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DISTRIBUTIONS OF AGENTS' CHARACTERISTICS*

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1. Introduction

In an exchange economy with l commodities an economic agent a is described by his preference relation $\succ_a$ and his initial endowment vector $e_a \in \mathbf{R}^l_+$.

Let $\mathscr{P}$ denote the set of all irreflexive and continuous binary relations on $\mathbf{R}^l_+$ with the following property: for every price vector $p \gg 0$ the set $\varphi(\succ, e, p)$ of *maximal elements* for $\succ$ in the *budget-set* $\{x \in \mathbf{R}^l_+ \mid px \leqslant pe\}$ is non-empty. For example, an irreflexive and continuous relation $\succ$ belongs to $\mathscr{P}$ if it is transitive or convex (i.e., the set $\{x \in \mathbf{R}^l_+ \mid x \succ z\}$ is convex for every $z \in \mathbf{R}^l_+$). In the latter case, a fixed point argument is used to establish the existence of maximal elements [Sonnenschein (1971) and Mas-Colell (1974)]. We endow the set $\mathscr{P}$ of preferences with Hausdorff's topology of closed convergence; then $\mathscr{P}$ becomes a separable metric space and the *demand-correspondence* φ is upper hemi-continuous at every point $(\succ, e, p) \in \mathscr{P} \times \mathbf{R}^l_+ \times \mathbf{R}^l_+$ where $p \gg 0$. [For details and concepts not explicitly defined, see e.g. Hildenbrand (1974)].

An *exchange economy* $\mathscr{E}$ is defined by a finite set A of economic agents and an assignment to every agent $a \in A$ of a preference relation $\succ_a \in \mathscr{P}$ and an endowment vector $e_a \in \mathbf{R}^l_+$. Hence an exchange economy is a mapping

$$\mathscr{E} : A \to \mathscr{P} \in \mathbf{R}^l_+.$$

The *distribution* $\mu_{\mathscr{E}}$ *of agents' characteristics* of the economy $\mathscr{E}$ is a measure on $\mathscr{P} \times \mathbf{R}^l_+$ defined by

$$\mu_{\mathscr{E}}(B) = \frac{\#\mathscr{E}^{-1}(B)}{\#A},$$

for every Borel subset B of $\mathscr{P} \times \mathbf{R}^l_+$.

The mapping $\mathscr{E}$ we call the individualistic (microscopic) description of an

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economy. The preference-endowment distribution $\mu_{\mathscr{E}}$ might be called the statistical (macroscopic) description of an economy. We will see later that for economies with a 'large' set A of agents the preference-endowment distribution $\mu_{\mathscr{E}}$ is the appropriate concept.

The outcome of any exchange process in the economy $\mathscr{E}$ is an *allocation*, i.e., a function $f\colon A \to \mathbf{R}^l_+$ such that

$$\sum_{a \in A} f_a = \sum_{a \in A} e_a .$$

Which allocations are likely to be observed as outcomes of the exchange process? We shall now define two a priori quite different concepts of economic equilibrium.

(a) A *Walras allocation* for the economy $\mathscr{E}$ is an allocation f which can be 'decentralized' by a suitably chosen price system, i.e., there exists a price vector $p^* \in \mathbf{R}^l$, $p^* \neq 0$, such that

$$f_a \in \varphi(\succ_a, e_a, p^*), \quad a \in A.$$

The set of all Walras allocations for the economy $\mathscr{E}$ is denoted by $W(\mathscr{E})$.

(b) A coalition S of agents in A can *improve upon* a proposed allocation f for the economy $\mathscr{E}$ if there exists an allocation g such that

(i) $\quad g_a \succ_a f_a$ for every $a \in S$,

(ii) $\quad \sum_{a \in S} g_a = \sum_{a \in S} e_a .$

The set of all allocations of the economy $\mathscr{E}$ that no coalition can improve upon is called the *core* of $\mathscr{E}$ and is denoted by $C(\mathscr{E})$.

