Non-separating cocircuits in binary matroids

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Abstract

In this note, we show that a 3-connected binary matroid $M$ has at least $r(M) - 1$ non-separating cocircuits avoiding a fixed element. As a consequence of this result, we get a lower bound for the number of non-separating cocircuits of a simple and a cosimple connected binary matroid that generalizes the bound obtained by McNulty and Wu.

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1. Introduction

We say that a cocircuit $C^*$ of a matroid $M$ is non-separating when $M \setminus C^*$ is connected. We denote the set of non-separating cocircuits of a matroid $M$ by $R^*(M)$. Note that a cocircuit of a matroid $M$ is non-separating if and only if its complement is a connected hyperplane of $M$. For a connected graphic matroid, a non-separating cocircuit corresponds to the star of a vertex whose deletion from the associated graph keeps it 2-connected. Non-separating circuits and cocircuits play an important role in studying the structure of graphic matroids (see [7,8,17,18]). There has been much
interest in the study of non-separating circuits and cocircuits in graphs and matroids lately (see [1,3–6,8,10,11,16–18]).

For matroid notation and terminology, we follow Oxley [14]. For definitions related to chains and fans, we use Oxley and Wu [15]. Kelmans [6] and, independently, Seymour (see [13]) proved that every simple and cosimple connected binary matroid has a non-separating cocircuit, that is,

**Theorem 1.** *If* $M$ *is a simple and cosimple connected binary matroid, then*

$$|\mathcal{R}^*(M)| \geq 1.$$  

Bixby and Cunningham [1] showed implicitly that a 3-connected binary matroid has a lot of non-separating cocircuits, namely:

**Theorem 2.** *If* $M$ *is a 3-connected binary matroid such that* $r(M) \geq 3$, *then*

$$|\mathcal{R}^*(M)| \geq r(M) + 1.$$  

Next, McNulty and Wu [12] improved the bound on the number of non-separating cocircuits for simple and cosimple connected binary matroids given on Theorem 1:

**Theorem 3.** *If* $M$ *is a simple and cosimple connected binary matroid, then*

$$|\mathcal{R}^*(M)| \geq 4.$$  

This bound also is the best possible: McNulty and Wu gave an infinite family of extremal examples. Note that there is a large gap between Bixby and Cunningham’s bound on the number of non-separating cocircuits for 3-connected binary matroids and McNulty and Wu’s bound for simple and cosimple connected binary matroids. In this note, we obtain a bound that is close to the bound presented in Theorem 2, in some sense, and generalize the bound given by Theorem 3. The main result of this note is the following:

**Theorem 4.** *If* $M$ *is a 3-connected binary matroid and* $a \in E(M)$, *then there are at least* $r(M) - 1$ *non-separating cocircuits of* $M$ *avoiding* $a$.

Theorem 4 is the best possible, since any 3-connected graphic matroid $M$ other than a loop has exactly $r(M) - 1$ cocircuits avoiding a fixed element.

An important tool used in this note is the following idea of decomposing a connected matroid $M$. Assume $|E(M)| \geq 3$. A tree decomposition of $M$ is a tree $T$ with edges labelled $e_1, e_2, \ldots, e_{k-1}$ and vertices labelled by matroids $M_1, M_2, \ldots, M_k$ such that

(i) each $M_i$ is 3-connected having at least four elements or is a circuit or cocircuit with at least three elements;
(ii) \(E(M_1) \cup E(M_2) \cup \cdots \cup E(M_k) = E(M) \cup \{e_1, e_2, \ldots, e_{k-1}\}\);

(iii) if the edge \(e_i\) joins the vertices \(M_{j_1}\) and \(M_{j_2}\), then \(E(M_{j_1}) \cap E(M_{j_2}) = \{e_i\}\);

(iv) if no edge joins the vertices \(M_{j_1}\) and \(M_{j_2}\), then \(E(M_{j_1}) \cap E(M_{j_2})\) is empty;

(v) \(M\) is the matroid that labels the single vertex of the tree \(T/e_1, e_2, \ldots, e_{k-1}\) at the conclusion of the following process: contract the edges \(e_1, e_2, \ldots, e_{k-1}\) of \(T\) one by one in order; when \(e_i\) is contracted, its ends are identified and the vertex formed by this identification is labelled by the 2-sum of the matroids that previously labelled the ends of \(e_i\).

