J. Math. Anal. Appl. 374 (2011) 374–384

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Study of weak solutions for parabolic equations with nonstandard growth conditions ☆

Bin Guo, Wenjie Gao *

Institute of Mathematics, Jilin University, Changchun 130012, PR China

ARTICLE INFO

Article history:
Received 2 August 2009
Available online 25 September 2010
Submitted by Goong Chen

Keywords:
Nonlinear parabolic equation
Nonstandard growth condition
Localization of solutions

ABSTRACT

The authors of this paper study the existence and uniqueness of weak solutions of the initial and boundary value problem for $u_t = \operatorname{div}((u^\sigma + d_0)|\nabla u|^{p(x,t)-2}\nabla u) + f(x,t)$. Localization property of weak solutions is also discussed.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded simply connected domain and $0 < T < \infty$. Consider the following quasilinear degenerate parabolic problem:

$$\begin{cases} u_t = \operatorname{div}\big(a(u)|\nabla u|^{p(x,t)-2}\nabla u\big) + f(x,t), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0,T]$, Γ_T denotes the lateral boundary of the cylinder Q_T, and $a(u) = u^\sigma + d_0$ with the assumption that σ and d_0 are two positive constants. It will be assumed throughout the paper that the exponent $p(x,t)$ is continuous in $Q = \overline{Q_T}$ with logarithmic module of continuity:

$$1 < p^- = \inf_{(x,t)\in Q} p(x,t) \leqslant p(x,t) \leqslant p^+ = \sup_{(x,t)\in Q} p(x,t) < \infty, \tag{1.2}$$

$$\forall z = (x,t) \in Q_T, \quad \xi = (y,s) \in Q_T, \quad |z-\xi| < 1, \quad \big|p(z) - p(\xi)\big| \leqslant \omega\big(|z-\xi|\big), \tag{1.3}$$

where

$$\limsup_{\tau\to 0^+} \omega(\tau)\ln\frac{1}{\tau} = C < +\infty.$$

In the case when p is a constant, there have been many results about the existence, uniqueness and the properties of the solutions, we refer to the bibliography [1–4].

☆ Supported by NSFC (10771085) and by Key Lab of Symbolic Computation and Knowledge Engineering of Ministry of Education and by the 985 program of Jilin University.

* Corresponding author.
E-mail addresses: bguo1029@gmail.com (B. Guo), wjgao@jlu.edu.cn (W. Gao).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.09.039

In the recent years, much attention has been paid to the study of mathematical models of electrorheological fluids. These models include parabolic or elliptic equations which are nonlinear with respect to gradient of the thought solution and with variable exponents of nonlinearity, see [5–8] and references therein. Besides, another important application is the image processing where the anisotropy and nonlinearity of the diffusion operator and convection terms are used to underline the borders of the distorted image and to eliminate the noise [9–11].

To the best of our knowledge, there are only a few works about parabolic equations with variable exponents of nonlinearity. In [6], applying Galerkin's method, S.N. Antontsev and S.I. Shmarev obtained the existence and uniqueness of weak solutions with the assumption that the function $a(u)$ in $\operatorname{div}(a(u)|\nabla u|^{p(x,t)-2}\nabla u)$ was bounded. However, we can't easily put the method in [6] generalized to the unbounded case. This paper applied the method of parabolic regularization to prove the existence of weak solutions to the problems mentioned. By making a sequence of estimates to weak solutions, the authors of this paper proved the weak convergence of the approximation solution sequence and hence testified the existence of weak solutions. Furthermore, making appropriate a priori estimates and calculating accurately, we also obtained the localization property of weak solutions.

The outline of this paper is the following: In Section 2, we shall introduce the function spaces of Orlicz–Sobolev type, give the definition of the weak solution to the problem and prove the existence of weak solutions with a method of regularization; Section 3 will be devoted to the proof of the uniqueness of the solution obtained in Section 2; in Section 4, we will get the localization property of the solution under suitable conditions.

2. Existence of weak solutions

We will study the existence of the weak solutions in this section. Let us introduce the Banach spaces

$$\mathbf{L}^{p(x,t)}(Q_T)=\left\{u(x,t)\,\middle|\,u \text{ is measurable in } Q_T,\ A_{p(.)}(u)=\iint\limits_{Q_T}|u|^{p(x,t)}\,dx\,dt<\infty\right\},$$
$$\|u\|_{p(.)}=\inf\bigl\{\lambda>0,\,A_{p(.)}(u/\lambda)\leqslant 1\bigr\},$$
$$\mathbf{V}_t(\Omega)=\bigl\{u\,\big|\,u\in L^2(\Omega)\cap W_0^{1,1}(\Omega),\ |\nabla u|\in L^{p(x,t)}(\Omega)\bigr\},$$
$$\|u\|_{\mathbf{V}_t(\Omega)}=\|u\|_{2,\Omega}+\|\nabla u\|_{p(.,t)\Omega},$$
$$\mathbf{W}(Q_T)=\bigl\{u:[0,T]\mapsto \mathbf{V}_t(\Omega)\,\big|\,u\in L^2(Q_T),\ |\nabla u|\in L^{p(x,t)}(Q_T),\ u=0 \text{ on } \Gamma_T\bigr\},$$
$$\|u\|_{\mathbf{W}(Q_T)}=\|u\|_{2,Q_T}+\|\nabla u\|_{p(x,t),Q_T}$$

and denote by $\mathbf{W}'(Q_T)$ the dual of $\mathbf{W}(Q_T)$ with respect to the inner product in $L^2(Q_T)$.