It follows trivially from the definition that

$$W(\mathscr{E}) \subset C(\mathscr{E}).$$

However, in general, the core is much larger than the set $W(\mathscr{E})$.

Since Edgeworth (1881) economists have argued that the difference between these two equilibrium concepts is small for an economy in which every individual agent has a negligible influence on collective actions. The following two basic results give a precise meaning to the above proposition in quite different settings. Both results relate the core to the set of Walras equilibria for 'large competitive economies'.

The Debreu–Scarf limit theorem

Denote by $\mathscr{E}_n$ the 'n-fold replica economy' of a given economy $\mathscr{E}$, i.e., $\mathscr{E}_n : A_n \to \mathscr{P} \times \mathbf{R}^l_+$ such that $\# A_n = n \cdot \# A$ and $\mu_{\mathscr{E}_n} = \mu_{\mathscr{E}}$ $(n = 1, 2, \ldots)$.

Let us measure the difference between $C(\mathscr{E})$ and $W(\mathscr{E})$ by the smallest number $\delta > 0$ for which the following holds: for every $f \in C(\mathscr{E})$ there exists $f^* \in W(\mathscr{E})$ such that

$$|f(a) - f^*(a)| \leqslant \delta \quad \text{for every } a \in A.$$

Call this number $\delta(C(\mathscr{E}), W(\mathscr{E}))$.

Since $W(\mathscr{E}) \subset C(\mathscr{E})$, $\delta(C(\mathscr{E}), W(\mathscr{E}))$ is the Hausdorff distance between the two sets $C(\mathscr{E})$ and $W(\mathscr{E})$ where the distance between two allocations f and g is given by

$$\operatorname*{Max}_{a \in A} |f(a) - g(a)|\,.$$

Theorem [*Debreu–Scarf* (*1963*)]. *Let $\mathscr{E}$ be an economy with monotonic and strongly convex preferences and strictly positive total endowments.*

If $(\mathscr{E}_n)$ is the sequence of replica economies of $\mathscr{E}$, then

$$\delta(C(\mathscr{E}_n), W(\mathscr{E}_n)) \to 0.$$

Aumann's equivalence theorem

In the definition of an economy one can substitute the finite set A of agents by any positive measure space $(A, \mathscr{A}, \nu)$ with $\nu(A) = 1$. Then an *economy with a measure space of economic agents* is a measurable mapping $\mathscr{E}$ of $(A, \mathscr{A}, \nu)$ into $\mathscr{P} \times \mathbf{R}^l_+$ such that the mean endowment vector $\int e \,\mathrm{d}\nu$ is finite. An allocation is now an integrable function of $(A, \mathscr{A}, \nu)$ into $\mathbf{R}^l_+$ with $\int f \,\mathrm{d}\nu = \int e \,\mathrm{d}\nu$. A coalition S is a measurable subset of A with $\nu(S) > 0$. The two equilibrium concepts $W(\mathscr{E})$ and $C(\mathscr{E})$ are extended in an obvious way. Let $\mathscr{P}_{\text{mo}}$ denote the set of monotonic preferences in $\mathscr{P}$.

Theorem [*Aumann* (*1964*)]. *Let $\mathscr{E} : (A, \mathscr{A}, \nu) \to \mathscr{P}_{\text{mo}} \times \mathbf{R}^l_+$ be an economy with an atomless measure space of economic agents and strictly positive mean endowments $\int e \,\mathrm{d}\nu$. Then*

$$W(\mathscr{E}) = C(\mathscr{E}).$$

The purpose of this survey is to relate these two fundamental results. For this purpose we introduce the concept of a 'purely competitive sequence' $(\mathscr{E}_n)$

of economies, which generalizes replica sequences, and we shall associate to every such sequence a 'limit economy' which may be represented as an economy with an *atomless* measure space of economic agents.

