Abstract

The purpose of this paper is to investigate the effects of strain-stiffening for the classical problems of axial and azimuthal shearing of a hollow circular cylinder composed of an incompressible isotropic non-linearly elastic material. For some specific strain-energy densities that give rise to strain-stiffening in the stress–stretch response, the stresses and resultant axial forces are obtained in explicit closed form. While such results are well known for classical constitutive models such as the Mooney–Rivlin and neo-Hookean models, our main focus is on materials that undergo severe strain-stiffening in the stress–stretch response. In particular, we consider in detail two phenomenological constitutive models that reflect limiting chain extensibility at the molecular level and involve constraints on the deformation. The amount of shearing that tubes composed of such materials can sustain is limited by the constraint. Numerical results are also obtained for an exponential strain-energy that exhibits a less abrupt strain-stiffening effect. Potential applications of the results to the biomechanics of soft tissues are indicated.

1. Introduction

The purpose of this paper is to investigate the effects of strain-stiffening for the classical problems of axial and azimuthal shearing of a hollow circular cylinder composed of an incompressible isotropic non-linearly elastic material. This problem has been widely investigated within the theory of finite hyperelasticity largely motivated by applications to rubber. Such shearing problems are also of considerable interest in the context of biomechanics of soft tissues. Our particular focus here is on investigation of the stress response for special classes of constitutive models that give rise to severe strain-stiffening in their stress–stretch curves at large strains. The constitutive models that we employ reflect limiting chain extensibility at the molecular level and thus are appropriate for modeling non-crystallizing elastomers and soft biological tissues.

In the next section, we discuss some preliminaries from the theory of non-linear hyperelasticity for isotropic incompressible solids. Two classes of phenomenological constitutive models that exhibit strain-stiffening at large strains are described. The first class reflects limiting chain extensibility at the molecular level and gives rise to severe strain-stiffening in the stress–stretch response. The second class exhibits a less abrupt strain-stiffening, for example, the exponential models widely used in biomechanics. In Section 3, we summarize results for the problems of axial and azimuthal shearing of a tube. These problems are classical problems solved by Rivlin (1949) for general incompressible isotropic elastic solids. In Section 4, we provide explicit expressions for the stresses and resultant axial force for the classical Mooney–Rivlin and neo-Hookean models. Our main focus is on results for strain-stiffening models and these are described in Section 5. Explicit analytic results are given for two limiting chain extensibility models that exhibit severe strain-stiffening. Such results are not available for...
the exponential model and so a numerical scheme is required in this case. These results are compared with one another and with those for the classical models in Section 6. Aside from its obvious application to shearing of rubber (see, e.g., Suh et al., 2007 for a discussion of rubber tube springs), one of the motivations for the present work arises from the demonstrated potential of the application of limiting chain extensibility models to the biomechanics of soft tissues (see, e.g., Horgan and Saccomandi, 2003b; Holzapfel, 2005). We refer to Humphrey (2002, 2003) and Taber (2004) for extensive discussions of the applications of non-linear elasticity theory to many aspects of cardiovascular mechanics. In particular, as pointed out by Humphrey (2002, p. 635), long cylindrical specimens are readily excised from papillary muscles and have been important sources for biomechanical data. While shearing problems in finite elasticity have received relatively little attention in the biomechanics literature, it is noted in Taber (2004, p. 257) that transverse shear of the heart wall may play a role in proper functioning of the left ventricle.

2. Preliminaries

The mechanical properties of elastomeric materials are described in terms of a strain-energy density function $W$ per unit undeformed volume. On denoting the left Cauchy–Green tensor by $B = FF^T$, where $F$ is the gradient of the deformation and $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, then, for an isotropic material, $W$ is a function of the stretch invariants

$$I_1 = \text{tr} B = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \frac{1}{2}[(\text{tr} B^2) - \text{tr}(B^2)] = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \quad I_3 = \text{det} B = \lambda_1^3 \lambda_2^3 \lambda_3^3.$$ (1)

For incompressible materials, the admissible deformations must be isochoric, i.e., $\text{det} F = 1$ so that $I_3 = 1$. The response of an incompressible isotropic elastic material is given by (see, e.g., Ogden, 1984; Beatty, 1987; Holzapfel, 2000)

$$T = -p I + 2 \frac{\partial W}{\partial I_1} B - 2 \frac{\partial W}{\partial I_2} B^{-1},$$ (2)

where $p$ is a hydrostatic pressure term associated with the incompressibility constraint and $T$ denotes the Cauchy stress.

The classical strain-energy density for incompressible rubber is the Mooney–Rivlin strain-energy

$$W^{\text{MR}} = \frac{1}{2} \mu [(I_1 - 3) + (1 - \alpha)(I_2 - 3)],$$ (3)

where $\mu > 0$ is the constant shear modulus and $0 < \alpha \leq 1$ is a dimensionless constant. When $\alpha = 1$ in (3), one obtains the neo-Hookean strain-energy

$$W^{\text{NH}} = \frac{\mu}{2}(I_1 - 3).$$ (4)

It is well known that the theoretical predictions based on (3) do not adequately describe experimental data for rubber at high values of strain. For example, the strain-energy (3) is not able to describe the significant rapid rise in the load versus stretch curve exhibited in simple tension experiments. Such severe strain-stiffening occurs even at moderate stretches for soft biological materials (see, e.g., Humphrey, 2002, 2003; Holzapfel, 2005), and so in this case classical models are completely inappropriate. To model such stiffening, a number of alternative models have been proposed, for example, models with limiting chain extensibility and strain-stiffening models of power-law or exponential type. Our emphasis is primarily on the former.

Some phenomenological models that have been shown to be particularly useful in modeling severe strain-stiffening phenomena are those reflecting a maximum achievable length of the polymeric molecular chains composing the material (see Horgan and Saccomandi, 2006 for a review). More recent papers are those of Beatty (2007, 2008) who uses the term “limited elastic” for such materials. For isotropic incompressible materials, these can be described by strain-energies of the form

$$W(I_1, I_2, \Gamma)$$

where $\Gamma$ is a limiting chain extensibility parameter. The function $W$ is such that the stress components are unbounded as $f(I_1, I_2, \Gamma) \to 0$ for some specified function $f$ and so one must impose the constraint

$$f(I_1, I_2, \Gamma) < 0$$ (5)

on admissible deformations.

One of the simplest of these models was first proposed by Gent (1996), namely

$$W^C = -\frac{\mu}{2} f_m \ln \left(1 - \frac{I_1 - 3}{f_m}\right), \quad I_1 < f_m + 3,$$ (6)

(henceforth called the basic Gent model), and one recovers the neo-Hookean model on taking the limit as $f_m \to \infty$ in (6). We note that $W^C$ depends only on the first invariant $I_1$ and so is a generalized neo-Hookean material. It is well known that for such models many problems in non-linear elasticity simplify considerably (see, e.g., Horgan and Saccomandi, 2006; Wineman, 2005). For rubber, typical values for the dimensionless parameter $f_m$ for simple extension range from 30 to 100, whereas for biological tissue, much smaller values of $f_m$ are appropriate. For example, for human arterial wall tissue, values on the order of 0.4–2.3 have been suggested by Horgan and Saccomandi (2003b). On using (2), we find that the Cauchy stress associated with (6) is given by