Definition 2.1. A function $u(x,t)\in\mathbf{W}(Q_T)\cap L^\infty(0,T;L^\infty(\Omega))$ is called a weak solution of problem (1.1) if for every test-function

$$\xi\in\mathcal{Z}\equiv\bigl\{\eta(z):\ \eta\in\mathbf{W}(Q_T)\cap\mathbf{L}^\infty\bigl(0,T;L^2(\Omega)\bigr),\ \eta_t\in\mathbf{W}'(Q_T)\bigr\},$$

and every $t_1,t_2\in[0,T]$ the following identity holds:

$$\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\bigl[u\xi_t-\bigl(u^\sigma+d_0\bigr)|\nabla u|^{p(x,t)-2}\nabla u\nabla\xi+f(x,t)\xi\bigr]\,dx\,dt=\int\limits_{\Omega}u\xi\,dx\Big|_{t_1}^{t_2}. \tag{2.1}$$

The main theorem in this section is:

Theorem 2.1. *Let $p(x,t)$ satisfy conditions* (1.2)–(1.3)*. If the following conditions hold*

$$(H_1)\quad \max\left\{1,\frac{2N}{N+2}\right\}<p^-<N,\quad 2\leqslant\sigma<\frac{2p^+}{p^+-1};$$

$$(H_2)\quad u_0\geqslant 0,\ f\geqslant 0,\quad \|u_0\|_{\infty,\Omega}+\int\limits_0^T\bigl\|f(x,t)\bigr\|_{\infty,\Omega}\,dt=K(T)<\infty,$$

then problem (1.1) *has at least one weak solution in the sense of Definition* 2.1.

Let us consider the following auxiliary parabolic problem

$$\begin{cases} u_t = \operatorname{div}\big(a_{\varepsilon,M}(u)|\nabla u|^{p(x,t)-2}\nabla u\big) + f(x,t), & (x,t) \in Q_T, \\ u(x,t) = 0, & (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.2}$$

Here M stands for a positive parameter to be chosen later and notice that

$$0 < d_0 \leqslant a_{\varepsilon,M}(u) = \big(\min\big(|u|^2, M^2\big) + \varepsilon^2\big)^{\frac{\sigma}{2}} + d_0 \leqslant \big(M^2+1\big)^{\frac{\sigma}{2}} + d_0, \quad 0 < \varepsilon < 1.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(x,t)}$ for $t \in (0,T)$, we may construct the sequence of approximate solutions $u^m(x,t) = \sum_{k=1}^m c_k^m \varphi_k(x)$, and with a similar method as in [6], we may prove that the regularized problem has a unique weak solution $u_\varepsilon(x,t) \in \mathbf{W}(Q_T) \cap \mathbf{L}^2(Q_T)$, $u_{\varepsilon t}(x,t) \in \mathbf{W}'(Q_T)$ satisfying the following integral identities

$$\int_{t_1}^{t_2}\int_\Omega \big[u_\varepsilon \xi_t - a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi + f(x,t)\xi\big]\,dx\,dt = \int_\Omega u_\varepsilon \xi\,dx\Big|_{t_1}^{t_2}, \tag{2.3}$$

and

$$\int_{t_1}^{t_2}\int_\Omega \big[u_{\varepsilon t}\xi + a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi - f(x,t)\xi\big]\,dx\,dt = 0. \tag{2.4}$$

In order to prove this theorem, we need the following lemmas.

Lemma 2.1. *The solution of problem* (2.2) *satisfies the estimate*

$$\|u_\varepsilon\|_{\infty,Q_T} \leqslant \|u_0\|_{\infty,\Omega} + \int_0^T \big\|f(x,t)\big\|_{\infty,\Omega}\,dt = K(T) < \infty. \tag{2.5}$$

Proof. Let us introduce the function

$$u_{\varepsilon M} = \begin{cases} M & \text{if } u_\varepsilon > M, \\ u_\varepsilon & \text{if } |u_\varepsilon| \leqslant M, \\ -M & \text{if } u_\varepsilon < -M. \end{cases}$$

The function $u_{\varepsilon M}^{2k-1}$, with $k \in \mathbb{N}$, can be chosen as a test-function in (2.4). Let in (2.4) $t_2 = t+h$, $t_1 = t$, with $t, t+h \in (0,T)$. Then

$$\frac{1}{2k}\int_t^{t+h}\frac{d}{dt}\left(\int_\Omega u_{\varepsilon M}^{2k}\,dx\right)dt + \int_t^{t+h}\int_\Omega (2k-1)a_{\varepsilon,M}(u_{\varepsilon M})u_{\varepsilon M}^{2(k-1)}|\nabla u_{\varepsilon M}|^{p(x,t)}\,dx\,dt = \int_t^{t+h}\int_\Omega f u_{\varepsilon M}^{2k-1}\,dx\,dt. \tag{2.6}$$

Dividing the last equality by h, letting $h \to 0$ and applying Lebesgue's dominated convergence theorem, we have that $\forall t \in (0,T)$