Definition. The sequence $(\mathscr{E}_n)$, $\mathscr{E}_n : A_n \to \mathscr{P} \times \mathbf{R}^l_+$, *is called purely competitive if*

(i) *the number* $\# A_n$ *of agents in the economy tends to infinity,*

(ii) *the sequence* $(\mu_{\mathscr{E}_n})$ *of preference-endowment distributions converges weakly to a limit* μ,

(iii) $\int e \, \mathrm{d}\mu_{\mathscr{E}_n} \to \int e \, \mathrm{d}\mu \gg 0$.

[For a detailed discussion of these sequences, see Hildenbrand (1974, ch. 2.1).]

It seems natural to consider as a 'limit economy $\mathscr{E}_\infty$' of a purely competitive sequence $(\mathscr{E}_n)$ any economy with an atomless measure space of agents; the only condition would be that the preference-endowment distribution $\mu_{\mathscr{E}_\infty}$ of this limit economy is equal to lim $\mu_{\mathscr{E}_n}$. If one takes this view, that is to say, the measure space $(A, \mathscr{A}, \nu)$ of agents has no intrinsic significance, then one has to show that two different atomless economies with the same distribution have the same set of equilibria. Since allocations are defined as functions on the space of agents one can not compare directly the allocations for different spaces of agents. We therefore consider their distributions on the commodity space $\mathbf{R}^l$, i.e.,

$$(\mathscr{D}f)(B) = \nu(f^{-1}(B)), \quad B \in \mathscr{B}(\mathbf{R}^l).$$

It is known [Kannai (1970)] that for two atomless economies $\mathscr{E}$ and $\mathscr{E}'$ with the same preference-endowment distribution the sets $\mathscr{D}W(\mathscr{E})$ and $\mathscr{D}W(\mathscr{E}')$ of distributions of Walras equilibria are not necessarily identical. However the difference of $\mathscr{D}W(\mathscr{E})$ and $\mathscr{D}W(\mathscr{E}')$ is not substantial; the two sets have the same closure. [For details see Hart–Hildenbrand–Kohlberg (1974).]

The arbitrariness in the choice of the atomless measure space of agents for the limit economy strongly suggests that one should define the equilibria for the limit economy only in terms of the limit preference-endowment distribution lim $\mu_{\mathscr{E}_n}$. In the limit the individual agent has so to speak lost his identity and it therefore seems artificial to keep the individualistic description of an economy in the limit.

In section 2 we define an equilibrium distribution for an arbitrary preference-endowment distribution μ. This concept is then used in section 3 to state a general limit theorem on the core.

2. Equilibrium distributions

In this section we describe an economy only by a distribution μ of agents' characteristics, without referring to the individual agents. In this context an allocation is described by the joint distribution of agents' characteristics $(\succ, e)$ and consumption vectors x, that is by a measure τ on the product $\mathscr{P} \times \mathbf{R}^l_+ \times \mathbf{R}^l_+$.

Definition. An equilibrium distribution for the distribution μ of agents' characteristics is a measure τ on $\mathscr{P} \times \mathbf{R}^l_+ \times \mathbf{R}^l_+$ with the properties:

(i) *the marginal distribution of τ on the characteristic space $T = \mathscr{P} \times \mathbf{R}^l_+$ is equal to the given measure μ, i.e., $\mu = \tau \circ \mathrm{proj}_T^{-1}$,*
where $\mathrm{proj}_T(\succ, e, x) = (\succ, e)$;

(ii) *mean supply equals mean demand, i.e. $\int \mathrm{proj}_e \, \mathrm{d}\tau = \int \mathrm{proj}_x \, \mathrm{d}\tau$,*
where $\mathrm{proj}_e(\succ, e, x) = e$ and $\mathrm{proj}_x(\succ, e, x) = x$;

(iii) *there exists a price vector $p \in \mathbf{R}^l$, $p \neq 0$, such that*
$\tau\{(\succ, e, x) \in \mathscr{P} \times \mathbf{R}^l_+ \times \mathbf{R}^l_+ \mid x \in \varphi(\succ, e, p)\} = 1.$

The set of all equilibrium distributions for μ is denoted by $W_{\mathscr{D}}(\mu)$.