$$\sigma_{ij} = \frac{\partial W^C}{\partial B_{ij}}$$
\[
T = -p\mathbf{I} + \mu \frac{J_m}{(J_1 - 3)} \mathbf{B}
\]
(7)
so that the stress has a singularity as \( J_1 \to J_m + 3 \), reflecting the rapid strain-stiffening observed in experiments. The basic Gent model (6) gives theoretical predictions similar to the more complicated eight-chain molecular model proposed by Arruda and Boyce (1993); see also Boyce (1996) and Boyce and Arruda (2000). A molecular basis for the basic Gent model was given by Horgan and Saccomandi (2002b). Thus the basic Gent model predicts similar behavior to the molecular models, has a clear microscopic interpretation for the constitutive coefficients, and is tractable analytically. See Horgan and Saccomandi (2005, 2006) for a summary of results on explicit solution to a wide variety of boundary-value problems. Details may be found in Horgan and Saccomandi (1999a,b, 2001a,b, 2003a), and Horgan et al. (2002). The Gent model was used by Horgan et al. (2004) to describe the Mullins effect in rubber. A relation between the Gent model and the worm-like chain model for long chain polymers was discussed in Ogden et al. (2006, 2007). Applications to the mechanics of arterial walls are described in Horgan and Saccomandi (2003b) and in Ogden and Saccomandi (2007). On using analysis involving homogeneous deformations, new results on the fracture of rubber were obtained by Horgan and Schwartz (2005) and an investigation of instabilities arising in cylindrical and spherical inflation of pressurized thin shells was carried out by Gent (1999, 2005) and by Kanner and Horgan (2007). See also Goriely et al. (2006) where instabilities for spherical inflation and for half-spaces under lateral compression are considered. The non-homogeneous deformations of plane strain bending of rectangular beams and torsion superimposed on extension of solid circular cylinders were investigated by Kanner and Horgan (2008a,b). Surface waves on an elastic half-space for non-homogeneous deformations of plane strain bending of rectangular beams and torsion superimposed on extension of a circular bar was investigated by Kanner and Horgan (2008b).

An alternative two-parameter limiting chain extensibility model with \( W(I_1, I_2, J) \) was proposed by Horgan and Saccomandi (2004) and by Horgan and Murphy (2007) where

\[
W^{HS} = -\frac{\mu}{2} \frac{(J - 1)^2}{J} \ln \left( \frac{J^2 - J^3 I_1 + J^2 I_2 - 1}{(J - 1)^3} \right), \quad J_1 - I_2 < \frac{J^2 - 1}{J}, \quad J > 1,
\]
(8)
or, on using the principal stretches of the deformation

\[
W^{HS} = -\frac{\mu}{2} \frac{(J - 1)^2}{J} \ln \left( \frac{1 - \frac{I_1}{J}}{1 - \frac{I_2}{J}} \right)^3, \quad \lambda_1 \lambda_2 \lambda_3 = 1.
\]
(9)

In (8) and (9), \( \mu \) is the shear modulus for infinitesimal deformations. On formally replacing \( J \) by \( J + 1 \), one recovers the form of \( W \) used in Horgan and Murphy (2007). Note that the definitions of \( W^{HS} \) here differ from those in Horgan and Saccomandi (2004, 2005, 2006) and in Horgan and Schwartz (2005) by a factor of \( (J - 1)^2 J^2 \). The limiting chain extensibility parameter \( J \) is the square of the maximum stretch allowed by the finite extensibility of the chains so that

\[
\max(\lambda_1^2, \lambda_2^2, \lambda_3^2) < J.
\]
(10)

Again, in the limit as \( J \to \infty \) in (8) or (9), we recover the neo-Hookean model (4). There is an important difference between the constraint (10) and the constraint \( I_1 < J_m + 3 \) arising in connection with the Gent model. As already pointed out in Horgan and Saccomandi (2003a, 2006), the limiting chain condition expressed in terms of the principal invariant is less physically accessible than (10). This alternative approach to constitutive model development reflecting limiting chain extensibility has been discussed by Horgan and Saccomandi (2002a), Murphy (2006), and Horgan and Murphy (2007). Furthermore, the absence of the dependence on the second invariant in the basic Gent model entails some physical limitations (see, e.g., Pucci and Saccomandi, 2002 and Ogden et al., 2004 for generalizations of the Gent model that include such dependence). Thus, the HS model has advantages over the basic Gent model. The response of the HS model in homogeneous deformations such as simple extension, simple shear and equibiaxial extension was examined in Horgan and Schwartz (2005), Horgan and Murphy (2007), and in Kanner and Horgan (2007). The non-homogeneous deformation of plane strain bending of a rectangular bar was discussed by Kanner and Horgan (2008a) and the problem of torsion superimposed on extension of a circular bar was investigated by Kanner and Horgan (2008b).

While our primary concern here is with limiting chain extensibility models such as the above that exhibit severe strain-stiffening, we note that there are numerous strain-hardening constitutive models that have been successfully employed to investigate the effects of a less abrupt strain-stiffening. A generalized neo-Hookean model of this type widely used in the biomechanics literature is the two-parameter exponential strain-energy density proposed by Demiray (1972) based on the basic idea of Fung (1967) namely

\[
W^f = \frac{\mu}{2B} \left( \exp[b(J_1 - 3)] - 1 \right),
\]
(11)
where the dimensionless constant \( b > 0 \). On taking the limit as \( b \to 0 \) in (11) we recover the neo-Hookean model (4).
3. Shearing of a hollow circular cylinder

The problems of axial and azimuthal shearing of an incompressible long hollow circular cylinder were considered by Rivlin (1949) for a general incompressible isotropic hyperelastic material. The solutions obtained by Rivlin are controllable, i.e., valid for all incompressible isotropic hyperelastic solids. Experimental results on these problems are described in Rivlin and Saunders (1951). For convenience of the reader, in this section we briefly present a summary of the general expressions for the stresses and resultant axial force arising in these problems. Some of our results are written in a somewhat different form than those obtained by Rivlin (1949).

3.1. Axial shear

We shall first consider the axial shear of a long circular cylindrical tube. Since the early work of Rivlin (1949) the axial shear problem and its generalizations for incompressible materials has received some attention (see e.g., Ogden, 1984; Horgan and Saccomandi, 1999b; Horgan et al., 2002; Wineman, 2005, and references cited therein). We assume the tube to be composed of an incompressible isotropic hyperelastic material with a strain-energy such as those presented in Section 2. The tube is subjected to a uniformly distributed axial shear traction applied to the outer surface, while the normal traction there is zero. The inner surface of the tube is bonded to a rigid cylindrical core. The deformation is thus that of pure axial shear described by

\[ r = R, \quad \theta = \Theta, \quad z = Z + w(R), \]  

where points in the undeformed configuration have cylindrical coordinates \((R, \Theta, Z)\), with \(A \leq R \leq B\), and \((r, \theta, z)\) denote points in the deformed configuration. Thus \(w(R)\) is the axial displacement. The deformation gradient \(F\), the left Cauchy–Green tensor \(B\), and its inverse are then given by

\[ F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ w' & 0 & 1 + w'^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & w' & 0 \\ 0 & 1 & 0 \\ w' & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 + w'^2 & 0 & -w' \\ 0 & 1 & 0 \\ -w' & 0 & 1 \end{pmatrix} \]  

with principal invariants \(I_1 = I_2 = 3 + w'^2\), \(I_3 = 1\), where \(w' = \frac{dw}{dR}\). On using the notation

\[ \tilde{W}_1 = \frac{\partial W}{\partial W} |_{I_1-I_2-3+w'^2}, \quad \tilde{W}_2 = \frac{\partial W}{\partial I_2} |_{I_1-I_2-3+w'^2}, \]  

the non-zero stresses are given as

\[ T_{rr} = -p + 2\tilde{W}_1 - 2\tilde{W}_2(1 + w'^2), \quad T_{\Theta \Theta} = -p + 2\tilde{W}_1 - 2\tilde{W}_2, \quad T_{zz} = -p + 2\tilde{W}_1(1 + w'^2) - 2\tilde{W}_2, \quad T_{rz} = 2(\tilde{W}_1 + \tilde{W}_2)w'. \]  

The equilibrium equations in absence of body forces \(\text{div}\mathbf{T} = 0\) reduce to

\[ \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\Theta \Theta}) = 0, \quad \frac{\partial T_{zz}}{\partial r} + \frac{T_{rz}}{r} = 0. \]  