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega u_{\varepsilon M}^{2k}\,dx + \int_\Omega (2k-1)a_{\varepsilon,M}(u_{\varepsilon M})u_{\varepsilon M}^{2(k-1)}|\nabla u_{\varepsilon M}|^{p(x,t)}\,dx = \int_\Omega f u_{\varepsilon M}^{2k-1}\,dx. \tag{2.7}$$

By Hölder's inequality

$$\left|\int_\Omega f u_{\varepsilon M}^{2k-1}\,dx\right| \leqslant \big\|u_{\varepsilon M}(\cdot,t)\big\|_{2k,\Omega}^{2k-1} \cdot \big\|f(\cdot,t)\big\|_{2k,\Omega}, \quad k = 1,2,\ldots$$

whence

$$\begin{aligned} &\|u_{\varepsilon M}\|_{2k,\Omega}^{2k-1}\frac{d}{dt}\big(\|u_{\varepsilon M}\|_{2k,\Omega}\big) + (2k-1)\int_\Omega a_{\varepsilon,M}(u_{\varepsilon M})u_{\varepsilon M}^{2(k-1)}|\nabla u_{\varepsilon,M}|^{p(x,t)}dx \\ &\quad \leqslant \|u_{\varepsilon,M}\|_{2k,\Omega}^{2k-1} \cdot \big\|f(\cdot,t)\big\|_{2k,\Omega}, \quad k = 1,2,\ldots. \end{aligned} \tag{2.8}$$

From (2.8) one gets, by integration over $(0,t)$, for all t,

$$\big\|u_{\varepsilon M}(.,t)\big\|_{2k,\Omega} \leqslant \big\|u_{\varepsilon M}(.,0)\big\|_{2k,\Omega} + \int_0^T \|f\|_{2k,\Omega}\,dt, \quad \forall k \in \mathbb{N}.$$

Then, as $k \to \infty$,

$$\big\|u_{\varepsilon M}(.,t)\big\|_{\infty,\Omega} \leqslant \big\|u_{\varepsilon M}(.,0)\big\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega}\,dt \leqslant \|u_0\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega}\,dt = K(T).$$

If we choose $M > K(T)$ then $u_{\varepsilon M}(.,t) \leqslant \sup|u_{\varepsilon M}(.,t)| \leqslant K(T) < M$ and therefore $u_{\varepsilon M}(.,t) = u_\varepsilon(.,t)$. □

Corollary 2.1. *According to the above text, we have*

$$\min\{u_\varepsilon^2, M^2\} = u_\varepsilon^2 \quad \text{and} \quad a_{\varepsilon,M}(u_{\varepsilon M}) = a_{\varepsilon,M}(u_\varepsilon) = \big(\varepsilon^2 + u_\varepsilon^2\big)^{\sigma/2} + d_0.$$

Corollary 2.2. *When $u_0 \geqslant 0$ and $f \geqslant 0$, the solution $u_\varepsilon(x,t)$ is nonnegative in Q_T.*

Proof. Set $u_\varepsilon^- = \min\{u_\varepsilon, 0\}$. Then $u_\varepsilon^-(x,0) = 0$, $u_\varepsilon^-|_{\Gamma_T} = 0$ and

$$\frac{1}{2}\frac{d}{dt}\big(\big\|u_\varepsilon^-(x,t)\big\|_{2,\Omega}^2\big) + \int_\Omega a_{\varepsilon,M}(u_\varepsilon)\big|\nabla u_\varepsilon^-\big|^{p(x,t)}\,dx \leqslant 0.$$

It follows that for every $t > 0$,

$$\big\|u_\varepsilon^-(x,t)\big\|_{2,\Omega} \leqslant \big\|u_\varepsilon^-(\cdot,0)\big\|_{2,\Omega} = 0.$$

The required assertion follows. □

Remark 2.1. It is clear that the constructed weak solution in this paper is nonnegative. But to the best of our knowledge, it is still not clear whether any solution of the problem is nonnegative if the given data are nonnegative.

Lemma 2.2. *The solution of problem* (2.2) *satisfies the estimates*

$$\iint_{Q_T} u_\varepsilon^\sigma |\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leqslant K(T)|\Omega|^{\frac{1}{2}}, \tag{2.9}$$

$$\varepsilon^\sigma \iint_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leqslant K(T)|\Omega|^{\frac{1}{2}}, \tag{2.10}$$

$$d_0 \iint_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leqslant K(T)|\Omega|^{\frac{1}{2}}. \tag{2.11}$$

Proof. To prove Lemma 2.2, we proceed as in the proof of Lemma 2.1 and in (2.8) we take $k = 1$. We then get

$$\frac{d}{dt}\big\|u_\varepsilon(.,t)\big\|_{2,\Omega} + \int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,dx \leqslant \|f\|_{2,\Omega}, \quad \forall t \in (0,T).$$

Therefore, integrating in time over $(0,t)$, $\forall t \in (0,T)$

$$\big\|u_\varepsilon(.,t)\big\|_{2,\Omega} + \int_0^t\int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leqslant \big\|u_\varepsilon(.,0)\big\|_{2,\Omega} + \int_0^T \|f\|_{2,\Omega}\,dt,$$

and since the first term on the left hand side is nonnegative and recalling the L^2-norm

$$\int_0^t\int_\Omega a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt \leqslant K(T)|\Omega|^{\frac{1}{2}}.$$