Clearly the definition of an equilibrium distribution τ is simply a reformulation of the usual concept of Walras equilibrium in terms of distributions. Indeed, if $\mathscr{E} : A \to \mathscr{P} \times \mathbf{R}^l_+$ is any economy with characteristic distribution $\mu_{\mathscr{E}}$ and if $f \in W(\mathscr{E})$, then the distribution of the mapping $(\mathscr{E}, f)$ is an equilibrium distribution for $\mu_{\mathscr{E}}$. However, if τ is an equilibrium distribution for the characteristic distribution $\mu_{\mathscr{E}}$ there is not necessarily a Walras allocation f for the economy $\mathscr{E}$ such that the distribution of $(\mathscr{E}, f)$ is equal to τ. The reason for this is that the space A of agents may be too small (even if it is atomless). One can assert only that for every $\tau \in W_{\mathscr{D}}(\mu)$ there exists a suitably chosen economy $\mathscr{E}$ with characteristic distribution μ and an $f \in W(\mathscr{E})$ such that τ is the distribution of $(\mathscr{E}, f)$.

Theorem 1. For every distribution μ of agents' characteristics in $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ with mean endowments $\int \mathrm{proj}_e \, \mathrm{d}\mu$ strictly positive there exists an equilibrium distribution τ. The correspondence

$$\mu \mapsto W_{\mathscr{D}}(\mu)$$

has the following continuity property:

Let the sequence (μ_n) be weakly convergent to μ on $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$, where

$$\lim \int \mathrm{proj}_e \, \mathrm{d}\mu_n = \int \mathrm{proj}_e \, \mathrm{d}\mu \gg 0.$$

Then for every sequence (τ_n), $\tau_n \in W_{\mathscr{D}}(\mu_n)$, there exists a weakly convergent subsequence whose limit belongs to $W_{\mathscr{D}}(\mu)$.

If one restricts all characteristic distributions to a compact subset T of $\mathscr{P}_{\text{mo}} \times \mathbf{R}^l_+$ then Theorem 1 means that the correspondence $\mu \mapsto W_{\mathscr{D}}(\mu)$ is compact-valued and u.h.c. at every characteristic distribution μ on T with strictly positive mean endowments.

We sketch now a proof for the compact case. The general situation is easily reduced to the compact case by using the tightness of the measures on $\mathscr{P}_{\text{mo}} \times \mathbf{R}^l_+$. [For details, see Hildenbrand (1974, Theorem 3, p. 159).]

The existence of equilibrium distributions τ will follow easily if we have established the u.h.c. of $\mu \mapsto W_{\mathscr{D}}(\mu)$. Let $\tau_n \in W_{\mathscr{D}}(\mu_n)$, $(n = 1, \ldots)$. We shall first show that the sequence (τ_n) is relative compact, so that there exists a converging subsequence, and then we shall prove that the limit of this subsequence belongs to $W_{\mathscr{D}}(\mu)$.

Let p_n be a normalized price vector corresponding to the equilibrium distribution τ_n. Since preferences are monotonic we must have $p_n \gg 0$. It follows that every limit point p of the sequence is strictly positive. Indeed, assume to the contrary that $p_n \to p$, where p is not strictly positive. It is well-known (and easily shown) that the sequence of individual demand sets $(\varphi(t, p_n))$ is unbounded, i.e., for every bounded set $B \subset \mathbf{R}^l$ we have $\varphi(t, p_n) \cap B = \emptyset$ for n large enough. By continuity it then follows that the same conclusion holds uniformly for all t in the compact set $T \subset \mathscr{P}_{\text{mo}} \times \mathbf{R}^l$. Thus, $\tau_n(T \times B) = 0$ for every bounded set $B \subset \mathbf{R}^l_+$ and n large enough. But this implies that the mean demand vector $\int \text{proj}_x \, d\tau_n$ will be larger than the mean supply $\int \text{proj}_e \, d\mu_n$; a contradiction.