Cunningham and Edmonds [2] proved the following result.

**Theorem 5.** Every connected matroid \(M\) has a tree decomposition \(T(M)\) in which no two adjacent vertices are both labelled by circuits or are both labelled by cocircuits. Furthermore, the tree \(T(M)\) is unique to within relabelling of its edges.

We shall call \(T(M)\) the canonical tree decomposition of \(M\) and let \(\Lambda_2(M)\) be the set of matroids that label vertices of \(T(M)\).

For a 3-connected binary matroid \(M\) and \(A \subseteq E(M)\), we define \(R^*_A(M)\) to be the set of non-separating cocircuits of \(M\) avoiding \(A\). When \(A = \{a\}\), we use \(R^*_a(M)\) instead of \(R^*_A(M)\).

**Theorem 6.** Suppose that \(M\) is a simple and cosimple connected binary matroid. If \(M\) is not 3-connected, then

\[ R^*_{a_1}(M_1) \cup R^*_{a_2}(M_2) \cup \cdots \cup R^*_{a_n}(M_n) \subseteq R^*(M), \]

where \(M_1, M_2, \ldots, M_n\) label the terminal vertices of \(T(M)\) and, for \(i \in \{1, 2, \ldots, n\}\), \(a_i\) labels the edge of \(T(M)\) incident to \(M_i\). Moreover,

\[ |R^*(M)| \geq \sum_{i=1}^{n} [r(M_i) - 1]. \]

As each matroid \(M_i\) that labels a terminal vertex of \(T(M)\) is not a circuit or a cocircuit, since \(M\) is simple and cosimple, it follows that \(r(M_i) \geq 3\) and so \(|R^*(M)| \geq 2n\). Theorem 3 follows from Theorem 6 because \(n \geq 2\). From Theorems 2 and 6, we conclude that, when the bound on Theorem 3 is attained for some simple and cosimple connected binary matroid \(M\), then \(T(M)\) is a path whose terminal vertices are labeled by matroids isomorphic to \(M(K_4)\).

For a simple and cosimple connected binary matroid \(M\), note that \(T(M)\) has only two vertices if and only if \(M\) has just one 2-separation. If \(M_1\) and \(M_2\) labels the two vertices of \(T(M)\), then \(M = M_1 \oplus M_2\) and so \(r(M) = r(M_1) + r(M_2) - 1\). Thus, from Theorem 6, we get:
Corollary 1. Suppose that $M$ is a simple and cosimple connected binary matroid. If $M$ has just one 2-separation, then

$$|\mathcal{R}^*(M)| \geq r(M) - 1.$$  

This bound is close to the bound given by Theorem 2 and much better than the bound given by Theorem 3. It is not possible to get a substantially better bound depending on the number of 2-separations of $M$, as we show in an example after the proof of Theorem 6: we construct an infinite family of simple and cosimple connected binary matroids each having just two 2-separations and only four non-separating cocircuits.

**Proof (Proof of Theorem 6 assuming Theorem 4).** Observe that each matroid $M_i$ that labels a terminal vertex of $T(M)$ is 3-connected. So, $|\mathcal{R}_{M_i}^*(M_i)| \geq r(M_i) - 1$, by Theorem 4. The result follows provided we show that

$$\mathcal{R}_{M_1}^*(M_1) \cup \mathcal{R}_{M_2}^*(M_2) \cup \cdots \cup \mathcal{R}_{M_n}^*(M_n) \subseteq \mathcal{R}^*(M).$$

Suppose that $C^* \in \mathcal{R}_{M_i}^*(M_i)$. Let $T$ be the connected component of $T(M) - e_1$ other than the vertex labeled by $M_i$. If $N_i$ is the matroid such that $T(N_i) = T$, then $M = M_i \oplus_2 N_i$. As $r(M_i \setminus C^*) = r(M_i) - 1 \geq 2$, for $C^* \in \mathcal{R}_{M_i}^*(M_i)$, it follows that $|E(M_i \setminus C^*)| \geq 3$ and so $M \setminus C^* = (M_i \setminus C^*) \oplus_2 N_i$. Thus $M \setminus C^*$ is connected and hence $C^* \in \mathcal{R}^*(M)$. \hfill \Box