On integration of (16)2 and use of the boundary condition

\[ T_{rz}(B) = T_s, \]  

where \(T_s > 0\) is the prescribed constant axial shear on the outer surface, one obtains

\[ T_{rz} = \frac{BT_s}{R}. \]  

Thus, for a given \(T_s\), the only non-zero shearing stress is easily determined. On substituting the last of (15) into (18) we obtain a first-order differential equation for \(w(R)\), namely

\[ 2(\tilde{W}_1 + \tilde{W}_2)w' = \frac{BT_s}{R}, \]  

which is subject to the boundary condition

\[ w(A) = 0. \]  

We may now determine \(T_r\) by substituting (15)1,2 into (16)1 to yield the differential equation

\[ \frac{dT_{rr}}{dr} + \frac{1}{r}(-2\tilde{W}_2w'^2) = 0, \]  

which is subject to the boundary condition

\[ T_{rr}(B) = 0. \]
Use of (21) and (22) yields the expression
\[ T_{rr} = -2 \int_{R}^{B} \frac{\hat{W}_2(s)[w'(s)]^2}{s} \, ds \] (23)
for \( T_{rr} \). Note that \( \hat{W}_1 \) and \( \hat{W}_2 \) are functions of \( w' \), which is determined from (19) and (20) for a given \( W \). From (23) and (15), we find that
\[ p(R) = 2 \int_{R}^{B} \frac{\hat{W}_2(s)[w'(s)]^2}{s} \, ds + 2\hat{W}_1 - 2\hat{W}_2(1 + w^2). \] (24)

Use of (15) yields expressions for the remaining non-zero stresses in terms of \( T_{rr} \), namely
\[ T_{\theta \theta} = T_{rr} + 2\hat{W}_2 w^2, \] (25)
\[ T_{zz} = T_{rr} + \frac{BT_4}{R} w'. \] (26)

The resultant axial force \( N \) at any fixed cross-section is given by
\[ N = \int_{0}^{2\pi} \int_{A}^{B} T_{zz} R \, dr \, d\theta. \] (27)
On using (26) and (23) and integrating by parts we obtain
\[ N = 2\pi \int_{A}^{B} \left( \frac{A^2}{R} - R \right) \hat{W}_2 w^2 \, dR + 2\pi BT_4 w(B). \] (28)

It is of interest to note that for strain-energy densities of the form \( W = W(l_1) \), i.e., generalized neo-Hookean models, the preceding expressions for the normal stresses and the resultant axial force may be greatly simplified and reduce to
\[ T_{rr} = T_{\theta \theta} = 0, \quad T_{zz} = \frac{BT_4}{R} w', \quad N = 2\pi BT_4 w(B), \] (29)
where \( w'(R) \) and \( w(R) \) are determined from
\[ \hat{W}_1 w' = \frac{BT_4}{2R} \] (30)
subject to \( w(A) = 0 \). These results were also obtained in Horgan and Saccomandi (1999b).

3.2. Azimuthal shear

We shall next consider the azimuthal (or circular) shear of a circular cylindrical tube composed of an incompressible isotropic hyperelastic material. Rivlin (1949) was the first to investigate this problem for general incompressible materials. Since then the problem and extensions thereof have been examined by many authors (see e.g., Ogden, 1984; Simmonds and Warne, 1992; Horgan and Saccomandi, 2001a; Tao et al., 1992). For the incompressible tube, with inner surface bonded to a rigid cylinder and a uniformly distributed azimuthal shear traction applied to the outer surface, the deformation is that of pure azimuthal shear described by
\[ r = R, \quad \theta = \Theta + g(R), \quad z = Z, \] (31)
where the material and spatial cylindrical polar coordinates are denoted by \( (R, \Theta, Z) \) and \( (r, \theta, z) \), respectively, with \( A \leq R \leq B \). Thus \( g(R) \) is the angular displacement. Here we have
\[ F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & R g' & 0 \\ R g' & 1 + R^2 g'^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 + R^2 g'^2 & -R g' & 0 \\ -R g' & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (32)
with principal invariants \( I_1 = I_2 = 3 + R^2 g'^2, \quad I_3 = 1 \), where \( g' = dg/dR \). Thus the non-zero stresses are
\[ T_{rr} = -p + 2\hat{W}_1 - 2\hat{W}_2(1 + R^2 g'^2), \quad T_{\theta \theta} = -p + 2\hat{W}_1(1 + R^2 g'^2) - 2\hat{W}_2, \quad T_{zz} = -p + 2\hat{W}_1 - 2\hat{W}_2, \quad T_{\theta \theta} = \frac{B T_4}{R} g', \] (33)
where we use the notation
\[ \hat{W}_1 = \frac{\partial W}{\partial I_1} |_{l_1, l_2, \gamma^2} , \quad \hat{W}_2 = \frac{\partial W}{\partial I_2} |_{l_1, l_2, \gamma^2} . \] (34)

On using a procedure similar to that outlined for axial shear, one can show that the only non-zero shearing stress is
\[ T_{r\theta} = \frac{B^2}{R^2} T_0 , \] (35)
where \( T_0 > 0 \) is the prescribed azimuthal shear stress on the outer boundary. Thus, the only non-zero shearing stress is easily determined for a given \( T_0 \). The governing equation for \( g(R) \) is then
\[ 2(\hat{W}_1 + \hat{W}_2) R g' = \frac{B^2 T_0}{R^2} . \] (36)

For a given \( W \), the angular displacement \( g(R) \) is to be obtained from (36), subject to the boundary condition
\[ g(A) = 0 . \] (37)

On using the equilibrium equations and (36) one finds that
\[ T_{rr}(R) = -B^2 T_0 \int_A^B \frac{g'(s)}{s^2} ds , \] (38)
\[ T_{r\phi} = T_{rr} + 2(\hat{W}_1 + \hat{W}_2) R^2 g'^2 = T_{rr} + B^2 T_0 \frac{g'}{R} , \] (39)
\[ T_{zz} = T_{rr} + 2 \hat{W}_2 R^2 g'^2 . \] (40)

The resultant axial force \( N \) necessary to maintain the deformation is given by
\[ N = \int_0^{2\pi} \int_A^B T_{zz} r dr d\theta . \] (41)

On using (40), (38), and integrating by parts we obtain
\[ N = \pi \int_A^B \left( \frac{B^2 A^2 T_0 g'}{R^2} + 4 \hat{W}_2 R^2 g'^2 \right) dR - \pi B^2 T_0 g(B) . \] (42)

For strain-energy densities of the form \( W = W(I_1) \), the preceding expressions for the normal stresses and the resultant axial force simplify considerably and reduce to
\[ T_{r\phi} = T_{rr} + 2 \hat{W}_1 R^2 g'^2 = T_{rr} + B^2 T_0 \frac{g'}{R} , \quad T_{zz} = T_{rr} , \quad N = \pi \int_A^B \frac{B^2 A^2 T_0 g'}{R^2} dR - \pi B^2 T_0 g(B) , \] (43)
where \( T_{rr} \) is given by (38), and \( g(R) \) and \( g'(R) \) are determined from
\[ \hat{W}_1 g' = \frac{B^2 T_0}{2R^2} \] (44)
subject to
\[ g(A) = 0 . \] (45)

4. Stress responses for classical constitutive models

4.1. Axial shear

It is instructive first to examine here the special case of a Mooney–Rivlin material with strain-energy given by (3) in which case (19) reads
\[ w' = \frac{BT_a}{\mu R} . \] (46)