Since $a_{\varepsilon,M}(u_\varepsilon) \geqslant d_0$, one gets inequality (2.11); since $a_{\varepsilon,M}(u_\varepsilon) \geqslant \varepsilon^\sigma$, one gets inequality (2.10); since $M > K(T)$, one gets $a_{\varepsilon,M}(u_\varepsilon) \geqslant u_\varepsilon^\sigma$, furthermore, we get inequality (2.9). □

Lemma 2.3. *The solution of problem* (2.2) *satisfies the estimate*

$$\|u_{\varepsilon t}\|_{W'(Q_T)} \leqslant C\big(\sigma, p^\pm, K(T), |\Omega|\big). \tag{2.12}$$

Proof. From identity (2.4), we get

$$\iint_{Q_T} u_{\varepsilon t}\xi\,dx\,dt = -\iint_{Q_T}\big[\big(u_\varepsilon^2+\varepsilon^2\big)^{\sigma/2}+d_0\big]|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,dx\,dt + \iint_{Q_T} f(x,t)\xi(x,t)\,dx\,dt$$
$$\leqslant \int_0^T\int_\Omega \big[\big(u_\varepsilon^2+\varepsilon^2\big)^{\sigma/2}+d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}|\nabla\xi|\,dx\,dt + \int_0^T\int_\Omega |f|\cdot|\xi|\,dx\,dt$$
$$\leqslant 2\big\|\big[\big(u_\varepsilon^2+\varepsilon^2\big)^{\sigma/2}+d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\big\|_{p'(x,t)}\|\nabla\xi\|_{p(x,t)} + 2\|f\|_{p'(x,t)}\cdot\|\xi\|_{p(x,t)}$$
$$\leqslant 2\max\Bigg\{\Bigg(\int_0^T\int_\Omega \big\{\big[\big(u_\varepsilon^2+\varepsilon^2\big)^{\sigma/2}+d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\big\}^{\frac{p(x,t)}{p(x,t)-1}}\,dx\,dt\Bigg)^{\frac{1}{p'^+}},$$
$$\Bigg(\int_0^T\int_\Omega \big\{\big[\big(u_\varepsilon^2+\varepsilon^2\big)^{\sigma/2}+d_0\big]|\nabla u_\varepsilon|^{p(x,t)-1}\big\}^{\frac{p(x,t)}{p(x,t)-1}}\,dx\,dt\Bigg)^{\frac{1}{p'^-}}\Bigg\}\|\nabla\xi\|_{p(x,t)}$$
$$+2\max\Bigg\{\Bigg(\int_0^T\int_\Omega |f|^{p'(x,t)}\,dx\,dt\Bigg)^{\frac{1}{p'^+}}, \Bigg(\int_0^T\int_\Omega |f|^{p'(x,t)}\,dx\,dt\Bigg)^{\frac{1}{p'^-}}\Bigg\}\|\xi\|_{p(x,t)}$$
$$\leqslant \Big(2\big(\big(K^2(T)+1\big)^{\sigma/2}+d_0\big)^{\frac{1}{p^\pm-1}}K(T)|\Omega| + 2|f|_\infty|T|\Big)\|\xi\|_{W(Q_T)}.$$

Then (2.12) follows from Lemma 2.2. □

From [6], we may get the following inclusions:

$$u_\varepsilon \in W(Q_T) \subseteq L^{p^-}\big(0,T;W_0^{1,p^-}(\Omega)\big),$$
$$u_{\varepsilon t} \in W'(Q_T) \subseteq L^{\frac{p^+}{p^+-1}}\big(0,T;V_+(\Omega)\big),$$
$$W_0^{1,p^-}(\Omega) \subset L^2(\Omega) \subset V'_+(\Omega)$$

with $V_+(\Omega)=\{u(x)\mid u\in L^2(\Omega)\cap W_0^{1,1}(\Omega),\ |\nabla u|\in L^{p^+}\}$.

These conclusions together with the uniform estimates in ε allow one to extract from the sequence $\{u_\varepsilon\}$ a subsequence (for the sake of simplicity, we assume that it merely coincides with the whole of the sequence) such that

$$\begin{cases} u_\varepsilon \to u & \text{a.e. in } Q_T;\\ \nabla u_\varepsilon \rightharpoonup \nabla u & \text{weakly in } L^{p(x,t)}(Q_T);\\ u_\varepsilon^\sigma|\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon \rightharpoonup A_i(x,t) & \text{weakly in } L^{p'(x,t)}(Q_T);\\ |\nabla u_\varepsilon|^{p(x,t)-2}D_iu_\varepsilon \rightharpoonup W_i(x,t) & \text{weakly in } L^{p'(x,t)}(Q_T), \end{cases} \tag{2.13}$$

for some functions $u\in W(Q_T)$, $A_i(x,t)\in L^{p'(x,t)}(Q_T)$, $W_i(x,t)\in L^{p'(x,t)}(Q_T)$.