One may think that Theorem 4 is a particular case of a more general result, that holds for graphic matroids: if $M$ is a 3-connected binary matroid and $A \subseteq E(M)$, then the number of non-separating cocircuits of $M$ that avoids $A$ is at least $r(M) + 1 - 2|A|$. But this assertion is false, even for $|A| = 2$, as the next example shows. For $m \geq 2$, let $S_m$ be the binary spike having legs $L_0, L_1, \ldots, L_m$ and tip $e$. Suppose that $L_0 = \{e, f, g\}$. Consider the 3-connected matroid $B_m = S_m \setminus g$. Now, we prove that $\mathcal{R}_{B_m}^*(M) = \emptyset$. If $C^*$ is a cocircuit of $B_m$ such that $C^* \cap \{e, f\} = \emptyset$, then $C^* \cap L_i \neq \emptyset$, for some $i \in \{1, 2, \ldots, m\}$, say $i = 1$. By orthogonality, $L_1 - e \subseteq C^*$. But $(L_1 - e) \cup f$ is a triad of $B_m$ and so $B_m \setminus C^*$ has $f$ as a coloop. Thus $\mathcal{R}_{B_m}^*(M) = \emptyset$.

For $n \geq 2$, let $M_1, M_2, \ldots, M_n$ be simple and cosimple connected binary matroids whose ground sets are pairwise disjoint. For $i \in \{1, 2, \ldots, n\}$, choose $a_i \in E(M_i)$. In this paragraph, we construct a matroid $H$ having arbitrary larger rank such that $M_1, M_2, \ldots, M_n$ label the terminal vertices of $T(H)$, $a_1, a_2, \ldots, a_n$ label the pendant edges of $T(H)$ and

$$\mathcal{R}_{M_1}^*(M_1) \cup \mathcal{R}_{M_2}^*(M_2) \cup \cdots \cup \mathcal{R}_{M_n}^*(M_n) = \mathcal{R}^*(H).$$

That is, Theorem 6 cannot be improved. Let $m, B_m$ and $\{e, f\}$ be as in the previous paragraph. We may assume that $E(B_m) \cap E(M_i) = \emptyset$, for every $i$. When $n = 2$, relabel the element $e$ of $B_m$ by $a_1$ and let $H'$ be the 2-sum of $M_1$ and $B_m$. When $n \geq 3$, let $H'$ be the 2-sum of $M_1, \ldots, M_{n-1}, B_m$ and $H'$, where $H'$ is a cocircuit such that $E(H') = \{e, a_1, a_2, \ldots, a_{n-1}\}$. Next, relabel the element $f$ of $B_m$ by $a_n$ and let $H$
be the 2-sum of $H'$ with $M_n$. Note that $H$ has the desired properties. When $n = 2$ and $M_1$ and $M_2$ are both isomorphic to $M(K_4)$, $H$ has only four non-separable cocircuits and two 2-separations.

To prove Theorem 4, we need the next lemma from Bixby and Cunningham [1].

**Lemma 1.** Suppose that $M$ is a 3-connected binary matroid such that $r^*(M) \geq 3$. If $e \in E(M)$ and $C^* \in R^*(M/e)$, then

(i) $C^* \in R^*(M)$; or
(ii) there are $C_1^*, C_2^* \in R^*(M)$ such that $C_1^* \cap C_2^* = \{e\}$, $C_1^* \cup C_2^* = C^* \cup e$ and $C_1^* \Delta C_2^* = C^*$.

Theorem 4 is a consequence of the next result. Instead of proving that the cardinality of $R^*_a(M)$ is at least $r(M) - 1$, we show that the dimension of the subspace of the cocycle space spanned by $R^*_a(M)$ is at least $r(M) - 1$.