Therefore there is a compact subset $K \subset \mathbf{R}^l_+$ such that $\varphi(t, p_n) \subset K$ for every $n = 1, \ldots$ and $t \in T$. Consequently, every measure τ_n is concentrated on the compact set $T \times K$ which proves the relative compactness of the sequence (τ_n).

Let $\tau_n \to \tau$ and $p_n \to p$. It remains to show that $\tau \in W_{\mathscr{D}}(\mu)$. Properties (i) and (ii) are easily verified. In order to prove property (iii), consider the correspondence

$$p \mapsto E_p := \{(t, x) \in T \times \mathbf{R}^l \mid x \in \varphi(t, p)\}.$$

The correspondence is compact-valued and u.h.c. at every $p \gg 0$. Thus E_{p_n} is contained in a closed neighborhood U of E_p for n large enough. Since $\tau_n(E_{p_n}) = 1$, we have $\tau_n(U) = 1$. Consequently, $\tau(U) \geqslant \limsup \tau_n(U) = 1$. It then follows that $\tau(E_p) = 1$.

In order to prove that $W_{\mathscr{D}}(\mu) \neq \emptyset$ it suffices to consider characteristic distributions μ with *finite support* since the set of these measures is dense and since we established the above continuity property of $W_{\mathscr{D}}(\cdot)$.

Let $\text{supp}\,(\mu) = \{t_1, t_2, \ldots, t_r\}$ and define

$$Z(p) := \sum_{i=1}^{r} (\text{co}\ \varphi(t_i, p) - e_i)\mu(t_i).$$

[Note that even for convex (but not complete) preferences the demand set $\varphi(t, p)$ is not necessarily convex.]

To prove the existence of an equilibrium distribution μ with an equilibrium price vector p it suffices to find a vector $p^* \neq 0$ such that $0 \in Z(p^*)$. But such a price vector p^* can be obtained by the well-known fixed point argument [Debreu (1959)]. The measure τ is then obtained in the following way:

$$0 = \sum_{i=1}^{r} z_i \cdot \mu(t_i),$$

where

$$z_i \in \text{co}\ \varphi(t_i, p^*) - e_i.$$

Thus,

$$z_i + e_i = \sum_{k=0}^{l} \lambda_i^k x_i^k,$$

where

$$x_i^k \in \varphi(t_i, p^*), \quad \sum_{k=0}^{l} \lambda_i^k = 1, \quad \lambda_i^k \geqslant 0.$$

Define now

$$\text{supp}\,(\tau) = \{(t_i, x_i^k) \mid i = 1, \ldots, r, k = 0, \ldots, l\},$$

and

$$\tau(t_i, x_i^k) = \mu(t_i) \cdot \lambda_i^k.$$

3. Limit theorems on the core

Let $(\mathscr{E}_n)$ be a purely competitive sequence of economies with characteristics in $\mathscr{P}_{\text{mo}} \times \mathbf{R}_+^l$ as defined in the introduction. Let $W_{\mathscr{D}}(\mu)$ denote the set of equilibrium distributions for the preference-endowment distribution $\mu = \lim \mu_{\mathscr{E}_n}$. If $f \in C(\mathscr{E})$, we denote by $\mathscr{D}(\mathscr{E}, f)$ the distribution of the mapping $a \mapsto (\mathscr{E}(a), f(a))$, i.e.,

$$\mathscr{D}(\mathscr{E}, f)(B) = \#\{a \in A \mid (\mathscr{E}(a), f(a)) \in B\} / \# A,$$

and let $C_{\mathscr{D}}(\mathscr{E}) = \{\mathscr{D}(\mathscr{E}, f) \mid f \in C(\mathscr{E})\}$. The set of measures on $\mathscr{P}_{\text{mo}} \times \mathbf{R}_+^l \times \mathbf{R}_+^l$ is endowed with the weak topology.