Lemma 2.4. *For almost all* $(x,t)\in Q_T$,

$$\lim_{\varepsilon\to0^+}\iint_{Q_T}\big(\big(u_\varepsilon^2+\varepsilon^2\big)^{\frac{\sigma}{2}}-u_\varepsilon^\sigma\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,dx\,dt = 0,\quad \forall\xi\in W(Q_T).$$

Proof.

$$I \triangleq \iint_{Q_T}\big(\big(u_\varepsilon^2+\varepsilon^2\big)^{\frac{\sigma}{2}}-u_\varepsilon^\sigma\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,dx\,dt$$
$$=\frac{\sigma}{2}\varepsilon^2\iint_{Q_T}\Bigg(\int_0^1\big(u_\varepsilon^2+s\varepsilon^2\big)^{\frac{\sigma-2}{2}}\,ds\Bigg)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla\xi\,dx\,dt$$
$$\leqslant \sigma\varepsilon^2\big(K^2(T)+1\big)^{\frac{\sigma-2}{2}}\big\||\nabla_\varepsilon|^{p(x,t)-1}\big\|_{p'(x,t)}\|\nabla\xi\|_{p(x,t)}$$

$$\leqslant C\varepsilon^2 \max\left\{\left(\iint\limits_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt\right)^{\frac{p^+-1}{p^+}}, \left(\iint\limits_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}\,dx\,dt\right)^{\frac{p^--1}{p^-}}\right\}\|\nabla\xi\|_{p(x,t)}.$$

By (2.10), we get

$$I \leqslant C\varepsilon^{2-\frac{\sigma(p^+-1)}{p^+}}\|\nabla\xi\|_{p(x,t)}.$$

Passing to the limit as $\varepsilon \to 0$, we obtain Lemma 2.4. □

Lemma 2.5. *For almost all $(x,t) \in Q_T$,*

$$A_i(x,t) = u^\sigma W_i(x,t), \quad i = 1, 2, \dots N.$$

Proof. In (2.13), letting $\varepsilon \to 0$, we have

$$\iint\limits_{Q_T} u_\varepsilon^\sigma |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi\,dx\,dt \to \Sigma \iint\limits_{Q_T} A_i(x,t) D_i\xi\,dx\,dt; \tag{2.14}$$

$$\iint\limits_{Q_T} |\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi\,dx\,dt \to \Sigma \iint\limits_{Q_T} W_i(x,t) D_i\xi\,dx\,dt. \tag{2.15}$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0}\sum_{i=1}^N \iint\limits_{Q_T} \left(u_\varepsilon^\sigma - u^\sigma\right) A_i(x,t) D_i\xi\,dx\,dt = 0. \tag{2.16}$$

So,

$$\lim_{\varepsilon\to 0}\sum \iint\limits_{Q_T} \left(u_\varepsilon^\sigma |\nabla u_\varepsilon|^{p(x,t)-2} D_i u_\varepsilon - u^\sigma W_i(x,t)\right) D_i\xi\,dx\,dt$$

$$= \lim_{\varepsilon\to 0}\sum \iint\limits_{Q_T} \left(\left(u_\varepsilon^\sigma - u^\sigma\right)|\nabla u_\varepsilon|^{p(x,t)-2} D_i u_\varepsilon + u^\sigma\left(|\nabla u_\varepsilon|^{p(x,t)-2} D_i u_\varepsilon - W_i(x,t)\right)\right) D_i\xi\,dx\,dt = 0.$$

By (2.14)–(2.16) and the above inequalities, this completes the proof of Lemma 2.5. □

Lemma 2.6. *For almost all $(x,t) \in Q_T$,*

$$W_i(x,t) = |\nabla u|^{p(x,t)-2} D_i u, \quad i = 1, 2, \dots, N.$$

Proof. In (2.4), choosing $\xi = (u_\varepsilon - u)\Phi$ with $\Phi \in W(Q_T)$, $\Phi \geqslant 0$, we have

$$\iint\limits_{Q_T} \left[u_{\varepsilon t}(u_\varepsilon - u)\Phi + \Phi\left(u_\varepsilon^\sigma + d_0\right)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)\right] dx\,dt$$

$$+ \iint\limits_{Q_T} \left[(u_\varepsilon - u)\left(u_\varepsilon^\sigma + d_0\right)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\Phi - f(x,t)(u_\varepsilon - u)\Phi\right] dx\,dt$$

$$+ \iint\limits_{Q_T} \left(\left(u_\varepsilon^2 + \varepsilon^2\right)^{\frac{\sigma}{2}} - u_\varepsilon^\sigma\right)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\xi\,dx\,dt = 0.$$

It follows that

$$\lim_{\varepsilon\to 0}\iint\limits_{Q_T} \Phi\left(u_\varepsilon^\sigma + d_0\right)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)\,dx\,dt = 0. \tag{2.17}$$

On the other hand, from $u_\varepsilon, u \in L^\infty(Q_T)$, $|\nabla u| \in L^{p(x,t)}(Q_T)$, we get

$$\lim_{\varepsilon\to 0}\iint_{Q_T} \Phi\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla(u_\varepsilon - u)\,dx\,dt = 0, \tag{2.18}$$

$$\lim_{\varepsilon\to 0}\iint_{Q_T} \Phi\big(u_\varepsilon^\sigma - u^\sigma\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla(u_\varepsilon - u)\,dx\,dt = 0. \tag{2.19}$$

Note that

$$\begin{aligned}
0 &\leqslant \big(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u\big)\nabla(u_\varepsilon - u)\\
&\leqslant \frac{1}{d_0}\big[\big(u_\varepsilon^\sigma + d_0\big)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - \big(u_\varepsilon^\sigma - u^\sigma\big)|\nabla u|^{p(x,t)-2}\nabla u\big]\nabla(u_\varepsilon - u)\\
&\quad - \frac{1}{d_0}\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla(u_\varepsilon - u).
\end{aligned} \tag{2.20}$$

By (2.17)–(2.20), we obtain

$$\lim_{\varepsilon\to 0}\iint_{Q_T} \Phi\big(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u\big)\nabla(u_\varepsilon - u)\,dx\,dt = 0. \tag{2.21}$$

The rest arguments are the same as those of Theorem 2.1 in [12], we omit the details. We complete the existence part by a standard limiting process. □

3. Uniqueness of weak solutions

In this section, we study the uniqueness of the weak solutions to problem (1.1). In order to obtain the main conclusion of this section, we need the following lemma.