Note that the parameter \( \alpha \) that appears in the definition (3) does not occur in (46) and thus (46) is also valid for the neo-Hookean model where \( \alpha = 1 \). On integration of (46) and use of the boundary condition \( w(A) = 0, w(A) = 0 \), we obtain the well-known result (see Rivlin, 1949)
\[ w(R) = \frac{BT_a}{\mu} \ln \left( \frac{R}{A} \right) . \] (47)
Thus we find the non-zero stresses to be

\[
T_{rr}^{\text{MR}} = \frac{T_0^2}{2\mu} (1 - \alpha) \left[ 1 - \left( \frac{B}{R} \right)^4 \right], \quad T_{\theta\theta}^{\text{MR}} = \frac{T_0^2}{2\mu} (1 - \alpha) \left[ 1 + \left( \frac{B}{R} \right)^2 \right],
\]

\[
T_{zz}^{\text{MR}} = \frac{T_0^2}{2\mu} \left[ (1 - \alpha) + (1 + \alpha) \left( \frac{B}{R} \right)^2 \right], \quad T_{rz}^{\text{MR}} = \frac{BT_0}{R},
\]

and the resultant axial force is found to be

\[
N^{\text{MR}} = 2\pi \frac{T_0^2}{\mu} \left[ \frac{1}{4} (1 - \alpha)(B^2 - A^2) + \frac{1}{2} (1 + \alpha)B^2 \ln \left( \frac{B}{A} \right) \right].
\]

The normal stresses depend on the prescribed axial shear stress \( T_s \) in a quadratic manner and illustrate the normal stress effect characteristic of non-linear elasticity theory. The shear stress depends linearly on \( T_s \). The hoop and axial stresses are tensile while the radial stress is compressive. The resultant axial force (50) is also tensile. When we set \( \alpha = 1 \) in the preceding, we recover the results for the neo-Hookean model so that

\[
T_{rr}^{\text{NL}} = T_{\theta\theta}^{\text{NL}} = 0, \quad T_{zz}^{\text{NL}} = \frac{T_0^2}{R^2}, \quad T_{rz}^{\text{NL}} = \frac{BT_0}{R}, \quad N^{\text{NL}} = 2\pi \frac{T_0^2}{\mu} \ln \left( \frac{B}{A} \right).
\]

### 4.2. Azimuthal shear

For the Mooney–Rivlin model, Eq. (36) reads

\[
g^\prime = \frac{B^2T_0}{R^2\mu},
\]

which is again independent of \( \alpha \). On integration of (52) and use of the boundary condition \( g(A) = 0 \), we obtain the well-known result (see, e.g., Rivlin, 1949; Ogden, 1984)

\[
g(R) = \frac{B^2T_0}{\mu} \int_A^R s^{-3} \, ds = \frac{T_0}{2\mu} \left[ \left( \frac{B}{A} \right)^2 - \left( \frac{B}{R} \right)^2 \right].
\]

Thus, we find the non-zero stresses as

\[
T_{rr}^{\text{MR}} = \frac{T_0^2}{4\mu} \left[ 1 - \left( \frac{B}{R} \right)^4 \right], \quad T_{\theta\theta}^{\text{MR}} = \frac{T_0^2}{4\mu} \left[ 1 + 3 \left( \frac{B}{R} \right)^4 \right],
\]

\[
T_{zz}^{\text{MR}} = T_{rr} + \frac{T_0^2}{\mu} (1 - \alpha) \left( \frac{B}{R} \right)^4, \quad T_{rz}^{\text{MR}} = \frac{BT_0}{R^2},
\]

and the resultant axial force as

\[
N^{\text{MR}} = \pi \frac{T_0^2}{\mu} \left\{ \frac{-1}{4} \left( \frac{B^2 - A^2}{A^2} \right)^2 + (1 - \alpha) \left( \frac{B}{A} \right)^2 (B^2 - A^2) \right\}.
\]

The hoop stress is tensile while the radial stress is compressive. The axial stress, however, may be compressive or tensile depending on \( \alpha \) and \( R \). This, in turn, implies that the resultant axial force (56) necessary to maintain the deformation may be compressive or tensile depending on \( \alpha \) and the aspect ratio of the tube. This important feature, which does not appear to have been noticed previously in the literature, will be discussed in more detail in Section 6. Note that the parameter \( \alpha \) that appears in the definition of the Mooney–Rivlin model only occurs in (55) and (56) so that except for these two equations, all the preceding results also hold for the neo-Hookean model. When we set \( \alpha = 1 \), we obtain results for the neo-Hookean model as

\[
T_{zz}^{\text{NL}} = T_{rr}^{\text{NL}} = \frac{T_0^2}{4\mu} \left[ 1 - \left( \frac{B}{R} \right)^4 \right], \quad T_{\theta\theta}^{\text{NL}} = \frac{T_0^2}{4\mu} \left[ 1 + 3 \left( \frac{B}{R} \right)^4 \right],
\]

\[
T_{rz}^{\text{NL}} = \frac{BT_0}{R^2}, \quad N^{\text{NL}} = -\pi \frac{T_0^2}{4\mu A^2} (B^2 - A^2)^2.
\]

Here, the axial stress and resultant axial force are always compressive.
5. Stress responses for strain-stiffening constitutive models

5.1. Axial shear

For the model defined in (8), we find that
\[
\tilde{W}_1 = \frac{\mu}{2} \left[ \frac{(J - 1)J}{J^2 - (2 + w^2)J + 1} \right], \quad \tilde{W}_2 = -\frac{1}{J} \tilde{W}_1, \tag{59}
\]
and so (19) reads
\[
2w' \left( 1 - \frac{J}{J} \right) \left[ \frac{(J - 1)J}{J^2 - (2 + w^2)J + 1} \right] = \frac{2BT_s}{\mu R}. \tag{60}
\]
Eq. (60) can be written as a quadratic equation in \( w' \) with positive solution
\[
w'(R) = \frac{J}{2} [f(R) - R], \tag{61}
\]
where we have introduced the notation
\[
\bar{f} \equiv \frac{(J - 1)^2}{J}, \quad \xi \equiv \frac{\mu}{BT_s}, \quad f(R) \equiv \sqrt{R^2 + \frac{4}{\xi^2} f}. \tag{62}
\]
Integration of (61) and use of the boundary condition (20) yields
\[
w(R) = \frac{J}{4} [Rf'(R) - Af(A) + A^2 - R^2] + \frac{1}{\xi} \ln \left[ \frac{f(R) + R}{f(A) + A} \right]. \tag{63}
\]
We may now calculate the non-zero stresses to be
\[
T_{rr}^{\text{HS}} = -\frac{\mu (J - 1)}{2J} \left\{ -\frac{f(R)}{R} + \frac{f(B)}{B} + \ln \left[ \frac{f(R) + R}{f(B) + B} \right] \right\}, \tag{64}
\]
\[
T_{\theta \theta}^{\text{HS}} = T_{rr}^{\text{HS}} - \frac{\mu (J - 1)}{2J} \left( \frac{f(R)}{R} - 1 \right) = -\frac{\mu (J - 1)}{2J} \left\{ -\frac{f(R)}{R} + \frac{f(B)}{B} + 1 + \ln \left[ \frac{f(R) + R}{f(B) + B} \right] \right\}, \tag{65}
\]
\[
T_{zz}^{\text{HS}} = T_{rr}^{\text{HS}} + \frac{\mu (J - 1)}{2J} \left( \frac{f(R)}{R} - 1 \right) = -\frac{\mu (J - 1)}{2J} \left\{ -\frac{f(R)}{R} + \frac{f(B)}{B} + J - 1 + \ln \left[ \frac{f(R) + R}{f(B) + B} \right] \right\}, \tag{66}
\]
\[
T_{rz}^{\text{HS}} = \frac{BT_s}{R}. \tag{67}
\]
and the resultant axial force is
\[
N^{\text{HS}} = \frac{\mu E}{2} \left\{ (J - 1) \left[ \frac{(2J - 1)}{2J} f'(B) - \frac{(2J - 1)}{2J} Af(A) + \frac{A}{2} f(B) + \frac{(2J - 1)}{2J} (A^2 - B^2) \right] \right\} + \frac{A}{2J} \ln \left( \frac{f(0) + A f(A)}{f(0) + A f(B)} \right) + \frac{A}{2J} \ln \left( \frac{f(0) + B f(A)}{f(0) + B f(B)} \right). \tag{68}
\]
It can be shown that the radial and axial stresses are tensile (and so \( N^{\text{HS}} \) is tensile) while the hoop stress is compressive.