Theorem 2. Let $(\mathscr{E}_n)$ be a purely competitive sequence of economies with characteristics in $\mathscr{P}_{\text{mo}} \times \mathbf{R}_+^l$ and let U be a neighborhood of $W_{\mathscr{D}}(\mu)$. Then for n large enough

$$C_{\mathscr{D}}(\mathscr{E}_N) \subset U.$$

The proof is too lengthy to be given here. The interested reader is referred to Hildenbrand (1974, Proposition 3, p. 200).

We now want to compare the assertion of Theorem 2 with the much stronger conclusion in Debreu–Scarf's theorem as stated in the introduction.

(a) Let us first remark that a statement in terms of distributions necessarily is a statement about *most* agents of the economy and not about *every* agent. This becomes clear if we give an equivalent formulation of Theorem 2 in terms of allocations as functions rather than in terms of their distributions. To avoid technical complications we restrict ourselves now to strongly convex preferences in order to obtain a unique maximal element $\varphi(\succ, e, p)$ [for a detailed discussion of the general case, see Hildenbrand (1974, ch. 3.3)].

Let $\mathscr{P}_{\text{mo,sco}}$ denote the subset of monotonic and strongly convex preferences in $\mathscr{P}$.

Theorem 3. Let $(\mathscr{E}_n)$ *be a purely competitive sequence on* $\mathscr{P}_{\text{mo,sco}} \times \mathbf{R}^l_+$. *Then for every* $\varepsilon > 0$ *and* $\eta > 0$ *there exists an* $\bar{n}$ *such that for every* $n \geqslant \bar{n}$ *and* $f \in C(\mathscr{E}_n)$ *there exists* $p \in \Pi(\mu)$ *with the properties:*

$$\frac{1}{\# A_n} \# \{a \in A_n \mid |f(a) - \varphi(\mathscr{E}_n(a), p)| \geqslant \eta\} \leqslant \varepsilon.$$

We emphasize that the price vector $p \in \Pi(\mu)$, which approximately decentralizes the core allocation f, for most agents is an equilibrium price vector for the limit characteristic distribution $\mu = \lim \mu_{\mathscr{E}_n}$.

In order to show that Theorem 2 implies Theorem 3 we remark that it suffices to show that for every sequence $(f_n), f_n \in C(\mathscr{E}_n)$, there is a subsequence (f_{n_q}) and a price vector $p \in \Pi(\mu)$ such that the sequence

$$|f_{n_q}(\cdot) - \varphi(\mathscr{E}_{n_q}(\cdot), p)|_{q=1,\dots}$$

converges in measure to zero. Let $f_n \in C(\mathscr{E}_n)$. By Theorem 2 we can assume that a subsequence $(\mathscr{E}_{n_q}, f_{n_q})$ converges in distribution to an equilibrium distribution $\tau \in W_{\mathscr{D}}(\mu)$. One now chooses a continuous representation of $(\mathscr{E}_{n_q}, f_{n_q})$ [see Hildenbrand (1974, Proposition 2, p. 139)] and then one applies the arguments of part (c) of the proof of Theorem 1 in Hildenbrand (1974, p. 189).

On the other hand one easily shows that Theorem 3 implies Theorem 2 in the case of strongly convex preferences.

Indeed, it suffices to show that for every sequence (f_n), $f_n \in C(\mathscr{E}_n)$, there is a subsequence (n_q) such that

$$\mathscr{D}(\mathscr{E}_{n_q}, f_{n_q}) \to \tau \in W_{\mathscr{D}}(\mu).$$

Theorem 3 implies that for every sequence $(f_n), f_n \in C(\mathscr{E}_n)$, there is a subsequence (n_q) and a price vector $p \in \Pi(p)$ such that