**Theorem 7.** Suppose that $M$ is a 3-connected binary matroid. If $a \in E(M)$, then

$$\dim_{GF(2)} R^*_a(M) \geq r(M) - 1.$$  

**Proof.** Suppose that Theorem 7 is not true and choose a counter-example $M$ such that $|E(M)|$ is minimum. Observe that $r(M) \geq 3$. Now, we divide the proof of this result in a sequence of lemmas.

**Lemma 2.** If $e \in E(M) - a$, then $M \setminus e$ is not 3-connected.

**Proof.** Suppose that $M \setminus e$ is 3-connected. By the choice of $M$, we have that

$$\dim_{GF(2)} R^*_a(M \setminus e) \geq r(M \setminus e) - 1 = r(M) - 1.$$ 

If $C^* \in R^*_a(M \setminus e)$, then $H = E(M) - (C^* \cup e)$ is a connected hyperplane of $M \setminus e$. Hence $H$ or $H \cup e$ is a connected hyperplane of $M$ and so $R^*_a(M) \cap \{C^*, C^* \cup e\} \neq \emptyset$. Thus $\dim_{GF(2)} R^*_a(M) \geq \dim_{GF(2)} R^*_a(M \setminus e)$; a contradiction.

**Lemma 3.** If $T$ is a triangle of $M$ and $e \in T - a$, then there is a triad $T^*$ of $M$ such that $e \in T^*$.

**Proof.** By Lemma 2, $M \setminus f$ is not 3-connected, for every $f \in T - a$. As $|T - a| \geq 2$, it follows, by Tutte’s triangle lemma (7.2 of [19]), that every element of $T - a$ belongs to a triad of $M$.

**Lemma 4.** If $e \in E(M) - a$ and $M/e$ is 3-connected, then

$$\dim_{GF(2)} R^*_a(M) = r(M) - 2$$

and $R^*_a(M) = R^*_{[a,e]}(M)$. 


Proof. By the choice of $M$, we have that
\[ \dim_{GF(2)} \mathcal{R}_n^*(M/e) \geq r(M/e) - 1 = r(M) - 2. \]

By Lemma 1, if $C^* \in \mathcal{R}_n^*(M/e)$, then

(i) $C^* \in \mathcal{R}_n^*(M)$; or
(ii) there is a partition $\{X_1, X_2\}$ of $C^*$ such that, for $i \in \{1, 2\}, C_i^* = X_i \cup e \in \mathcal{R}_n^*(M)$.

Let $\mathcal{R}$ be the family of cocircuits obtained from $\mathcal{R}_n^*(M/e)$ by replacing each $C^* \in \mathcal{R}_n^*(M/e)$ for which (ii) happens by $C_1^*$ and $C_2^*$. If (ii) happens to some $C^* \in \mathcal{R}_n^*(M/e)$, then the space spanned by $\mathcal{R}_n^*(M/e)$ is properly contained in the space spanned by $\mathcal{R}$ and so $\dim_{GF(2)} \mathcal{R}_n^*(M) > \dim_{GF(2)} \mathcal{R}_n^*(M/e)$; a contradiction. Hence (i) occurs for every $C^* \in \mathcal{R}_n^*(M/e)$ and so $\mathcal{R}_n^*(M/e) \subseteq \mathcal{R}_n^*(M)$. But $\dim_{GF(2)} \mathcal{R}_n^*(M) \leq r(M) - 2 \leq \dim_{GF(2)} \mathcal{R}_n^*(M/e)$.

So equality holds along this inequality and the first part of this lemma follows. Moreover, $\mathcal{R}_n^*(M/e)$ spans $\mathcal{R}_n^*(M)$ and hence every cocircuit of $\mathcal{R}_n^*(M)$ does not include $e$. The second part of this result also follows. □

Lemma 5. If $T_1, T_2, \ldots, T_n$ is a fan of $M$ such that $n \geq 2$, then $n = 3, T_2$ is a triangle and $T_1 \cap T_2 \cap T_3 = \{a\}$.

Proof. There are two possibilities for $T_1$:

(i) If $T_1$ is a triangle, then $T_1 - T_2 = \{a\}$, by Lemma 3.
(ii) If $T_1$ is a triad, then $a \in T_1$. Suppose that $a \notin T_1$. As $M/e$ is 3-connected, for $e \in T_1 - T_2$, it follows that $\mathcal{R}_n^*(M) = \mathcal{R}_{n,e}^*(M)$, by Lemma 4. But $T_1 - e$ is a series class of $M \setminus e$ contained in a triangle and so $M \setminus T_1$ is connected; a contradiction. So $a \in T_1$.