For comparison purposes, we now list the corresponding results for the Gent model obtained in Horgan and Saccomandi (1999b) as
\[
w(R) = \frac{J}{4} [R f'(R) - Af(A) + A^2 - R^2] + \frac{1}{\xi} \ln \left[ \frac{f(R) + R}{f(A) + A} \right], \tag{69}
\]
\[
T_{rr}^{\text{HS}} = 0, \quad T_{\theta \theta}^{\text{HS}} = \frac{\mu J f_m}{2} \left[ \frac{f_m(R)}{R} - 1 \right], \quad T_{zz}^{\text{HS}} = \frac{BT_s}{R}, \tag{70}
\]
\[
N^{\text{HS}} = \frac{\mu E}{2} \left\{ f_m [B f_m(B) - Af_m(A) + A^2 - B^2] + \frac{4}{\xi^2} \ln \left( \frac{f_m(0) + B}{f_m(0) + A} \right) \right\}, \tag{71}
\]
where \( f_m(R) \) is the same as \( f(R) \), defined in (62), upon formally replacing \( \bar{f} \) with \( f_m \). On comparing (69) with (63), we see that the displacements for both models are identical on making this formal replacement. The radial and hoop stresses are now zero, while the axial stress and its resultant are tensile.

For a material with strain-energy density given by the Fung exponential model (11), the differential equation (19) for \( w(R) \) is
\[
w'(R) = \frac{\mu \sigma_f}{R}, \tag{62}
\]
and so (19) reads
\[
2w' \left( 1 - \frac{J}{J} \right) \left[ \frac{(J - 1)J}{J^2 - (2 + w^2)J + 1} \right] = \frac{2BT_s}{\mu R}. \tag{63}
\]
subject to \( w(A) = 0 \). This problem does not appear amenable to analytic solution and so a numerical scheme was employed to generate results that will be described in Section 6.

5.2. Azimuthal shear

For the model defined in (8), we find that

\[
\tilde{W}_1 = \frac{\mu}{2} \left[ \frac{(J - 1)J}{J^2 - (2 + R^2 g^2)J + 1} \right], \quad \tilde{W}_2 = -\frac{1}{J} \tilde{W}_1, \tag{73}
\]

and so (36) reads

\[
\mu \left[ \frac{(J - 1)^2}{J^2 - (2 + R^2 g^2)J + 1} \right] R^3 g' = B^2 T_0. \tag{74}
\]

Eq. (74) can be written as a quadratic equation in \( Rg' \) with positive solution

\[
Rg' = \frac{\delta J}{2} [h(R) - R^2], \tag{75}
\]

where

\[
\delta J = \frac{(J - 1)^2}{J}, \quad \delta = \frac{\mu}{B^2 T_0} > 0, \quad h(R) \equiv \sqrt{R^4 + \frac{4}{\delta^2 J}}. \tag{76}
\]

Notice that this definition of \( \delta J \) is identical to (62), which was used for the axial shear problem. On integration of (75) and using the boundary condition \( g(A) = 0 \), we obtain

\[
g(R) = \frac{\delta J}{4} [h(R) - h(A) + A^2 - R^2] - \frac{\delta J}{2} \ln \left( \frac{A^2}{R} \right) + \frac{2}{\delta J^{1/2}} \ln \left( \frac{A^2 + 2}{h(A) + \frac{2}{\delta J^{1/2}}} \right). \tag{77}
\]

On taking the limit as \( J \to \infty \), i.e., as \( \delta J \to \infty \), in (77), we recover the result (53) for the neo-Hookean material. On using (75) in (38) we find that

\[
T_n(R) = -\frac{B^2 T_0}{2} \int_R^b \left[ -\frac{\delta J}{s} + \frac{2\delta J^2 s^4}{s^4} \right] ds \tag{78}
\]

so that

\[
T_n^{HS}(R) = \frac{\mu J}{4} \left\{ -\frac{h(R)}{R^2} + \frac{h(B)}{B^2} + \ln \left( \frac{B^2 h(R) + B^2 R^2}{R^2 h(B) + h (R)^2} \right) \right\}. \tag{79}
\]

The remaining non-zero stresses are

\[
T_{\theta\theta}^{HS} = \frac{T_n^{HS} + \frac{B^2 T_0}{R^2} g'}{R^2} = \frac{\mu J}{4} \left\{ \frac{h(R)}{R^2} + \frac{h(B)}{B^2} \right\} - 2 + \ln \left( \frac{B^2 h(R) + B^2 R^2}{R^2 h(B) + h (R)^2} \right), \tag{80}
\]

\[
T_{zz}^{HS} = -\frac{\mu (J - 1)}{2J} \left[ \frac{h(R)}{R^2} - 1 \right],
\]

and the resultant axial force is found to be

\[
N^{HS} = \frac{\mu \pi (J - 1)}{4} \left\{ \frac{-J}{J} h(B) + 2h(R) - \frac{J - 1}{J} A^2 B h(B) + \frac{J + 1}{J} (B^2 - A^2) + \frac{J - 1}{J} A^2 \ln \left( \frac{B^2 h(B) + A^2 B^2}{B^2 h(A) + A^2 B^2} \right) \right\}
\]

\[
+ \frac{\pi (J + 1) T_0 B^2}{2 \sqrt{J}} \ln \left( \frac{h(B) + \frac{2}{\delta J^{1/2}}}{h(A) + \frac{2}{\delta J^{1/2}}} \right). \tag{81}
\]

It can be shown that the hoop stress is tensile while the radial and axial stresses are compressive and so \( N^{HS} \) is compressive.

For comparison purposes, we now list the corresponding results for the Gent model obtained in Horgan and Saccomandi (2001a), namely
\[ g(R) = \frac{\delta_m}{4} \left[ h_m(R) - h_m(A) + A^2 - R^2 \right] - \frac{J_m}{2} \ln \left( \frac{A}{R} \right) \frac{2 h_m(R)}{h_m(A) + \frac{2}{J_m}}. \]  

(82)

\[ T_{rr}' = T_{zz}' = \frac{\mu h_m}{4} \left\{ -h_m(R) + \frac{h_m(B)}{R^2} + \ln \left( \frac{B^2 h_m(R) + B^2 R^2}{R^2 h_m(B) + B^2 R^2} \right) \right\}, \]  

(83)

\[ T_{\phi\phi}' = \frac{\mu h_m}{4} \left\{ \frac{h_m(R)}{R^2} + h_m(B) \frac{B^2}{R^2} - 2 + \ln \left( \frac{B^2 h_m(R) + B^2 R^2}{R^2 h_m(B) + B^2 R^2} \right) \right\}, \]  

(84)

\[ T_{r\phi}' = \frac{B^2 T_0}{R^2}. \]  

(85)

We calculate the resultant axial force to be

\[ N^C = \frac{\mu R}{4} \left\{ -h_m(B) + 2h_m(A) - \frac{A^2}{B^2} h_m(B) + B^2 - A^2 + A^4 \ln \left( \frac{A^2 B^2 + A^4 h_m(B)}{A^2 B^2 + B^2 h_m(A)} \right) \right\} + \frac{\pi \sqrt{f_m} T_0 B^2}{2} \ln \left( \frac{J_m}{B} \right) \frac{2 h_m(B) + 2}{h_m(A) + \frac{2}{\sqrt{f_m}}}, \]  

(86)

where \( h_m(R) \) is the same as \( h(R) \) defined in (76) upon formally replacing \( \bar{\epsilon} \) with \( J_m \). On comparing (82) with (77), we see that the displacements for both models are identical on making this formal replacement. It may be shown that the hoop stress is tensile while the radial and axial stresses are compressive so that \( N^C \) is also compressive.