Lemma 3.1. *Let $M(s) = |s|^{p(x,t)-2}s$, then $\forall \xi, \eta \in R^N$*

$$\big(M(\xi) - M(\eta)\big)(\xi - \eta) \geqslant \begin{cases} 2^{-p(x,t)}|\xi - \eta|^{p(x,t)} & \text{if } 2 \leqslant p(x,t) < \infty,\\ (p(x,t) - 1)|\xi - \eta|^2(|\xi|^{p(x,t)} + |\eta|^{p(x,t)})^{\frac{p(x,t)-2}{p(x,t)}} & \text{if } 1 < p(x,t) < 2. \end{cases}$$

The main result is:

Theorem 3.1. *Suppose that the conditions in Theorem 2.1 are fulfilled and $2 < \sigma < \frac{2p^+}{p^+-1}$, $p^+ \geqslant 2$. Then the nonnegative solution of problem (1.1) is unique within the class of all nonnegative weak solutions.*

Proof. We argue by contradiction. Suppose $u(x,t)$ and $v(x,t)$ are two nonnegative weak solutions of problem (1.1) and there is a $\delta > 0$ such that for some $0 < \tau \leqslant T$, $w = u - v > \delta$ on the set $\Omega_\delta = \Omega \cap \{x\colon\ w(x,\tau) > \delta\}$ and $\mu(\Omega_\delta) > 0$. Let

$$F_\varepsilon(\xi) = \begin{cases} \frac{1}{\alpha-1}\varepsilon^{1-\alpha} - \frac{1}{\alpha-1}\xi^{1-\alpha} & \text{if } \xi > \varepsilon,\\ 0 & \text{if } \xi \leqslant \varepsilon, \end{cases}$$

where $\delta > 2\varepsilon > 0$ and $\alpha = \frac{\sigma}{2}$.

By the definition of weak solutions, let a test-function $\xi = F_\varepsilon(w) \in \mathcal{Z}$,

$$\begin{aligned}
0 &= \iint_{Q_\tau} \big[w_t F_\varepsilon(w) + \big(v^\sigma + d_0\big)\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla F_\varepsilon(w)\big]\,dx\,dt\\
&\quad + \iint_{Q_\tau} \big(u^\sigma - v^\sigma\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla F_\varepsilon(w)\,dx\,dt\\
&= \iint_{Q_{\varepsilon,\tau}} w_t F_\varepsilon(w)\,dx\,dt + \iint_{Q_{\varepsilon,\tau}} \big(v^\sigma + d_0\big)w^{-\alpha}\big(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\big)\nabla w\,dx\,dt\\
&\quad + \iint_{Q_{\varepsilon,\tau}} \big(u^\sigma - v^\sigma\big)w^{-\alpha}|\nabla u|^{p(x,t)-2}\nabla u\nabla w\,dx\,dt := J_1 + J_2 + J_3,
\end{aligned} \tag{3.1}$$

with $Q_{\varepsilon,\tau} = Q_\tau \cap \{(x,t) \in Q_\tau\colon\ w > \varepsilon\}$.

Now, let $t_0 = \inf\{t \in (0,\tau]:\ w > \varepsilon\}$, then we estimate J_1, J_2, J_3 as follows:

$$J_1 = \iint_{Q_{\varepsilon,\tau}} w_t F_\varepsilon(w)\,dx\,dt = \int_\Omega \left(\int_0^{t_0} w_t F_\varepsilon(w)\,dt + \int_{t_0}^{\tau} w_t F_\varepsilon(w)\,dt \right) dx$$
$$\geqslant \int_\Omega \int_\varepsilon^{w(x,\tau)} F_\varepsilon(s)\,ds\,dx \geqslant \int_{\Omega_\varepsilon} \int_\varepsilon^{w(x,\tau)} F_\varepsilon(s)\,ds\,dx$$
$$\geqslant \int_{\Omega_\delta} (w - 2\varepsilon) F_\varepsilon(2\varepsilon)\,dx \geqslant (\delta - 2\varepsilon) F_\varepsilon(2\varepsilon)\mu(\Omega_\delta). \tag{3.2}$$

Let us consider first the case $p^- \geqslant 2$. By virtue of the first inequality of Lemma 3.1, we get

$$J_2 = \iint_{Q_{\varepsilon,\tau}} \left(v^\sigma + d_0\right) w^{-\alpha} \left(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\right)\nabla w\,dx\,dt$$
$$\geqslant \iint_{Q_{\varepsilon,\tau}} \left(v^\sigma + d_0\right) w^{-\alpha} 2^{-p(x,t)} |\nabla w|^{p(x,t)}\,dx\,dt$$
$$\geqslant 2^{-p^+} \iint_{Q_{\varepsilon,\tau}} \left(v^\sigma + d_0\right) w^{-\alpha} |\nabla w|^{p(x,t)}\,dx\,dt \geqslant 0. \tag{3.3}$$