$$|f_{n_q}(\cdot) - \varphi(\mathscr{E}_{n_q}(\cdot), p)|$$

converges in measure to zero. Therefore, if $(\mathscr{E}_{n_q}, \varphi(\mathscr{E}_{n_q}(\cdot), p))$ converges in distribution to a measure, say τ, then $(\mathscr{E}_{n_q}, f_{n_q})$ also converges to the same measure [e.g., Billingsley (1968, Theorem 4.1, p. 25)]. But $(\mathscr{E}_{n_q}, \varphi(\mathscr{E}_{n_q}(\cdot), p))$, where $p \in \Pi(\mu)$, converges in distribution to the equilibrium distribution $\tau \in W_{\mathscr{D}}(\mu)$ which corresponds to the equilibrium price p.

(b) In Debreu–Scarf's limit theorem one compares the core $C(\mathscr{E}_n)$ with the set $W(\mathscr{E}_n)$ of Walras allocations of the *same* economy while in Theorem 2 we compare the core of $\mathscr{E}_n$ – more precisely the set $C_{\mathscr{D}}(\mathscr{E}_n)$ of measures – with the set $W_{\mathscr{D}}(\mu)$ of equilibrium distributions of the limit characteristic distribution $\mu = \lim \mu_{\mathscr{E}_n}$. [Of course, for replica sequences there is no difference, since $W_{\mathscr{D}}(\mu_n) = W_{\mathscr{D}}(\mu)$!]

If we endow the space of measures on $\mathscr{P} \times \mathbf{R}^l \times \mathbf{R}^l$ with the Prohorov metric, then we can introduce the Hausdorff distance δ for subsets of measures. Debreu–Scarf's limit theorem then implies that

$$\delta(C_{\mathscr{D}}(\mathscr{E}_n), W_{\mathscr{D}}(\mu)) \to 0.$$

In Theorem 2 we assert only that the Hausdorff semi-distance

$$\sup_{\tau \in C_{\mathscr{D}}\mathscr{E}_n} \operatorname{dist}(\tau, W_{\mathscr{D}}(\mu))$$

converges to zero. The reason for the stronger conclusion to be true is again the continuity of the set $W_{\mathscr{D}}(\cdot)$ when $\mu_{\mathscr{E}n} \to \mu$. Indeed, one easily shows:

Let $(\mathscr{E}_n)$ be a purely competitive sequence on $\mathscr{P}_{\mathrm{mo}} \times \mathbf{R}^l_+$ such that

$$\delta(\mathscr{D}W(\mu_{\mathscr{E}_n}), \mathscr{D}W(\mu)) \to 0.$$

Then

(i) $$\delta(C_{\mathscr{D}}(\mathscr{E}_n), W_{\mathscr{D}}(\mu)) \to 0,$$

(ii) $$\delta(C_{\mathscr{D}}(\mathscr{E}_n), W_{\mathscr{D}}(\mu_n)) \to 0.$$

We emphasize that the continuity of $W_{\mathscr{D}}(\cdot)$ is essential. Without it, one can easily construct examples such that (i) or (ii) does not hold. Therefore, in order to obtain stronger conclusions than in Theorem 2 or 3 either one has to restrict the analysis to special sequences – e.g., the replica sequences – or one has to specify those characteristic distributions μ for which the correspondence $\mu \mapsto W_{\mathscr{D}}(\mu)$ is continuous.

The latter approach has been taken by H. Dierker in a paper appearing in this issue.

Once one has the stronger conclusion

$$\delta(C_{\mathscr{D}}(\mathscr{E}_n),\ W_{\mathscr{D}}(\mu_n)) \to 0,$$

one can ask a much stronger question: how fast is this convergence? For replica economies the answer has recently been given by Debreu (1975): for a regular economy $\mathscr{E}$ the distance between the core of $\mathscr{E}_n$ and its set of Walras allocations converges to zero at least as fast as the inverse of the number of agents. The same speed of convergence for the general case of purely competitive sequences with regular limit distribution μ was shown by B. Grodal, whose paper is also presented in this issue.

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