For the exponential model (11), the differential equation (36) for \( g(R) \) is

\[ R G' \exp(b R^2 g'^2) = \frac{B^2 T_0}{\mu R^2} \]  

subject to \( g(A) = 0 \). This problem does not appear amenable to analytic solution and so a numerical scheme was employed to generate results that will be described in Section 6.

6. Discussion

6.1. Axial shear

We begin by examining the axial displacement \( w(R) \) predicted by each of the models. For the Mooney–Rivlin and neo-Hookean models we obtain the non-dimensional axial displacement from (47) as

\[ \frac{w(R)}{A} = \eta T_{a s} \ln R, \]  

(88)

where we have defined the non-dimensional quantities as

\[ R = \frac{R}{A}, \quad \eta = \frac{B}{A} (> 1), \quad T_{a s} = \frac{T_a}{\mu}. \]  

(89)

For the model (8), we find from (63) that

\[ \frac{w(R)}{A} = \frac{\bar{J}}{4T_\eta} \left[ \left( 1 - R^2 \right) + R \sqrt{R^2 + 4 T_\eta^2 \bar{J}^2} - \sqrt{1 + 4 T_\eta^2 \bar{J}^2} \right] + T_{a s} \eta \ln \left( \frac{R + \sqrt{R^2 + 4 T_\eta^2 \bar{J}^2}}{1 + \sqrt{1 + 4 T_\eta^2 \bar{J}^2}} \right), \]  

(90)

where we recall from (62) that \( \bar{J} = (J - 1)/J \). Furthermore, (90) is valid for the Gent model if we formally replace \( \bar{J} \) with \( J_m \). In Fig. 1a, the non-dimensional axial displacement is plotted versus \( R \) over a range of values of \( T_{a s} \) for the Mooney–Rivlin and neo-Hookean models, while a similar plot is shown in Fig. 1b for the model (9) with \( J = 99.2 \), which is identical to the Gent model with \( J_m = 97.2 \). The value \( J_m = 97.2 \) was suggested by Gent (1996) on the basis of experiments on rubber. For the Fung model (11), the initial value problem corresponding to the first-order ordinary differential equation (72) was solved on using a numerical scheme developed by Warne et al. (2006) based on conversion to an autonomous initial value problem for a system with polynomial form, and the results1 are plotted in Fig. 1c for \( b = 0.01 \). The aspect ratio of the tube in Figs. 1a–c is taken to be \( \eta = B/A = 2 \). It is seen from Fig. 1b and c that the predictions for both classes of strain-stiffening materials are similar. On comparing with Fig. 1a, we see that at a given distance \( R \) from the inner boundary the axial displacement at each value of loading is smaller for the strain-stiffening models, as might be anticipated.

To further compare the predictions of the different material models, we now consider the relative axial displacement, defined by

\[ d = \frac{w(B)}{w}. \]  

(91)

1 We are grateful to Professors Paul Warne and Debra Polignone Warne for generating these numerical results.
Since the inner boundary is fixed, this provides a measure of the total displacement of the annular region. For the Mooney–Rivlin and neo-Hookean materials, we thus obtain from (91) and (47) the well-known result

Fig. 1. Non-dimensional axial displacement versus $R = r/A$ over a range of values of $T_*$ for (a) the Mooney–Rivlin or neo-Hookean model, (b) the HS model with $J = 99.2$, which is identical to the Gent model with $J_m = 97.2$, and (c) the Fung exponential model with $b = 0.01$. The aspect ratio of the tube is $\eta = B/A = 2$.

Since the inner boundary is fixed, this provides a measure of the total displacement of the annular region. For the Mooney–Rivlin and neo-Hookean materials, we thus obtain from (91) and (47) the well-known result
\[
\frac{d}{\bar{A}} = \frac{T_a}{2} \eta \ln(\eta),
\]

exhibiting a linear relationship between \(d\) and \(T_a\).

For the HS model, we obtain the counterpart of (92) as
\[
\frac{d}{\bar{A}} = \frac{j}{4T_a} \left[ (1 - \eta^2) + \eta^2 \sqrt{1 + 4T_a^2/j} - \sqrt{1 + 4T_a^2\eta^2/j} \right] + T_a \eta \ln \left[ \eta \left( 1 + \sqrt{1 + 4T_a^2/j} \right) \right].
\]

We find that (93) also holds for the Gent model if we formally replace \(j\) with \(j_m\). It is important to observe the following asymptotic results for the HS and Gent models, namely
\[
\lim_{T_a \to \infty} \frac{d}{\bar{A}} = \frac{j - 1}{j^{1/2}} (\eta - 1) \quad \text{and} \quad \lim_{T_a \to \infty} \frac{d}{\bar{A}} = j_m^{1/2} (\eta - 1),
\]

respectively. The finite value of these limits reflects the limiting chain extensibility character of the HS and Gent models and shows that the relative axial displacement is limited by the constraints inherent in these models. In Fig. 2a, we plot \(d/\bar{A}\) versus \(T_a\) (with \(\eta = 2\)) for the Mooney–Rivlin or neo-Hookean model (in which case we have a straight line) and for the HS model with \(j = 99.2\) and \(4.042\), which is identical to the Gent model with \(j_m = 97.2\) and 2.289, respectively. The value \(j_m = 97.2\) was suggested by Gent (1996) on the basis of experiments on rubber, while the value \(j_m = 2.289\) was proposed by Horgan and Saccomandi (2003b) based on a study of longitudinal oscillations of soft tissue strips. For small \(T_a\), the results for all the models are similar but as \(T_a\) increases, we see a significant difference in the behavior of the strain-stiffening models. From (94) with \(\eta = 2\), we see that as \(T_a \to \infty\), the latter curves tend to a horizontal asymptote \(d/\bar{A} = (j - 1)/j^{1/2} = j_m^{1/2}\). In Fig. 2b the corresponding results for the Fung exponential model are given for a range of values of the parameter \(b\). These results were again obtained on using the numerical scheme developed by Warne et al. (2006). While these results are qualitatively similar
to one another, it should be noted that asymptotic results of the type (94) do not hold for the exponential model and so the curves in Fig. 2b all tend to infinity as \( T_a \to \infty \).

We turn now to the resultant axial force \( N \). We find that

\[
\begin{align*}
\frac{N_{MR}}{2\pi \mu A^2} &= T_a \left[ \frac{1}{4} (1 - \alpha) (\eta^2 - 1) + \frac{1}{2} (1 + \alpha) \eta^2 \ln(\eta) \right], \\
\frac{N_{GH}}{2\pi \mu A^2} &= T_a^2 \eta^2 \ln(\eta), \\
\frac{N_{HS}}{2\pi \mu A^2} &= \frac{(j - 1)}{4} \left( \frac{2(j - 1)}{2j} (1 - \eta^2) + \frac{2(j - 1)}{2j} \eta^2 \sqrt{1 + 4T_a^2/j} - \frac{(2j + 1)}{2j} \sqrt{1 + 4T_a^2\eta^2/j} \right) \right.
+ \left. \frac{1}{j} \sqrt{1 + 4T_a^2/j} \ln \left( \frac{1 + \sqrt{1 + 4T_a^2/j}}{1 + \sqrt{1 + 4T_a^2\eta^2/j}} \right) \right) + \frac{T_a^2 \eta^2}{2} \left( \frac{2(j - 1)}{j - 1} \right) \ln \left( \eta \frac{1 + \sqrt{1 + 4T_a^2/j}}{1 + \sqrt{1 + 4T_a^2\eta^2/j}} \right). \\
\frac{N_{G}}{2\pi \mu A^2} &= \frac{J_m}{4} \left[ (1 - \eta^2) + \eta^2 \sqrt{1 + 4T_a^2/J_m} - \sqrt{1 + 4T_a^2\eta^2/J_m} \right] + T_a^2 \eta^2 \ln \left( \frac{1 + \sqrt{1 + 4T_a^2/j}}{1 + \sqrt{1 + 4T_a^2\eta^2/j}} \right).
\end{align*}
\]