Noting that $\frac{p(x,t)}{p(x,t)-1} \geqslant \frac{p^+}{p^+-1} \geqslant \frac{\sigma}{2} = \alpha > 1$ and applying Young's inequality, we may estimate integrand of J_3 in the following way

$$\left|\left(u^\sigma - v^\sigma\right) w^{-\alpha} |\nabla u|^{p(x,t)-2}\nabla u \nabla w\right| = \left|\sigma w \int_0^1 \left(\theta u + (1-\theta)v\right)^{\sigma-1} d\theta\, w^{-\alpha} |\nabla u|^{p(x,t)-2}\nabla u \nabla w\right|$$
$$\leqslant \frac{C}{w^\alpha}\left[\frac{v^\sigma + d_0}{C}|\nabla w|^{p(x,t)} + C_1\left(\sigma, d_0, K(T), p^\pm\right)|w|^{p'(x,t)}|\nabla u|^{p(x,t)}\right]$$
$$= \frac{(v^\sigma + d_0)}{2^{p^+ + 1} w^\alpha}|\nabla w|^{p(x,t)} + C_1\left(\sigma, d_0, K(T), p^\pm\right)|w|^{p'(x,t)-\alpha}|\nabla u|^{p(x,t)}$$
$$\leqslant \frac{(v^\sigma + d_0)}{2^{p^+ + 1} w^\alpha}|\nabla w|^{p(x,t)} + C_1\left(\sigma, d_0, K(T), p^\pm\right)|\nabla u|^{p(x,t)}. \tag{3.4}$$

Substituting (3.4) into J_3, we get

$$J_3 \leqslant \frac{1}{2} J_2 + C \iint_{Q_{\varepsilon,\tau}} |\nabla u|^{p(x,t)}\,dx\,dt. \tag{3.5}$$

Secondly, we consider the case $1 < p^- \leqslant p(x,t) < 2$, $p^+ \geqslant 2$. According to the second inequality of Lemma 3.1, it is easily seen that the following inequalities hold

$$J_2 = \iint_{Q_{\varepsilon,\tau}} \left(v^\sigma + d_0\right) w^{-\alpha} \left(|\nabla u|^{p(x,t)-2}\nabla u - |\nabla v|^{p(x,t)-2}\nabla v\right)\nabla w\,dx\,dt$$
$$\geqslant \left(p^- - 1\right) \iint_{Q_{\varepsilon,\tau}} \left(v^\sigma + d_0\right) w^{-\alpha} \left(|\nabla u| + |\nabla v|\right)^{p(x,t)-2} |\nabla w|^2\,dx\,dt \geqslant 0. \tag{3.6}$$

Using the conditions $1 < \alpha \leqslant \frac{p^+}{p^+-1} \leqslant 2$ and Young's inequality, we may evaluate integrand of J_3 as follows

$$\left|\left(u^\sigma - v^\sigma\right) w^{-\alpha} |\nabla u|^{p(x,t)-2}\nabla u \nabla w\right|$$
$$= \left|\sigma w \int_0^1 \left(\theta u + (1-\theta)v\right)^{\sigma-1} d\theta\, w^{-\alpha} |\nabla u|^{p(x,t)-2}\nabla u \nabla w\right|$$

$$\leqslant \frac{(v^\sigma + d_0)(p^- - 1)}{2w^\alpha}\big(|\nabla u| + |\nabla v|\big)^{p(x,t)-2}|\nabla w|^2 + C_1\big(\sigma, d_0, K(T), p^\pm\big)|w|^{2-\alpha}\big(|\nabla u| + |\nabla v|\big)^{p(x,t)}$$
$$\leqslant \frac{(v^\sigma + d_0)(p^- - 1)}{2w^\alpha}\big(|\nabla u| + |\nabla v|\big)^{p(x,t)-2}|\nabla w|^2 + C_1\big(\sigma, d_0, K(T), p^\pm\big)\big(|\nabla u| + |\nabla v|\big)^{p(x,t)}. \tag{3.7}$$

Plugging (3.7) into J_3, we get

$$J_3 \leqslant \frac{1}{2}J_2 + C\iint\limits_{Q_{\varepsilon,\tau}} \big(|\nabla u| + |\nabla v|\big)^{p(x,t)}\,dx\,dt. \tag{3.8}$$

Plugging the above estimates (3.2), (3.3), (3.5) and (3.2), (3.6), (3.8) into (3.1) and dropping the nonnegative terms, we arrive at the inequality

$$(\delta - 2\varepsilon)\big(1 - 2^{1-\alpha}\big)\varepsilon^{1-\alpha}\mu(\Omega_\delta) \leqslant \widetilde{C}, \tag{3.9}$$

with a constant $\widetilde{C}$ independent of ε.