For $T_n$, we have a similar result, namely.

(iii) $T_n$ is a triangle and $T_n - T_{n-1} = \{a\}$; or
(iv) $T_n$ is a triad and $a \in T_n$.

So (ii) and (iv) hold. Hence $T_1$ and $T_n$ are triads such that $a \in T_1 \cap T_n$. Thus $n = 3$ and $a \in T_2$. □

Lemma 6. If $T$ is a triangle of $M$, then there are triads $T_1^*$ and $T_2^*$ of $M$ such that $T_1^*, T, T_2^*$ is a fan of $M$ and $a \in T_1^* \cap T \cap T_2^*$.

Proof. Observe that $M$ is not isomorphic to a wheel because the result holds for graphic matroids. Thus every chain of $M$ is contained in a fan of $M$, by Theorem
Lemma 7. If \( C^* \) is a cocircuit of \( M \) such that \( a \notin C^* \), then there is \( e \in C^* \) such that \( M/e \) is 3-connected.

Proof. Suppose that \( M/e \) is not 3-connected, for every \( e \in C^* \). By the dual of Theorem 1 of Lemos [9], there are distinct triangles \( T_1 \) and \( T_2 \) of \( M \) meeting \( C^* \). By Lemma 6, \( a \in T_1 \cap T_2 \). Moreover, there are triads \( T_1^* \) and \( T_2^* \) of \( M \) such that \( a \in T_1^* \cap T_2^* \) and \( T_1^*, T_1, T_2^* \) is a fan of \( M \); a contradiction because \( T_2 \) cannot intersect \( T_1^* \) or \( T_2^* \). □

By Corollary 3.5 of Oxley and Wu [15], \( M \) must have at least two elements which are non-essential, since it is not isomorphic to a wheel. Let \( e \) be a non-essential element of \( M \) other than \( a \). By Lemma 2, \( M/e \) is 3-connected. By Lemma 4, \( \dim_{GF(2)} \mathcal{R}_a^*(M) = r(M) - 2 \geq 1 \). So \( \mathcal{R}_a^*(M) \neq \emptyset \). Let \( C^* \) be a cocircuit of \( M \) such that \( M \setminus C^* \) is connected and \( a \notin C^* \). By Lemma 4, \( M/f \) is not 3-connected, for every \( f \in C^* \) because \( C^* \in \mathcal{R}_a^*(M) - \mathcal{R}_{[a,f]}^*(M) \); a contradiction to Lemma 7.

So Theorem 4 follows.

From Theorem 4, we have the next result which generalizes the bound obtained by McNulty and Wu [12] for the number of non-separating cocircuits avoiding an element of a simple and cosimple connected binary matroid. We omit its proof because it is similar to the proof of Theorem 6.

**Corollary 2.** Suppose that \( a \) is an element of a simple and cosimple connected binary matroid \( M \). If \( M \) is not 3-connected, then

\[
\mathcal{R}_a^*(M_1) \cup \mathcal{R}_a^*(M_2) \cup \cdots \cup \mathcal{R}_a^*(M_t) \subseteq \mathcal{R}_a^*(M),
\]

where \( M_1, M_2, \ldots, M_t \) label the terminal vertices of \( T(M) \) avoiding the element \( a \) and, for \( i \in \{1, 2, \ldots, t\} \), \( a_i \) labels the edge of \( T(M) \) incident to \( M_i \). Moreover,

\[
|\mathcal{R}_a^*(M)| \geq \sum_{i=1}^{t} [r(M_i) - 1].
\]

This corollary is the best possible as the next examples show. Let \( m, B_m, H, H' \) and \( f \) be as in the two paragraphs after the proof of Theorem 4. When \( a \) is chosen in \( E(B_m) - \{e, f\} \), the matroid \( H \) attains equality in (1). When \( a \) is chosen to be equal to \( f \), the equality in (1) holds for \( H' \).

**References**