This result for the Gent model was obtained previously by Horgan and Saccomandi (1999b). For fixed aspect ratios \( \eta = B/A \), we see that the resultant axial force for the Mooney–Rivlin and neo-Hookean models is quadratic in \( T_a \) while for the HS and Gent models, the dependence on \( T_a \) is more complicated. The results (95)–(98) are plotted in Fig. 3 for \( \alpha = 0.5, J = 99.2 \) and 4.042, and \( J_m = 97.2 \) and 2.289, respectively. For the larger values of \( J \) and \( J_m \), the curves for the HS and Gent models are virtually coincident while for the smaller values of the extensibility parameters, the HS model yields a slightly larger resultant force. We recall that the higher values of the extensibility parameters correspond to a rubbery material, while the lower values correspond to a relatively stiff material such as biological tissue. The aspect ratio of the tube is taken to be \( \eta = 2 \). For fixed values of the applied traction, the resultant axial force is seen to be smaller for the strain-stiffening models. The rate of increase of the resultant axial force with applied traction is also smaller for these models.

6.2. Azimuthal shear

We begin by examining the angular displacement function \( g(R) \) predicted by each of the models. For the Mooney–Rivlin and neo-Hookean models we obtain from (53) the angular displacement as

\[
g(R) = \frac{T_0^2 \eta^2}{2} \left[ 1 - \left( \frac{1}{R} \right)^2 \right].
\]

where the non-dimensional quantities are defined as in (89). For the HS model, we find that

\[
g(R) = \frac{J^{1/2}}{2} \left\{ \frac{J^{1/2}}{2T_0^2 \eta^2} (1 - R^2) + \sqrt{1 + \frac{J R^2}{4T_0^2 \eta^2}} - \sqrt{1 + \frac{J}{4(T_0^2 \eta^2)^2}} \right\} \ln \left[ \frac{1}{R} \left( \sqrt{1 + \frac{J R^2}{4(T_0^2 \eta^2)^2}} + 1 \right) \right].
\]

--Fig. 3. Resultant axial force \( N \) for the neo-Hookean, Mooney–Rivlin, HS, and Gent models. Results for the Mooney–Rivlin model are shown with \( \alpha = 0.5 \), for the HS model with \( J = 99.2 \) and 4.042, and for the Gent model with \( J_m = 97.2 \) and 2.289. For the larger values of these parameters the curves are virtually coincident. The aspect ratio of the tube is \( \eta = B/A = 2 \).
Furthermore, (100) is valid for the Gent model if we formally replace $J$ with $J_m$. The angular displacement is plotted versus $\bar{R}$ over a range of values of $T_0$ for the Mooney–Rivlin or neo-Hookean model in Fig. 4a, the HS model with $J = 99.2$, which is identical to the Gent model with $J_m = 97.2$ in Fig. 4b, and for the Fung exponential model in Fig. 4c with $b = 0.01$. The latter

Fig. 4. Angular displacement versus $\bar{R} = R/A$ over a range of values of $T_0$ for (a) Mooney–Rivlin or neo-Hookean model, (b) the HS model with $J = 99.2$, which is identical to the Gent model with $J_m = 97.2$, and (c) the Fung exponential model with $b = 0.01$. The aspect ratio of the tube is $\eta = B/A = 2$. 
results were obtained numerically. The aspect ratio is taken to be \( \eta = B/A = 2 \). It is seen from Fig. 4b and c that the predictions for both classes of strain-stiffening materials are similar. On comparing these results with those in Fig. 4a, we see that, at a given distance \( R \) from the inner boundary, the angular displacement at each value of loading is smaller for the strain-stiffening models, as might be anticipated.

We next consider the dependence of the relative angle of twist, defined by

\[
\Psi = g(B),
\]

on the prescribed azimuthal shear \( T_0 \). For the Mooney–Rivlin and neo-Hookean materials, (53) yields

\[
\Psi = \frac{T_0}{2}(\eta^2 - 1),
\]

which is the well-known linear relation between \( \Psi \) and \( T_0 \).

For the HS model we obtain

\[
\Psi = \frac{j^{1/2}}{2} \left\{ \frac{j^{1/2}}{2} \left( \frac{1}{\eta^2} - 1 \right) + \sqrt{1 + j/4T_0^2} - \sqrt{1 + j/4T_0^2}\eta^4 - \ln \left[ \frac{1}{\eta^2} \left( \frac{1 + \sqrt{1 + j/4T_0^2}}{1 + \sqrt{1 + j/4T_0^2}\eta^4} \right) \right] \right\}.
\]

This coincides with the result obtained by Horgan and Saccomandi (2001a) for the Gent model if \( J_m \) is formally replaced by \( \bar{J} \).

We see from (103) that

\[
\lim_{T_0 \to \infty} \Psi = \frac{j}{j^{1/2}} \ln \eta \quad \text{and} \quad \lim_{T_0 \to \infty} \Psi = \frac{j^{1/2}}{j} \ln \eta
\]

for the HS and Gent models, respectively. These asymptotic results reflect the limiting chain extensibility inherent in the two models. In Fig. 5a, we plot \( \Psi \) versus \( T_0 \) (with \( \eta = 2 \)) for the Mooney–Rivlin or neo-Hookean model (in which case we have a

![Fig. 5](image-url)
straight line) and for the HS model with \( J = 99.2 \) and 4.042, which is identical to the Gent model with \( J_m = 97.2 \) and 2.289. In Fig. 5b, the corresponding results for the Fung exponential model are given for a range of values of the parameter \( b \). These results were again obtained on using the numerical scheme developed by Warne et al. (2006). While these results are qualitatively similar to one another, it should be noted that asymptotic results of the type (104) do not hold for the exponential model and so all the curves in Fig. 5b tend to infinity as \( T_0 \rightarrow \infty \).

We turn now to the resultant axial force \( N \) necessary to maintain the assumed azimuthal shear deformation. We obtain

\[
\begin{align*}
\frac{N^{AR}}{2\pi J^2} &= -T_0^4 \left[ \frac{1}{8} (\eta^2 - 1) - \frac{(1 - \xi)}{2} \eta^2 \right] (\eta^2 - 1), \\
\frac{N^{NS}}{2\pi J^2} &= -\frac{T_0^2}{8} (\eta^2 - 1)^2, \\
\frac{N^{HS}}{2\pi J^2} &= J - 1 \left\{ \frac{-2}{J} \eta^2 \sqrt{1 + \frac{4T_0^2}{J}} + 2 \frac{1 + \frac{4T_0^2}{J} \eta^4}{J} - J - 1 \frac{1 + \frac{4T_0^2}{J} (J + 1)}{(\eta^2 - 1)} \right\} + \frac{J - 1}{J} \ln \left[ \frac{\sqrt{1 + \frac{4T_0^2}{J} + 1}}{\sqrt{1 + \frac{4T_0^2}{J} \eta^4 / J + 1}} \right], \\
\frac{N^{G}}{2\pi J^2} &= J_m \left\{ \frac{-\eta^2}{J_m} \sqrt{1 + \frac{4T_0^2}{J_m} + 2 \frac{1 + \frac{4T_0^2}{J_m} \eta^4}{J_m} - \sqrt{1 + \frac{4T_0^2}{J_m} + (\eta^2 - 1)}} \right\} + \frac{\eta^2}{J_m} \ln \left[ \frac{\sqrt{1 + \frac{4T_0^2}{J_m} + 1}}{\sqrt{1 + \frac{4T_0^2}{J_m} \eta^4 / J_m + 1}} \right].
\end{align*}
\]

The results (106)–(108) are plotted in Fig. 6 for \( \eta = 2 \) for \( J = 99.2 \) and 4.042, and \( J_m = 97.2 \) and 2.289, respectively. For the larger values of these parameters, the curves for the HS and Gent models are close to one another while for the smaller values (appropriate for biological tissue) the curves are distinct. For fixed values of the applied traction, the magnitude of the resultant axial force is seen to be smaller for the strain-stiffening models than for the neo-Hookean model. The rate of increase of the magnitude of the resultant axial force with applied traction is also smaller for these models.

