resp. of MOD($\mathscr{E}$ | $\mathscr{D}$)), then $\mathscr{W} \otimes_{\mathscr{E}} (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B})$ is an object of MOD($\mathscr{B}$ | weak $\mathscr{M}$) (resp. of MOD($\mathscr{B}$ | $\mathscr{M}$)).

We conclude from the above paragraphs that:

THEOREM 5.5. *The restrictions of the additive functors* $\mathrm{HOM}_{\mathscr{B}}(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B}, \cdot)$ *and* $\cdot \otimes_{\mathscr{E}} (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{B})$ *form an equivalence*

$$\mathrm{MOD}(\mathscr{B} \mid \text{weak } \mathscr{M}) \approx \mathrm{MOD}(\mathscr{E} \mid \text{weak } \mathscr{D}).$$

*If* $\mathscr{M}$ *is of finite type* (*or small*) *then the restrictions of those functors form an equivalence*

$$\mathrm{MOD}(\mathscr{B} \mid \mathscr{M}) \approx \mathrm{MOD}(\mathscr{E} \mid \mathscr{D}).$$

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