Note from (106) that for the neo-Hookean material, \( N \) is always negative and so a compressive force is required to sustain pure azimuthal shear. The same result can also be shown for the HS and Gent models on using (107) and (108), respectively. In the absence of such a compressive force, the tube would undergo an axial extension. In fact, for all generalized neo-Hookean materials \( W(t) \) for which \( W_t > 0 \), it follows from (44) that \( g(R) > 0 \) and so (38) shows that \( T_{\text{ns}}(R) < 0 \). By virtue of (43)2, we thus have \( T_{\text{ns}}(R) < 0 \) and so \( N < 0 \). In particular, for the generalized neo-Hookean exponential model (11), for which analytical results are not available, we see that \( N \) is compressive.

For the Mooney–Rivlin model, however, the result is more complicated. From (105) it can be shown that \( N < 0 \) if and only if

\[ \eta > \frac{1}{\sqrt{4\alpha - 3}} \quad \text{and} \quad \alpha > 3/4. \]

\[ \text{Fig. 6. Resultant axial force } N \text{ for the neo-Hookean, HS, and Gent models. The HS model is shown with } J = 99.2 \text{ and 4.042, and the Gent model is shown with } J_m = 97.2 \text{ and 2.289. The aspect ratio of the tube is } \eta = B/A = 2. \]
Thus, for a given \( a \) such that \( 3/4 < a < 1 \), one requires that \( g \) be sufficiently large, i.e., the tube must be sufficiently thick in order for the axial force to be compressive. Experimental work of Rivlin and Saunders (1951) suggest that \( a = 7/8 \) for vulcanized rubber, which implies that \( N < 0 \) for this particular rubber tube if and only if
\[
\sqrt{\frac{2}{\pi}} \gg g.
\]
(110)

Alternatively, the condition (109) shows that \( N < 0 \) if and only if
\[
a > \frac{3}{4} + \frac{1}{4\eta^2}.
\]
(111)

Thus, for a given \( \eta > 1 \), (111) restricts the range of the material parameter \( a \) for which \( N < 0 \). For a thin-walled tube, we have \( \eta = 1 + \varepsilon (\varepsilon \ll 1) \) and so (111) implies that \( N < 0 \) if and only if \( a \) is close to 1, i.e., the material is “close to being” neo-Hookean.

We also note from (105) that when
\[
a = \frac{3}{4} + \frac{1}{4\eta^2},
\]
(112)
i.e.,
\[
\eta = \frac{1}{\sqrt{4a - 3}} \quad (a > 3/4),
\]
(113)
then \( N = 0 \). Thus, for a given aspect ratio \( \eta \), one can obtain from (112) a value of the Mooney–Rivlin constant \( a \) such that there is no axial force required to sustain pure azimuthal shear. Conversely, for a given Mooney–Rivlin material such that \( a > 3/4 \), there is always a specific aspect ratio, i.e. that given by (113), for which \( N = 0 \).

The results (109) and (111) can be further explained from an analysis of the axial stress (55). On using (54), we write (55) for the Mooney–Rivlin model as
\[
T^\text{MR}_{zz} = T_0^2 \left[ 1 + \left( \frac{B}{R} \right)^4 (3 - 4\varepsilon) \right],
\]
(114)
where we recall that \( B \) is the outer radius of the tube. It follows from (114) that \( T^\text{MR}_{zz} > 0 \) if
\[
\begin{cases}
R > B\sqrt{4\varepsilon - 3}, & a > \frac{3}{4},
\text{or} & \forall R, a \leq \frac{3}{4}.
\end{cases}
\]
(115)

For \( a < 1 \), we have \( 4\varepsilon - 3 < 1 \). Thus, if \( 3/4 < a < 1 \), the axial stress is tensile on the outer part of the tube and compressive on the inner part, and when \( a \leq 3/4 \) the stress is always tensile. These results for the Mooney–Rivlin model and their implications for the resultant axial force are somewhat surprising and to the best of our knowledge have not been noticed previously in the literature. In Fig. 7, the resultant axial force \( N \) is plotted versus \( T_0 \) for the Mooney–Rivlin model with \( \eta = 2 \) for various values of \( a \). Notice that when \( \eta = 2 \), as in Fig. 7, the result (111) shows that \( N < 0 \) when \( a = 13/16 \). In Fig. 7, the values of \( a \) for the Mooney–Rivlin model were chosen to show positive, zero, and negative values of \( N \).

7. Concluding remarks

Our objective in this paper was to investigate the effects of strain-stiffening for the classical problems of axial and azimuthal shearing of a long circular tube composed of an incompressible isotropic non-linearly elastic material. While such problems...
have direct applications to the deformations of rubber, it has also been recognized that such inhomogeneous shearing problems are of interest in the context of biomechanics of soft tissues. Our particular focus here was on obtaining explicit results for the stress response for special classes of constitutive models that give rise to strain-stiffening in their stress–stretch curves at large strains. The first class of constitutive models that we employed reflects limiting chain extensibility at the molecular level and gives rise to severe strain-stiffening. The second class exhibits a less abrupt strain-stiffening, for example, the exponential models widely used in biomechanics. For the two limiting chain extensibility models considered in detail, the second of which depends on both invariants of the Cauchy–Green tensor, one recovers the classical neo-Hookean model as the limiting chain parameter tends to infinity, while for the exponential model, the neo-Hookean model arises in the limit as a hardening parameter tends to zero. For both shearing problems, it was possible in the case of the limiting chain extensibility models to obtain closed form analytic solutions for the displacements, stresses and resultant axial force. In contrast, for the Fung exponential model, a numerical scheme was required. Thus the limiting chain models offer advantages over the exponential models for application to the biomechanics of soft tissues (see, e.g., Horgan and Saccomandi, 2003b, 2006 for a discussion of further contrasts between these models). Furthermore, since the second limiting chain model depends on both invariants of the Cauchy–Green tensor, it was possible to examine explicitly the effects of this dependence. While the pointwise and total displacement profiles obtained for both classes of strain-stiffening models were qualitatively similar, the total shear that the tube can sustain was shown to be limited by the constraints inherent in the limiting chain models whereas no such limit arises for the exponential model. For the azimuthal shear problem, it was shown that the resultant axial force necessary to maintain the deformation is compressive for all the models considered except for the Mooney–Rivlin material. In the absence of such a force, the tube would elongate in the axial direction. However, it was shown that for a Mooney–Rivlin material, the resultant axial force is compressive only if certain conditions on the geometry and on the material parameter are satisfied. To the best of our knowledge, this surprising result with important implications for experimental work has not been previously noticed in the literature and illustrates yet again the unexpected features that can arise in problems of finite elasticity.

Acknowledgements

The research of L.M.K. was supported by Graduate Teaching Assistantships, a Ballard and VEF Fellowship from the University of Virginia and a Virginia Space Grant Consortium Fellowship. The work of C.O.H. was supported by the US National Science Foundation under Grant CMMI 0754704. We are grateful to Professors Paul Warne and Debra Polignone Warne for implementation of the numerical scheme developed in Warne et al. (2006) to generate some of the results presented here for the exponential model.

References