Notice that $\lim_{\varepsilon\to 0}(\delta - 2\varepsilon)(1 - 2^{1-\alpha})\varepsilon^{1-\alpha}\mu(\Omega_\delta) = +\infty$, we obtain a contradiction. This means $\mu(\Omega_\delta) = 0$ and $w \leqslant 0$, a.e. in Q_τ. □

Corollary 3.1 *(Comparison principle). Let $u, v \in W(Q_T) \cap L^\infty(0,T; L^\infty(\Omega))$ be two nonnegative solutions of problem* (1.1) *such that $u(x,0) \leqslant v(x,0)$ a.e. in Ω. If the coefficients and nonlinearity exponents satisfy the conditions of Theorem* 3.1*, then $u(x,t) \leqslant v(x,t)$ a.e. in Q_T.*

4. Localization of weak solutions

In this section, we shall concentrate on the study of localization of weak solutions to problem (1.1). Our main result is:

For a function $w : \Omega \mapsto [0,\infty)$, we define

$$\operatorname{supp} w = \overline{\left\{x \in G;\ \lim_{\rho\to 0}\frac{\mu(G \cap B_\rho(x))}{\mu(B_\rho(x))} > 0\right\}},$$

where $G = \{x \in \Omega;\ w > 0\}$, $B_\rho(x) = \{y \in \Omega;\ |x - y| < \rho\}$.

It is easy to see that if $w \in C(\Omega)$, then $\operatorname{supp} w = \overline{G}$.

Theorem 4.1. *Assume that the conditions of Theorem* 3.1 *are fulfilled and $2 < \sigma < \frac{2(p^+ - p^-)}{p^-(p^+ - 1)}$, $\operatorname{supp} u_0 \subset \Omega$. If u is a nonnegative solution of problem* (1.1) *and $f \equiv 0$, then*

$$\operatorname{supp} u \subset \operatorname{supp} u_0 \quad \text{a.e. in } Q_T.$$

Proof. By Definition 2.1, it is easily seen that there holds

$$\iint\limits_{Q_\tau}\big[u_\tau\xi + \big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla\xi\big]\,dx\,dt = 0, \tag{4.1}$$

with $\forall\tau \in (0,T)$.

Let $\Psi = \inf\{\operatorname{dist}(x, \operatorname{supp} u_0 \cup \partial\Omega)/\lambda, 1\}$ $(0 < \lambda < 1)$, and

$$F_\varepsilon(\xi) = \begin{cases} \frac{1}{\alpha-1}\varepsilon^{1-\alpha} - \frac{1}{\alpha-1}\xi^{1-\alpha} & \text{if } \xi > \varepsilon,\\ 0 & \text{if } \xi \leqslant \varepsilon,\end{cases}$$

with $\alpha = \frac{\sigma}{2}$.

Taking $\xi = \Psi F_\varepsilon(u)$ $(0 < \varepsilon < 1)$ and substituting it into (4.1) yields

$$0 = \iint\limits_{Q_{\varepsilon,\tau}} u_t\Psi F_\varepsilon(u)\,dx\,dt + \iint\limits_{Q_{\varepsilon,\tau}} \Psi\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla F_\varepsilon(u)\,dx\,dt$$
$$+ \iint\limits_{Q_{\varepsilon,\tau}} F_\varepsilon(u)\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u\nabla\Psi\,dx\,dt := I_1 + I_2 + I_3, \tag{4.2}$$

with $Q_{\varepsilon,\tau} = Q_\tau \cap \{(x,t) \in Q_\tau:\ u > \varepsilon\}$.

Denote $E = \{x \in \{\Psi = 1\};\ u(x,\tau) > \delta\}$ with $\delta > 2\varepsilon > 0$, then

$$I_1 = \iint\limits_{Q_{\varepsilon,\tau}} \Psi u_t F_\varepsilon(u)\,dx\,dt \geqslant \int\limits_{\Omega_\varepsilon} \chi_{\operatorname{supp}\Psi}\Psi \int\limits_{\varepsilon}^{u} F_\varepsilon(s)\,ds\,dx$$

$$\geqslant \int\limits_{\Omega_\delta} \chi_{\operatorname{supp}\Psi}\Psi(u-\varepsilon)F_\varepsilon(\delta)\,dx \geqslant (\delta - 3\varepsilon/2)F_\varepsilon(3\varepsilon/2)\mu(E), \tag{4.3}$$

$$I_2 = \iint\limits_{Q_{\varepsilon,\tau}} \Psi\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u \nabla \frac{1}{\alpha-1}\big(-u^{1-\alpha}\big)\,dx\,dt$$

$$\geqslant \iint\limits_{Q_{\varepsilon,\tau}} \Psi\big(u^\sigma + d_0\big)u^{-\alpha}|\nabla u|^{p(x,t)}\,dx\,dt \geqslant 0. \tag{4.4}$$

Applying Young's inequality with η and choosing $\eta = (\varepsilon^\beta)^{1-p(x,t)}$, we may estimate $|I_3|$

$$|I_3| = \left|\iint\limits_{Q_{\varepsilon,\tau}} F_\varepsilon(u)\big(u^\sigma + d_0\big)|\nabla u|^{p(x,t)-2}\nabla u \nabla\Psi\,dx\,dt\right|$$

$$\leqslant C \iint\limits_{Q_{\varepsilon,\tau}} \varepsilon^{1-\alpha}|\nabla u|^{p(x,t)-1}|\nabla\Psi|\,dx\,dt$$

$$\leqslant C\big(\sigma, d_0, K(T), p^\pm\big)\varepsilon^{\beta+\frac{(1-\alpha)p^-}{p^--1}} \iint\limits_{Q_{\varepsilon,\tau}} |\nabla u|^{p(x,t)}\,dx\,dt + \varepsilon^{\beta(1-p^+)} \iint\limits_{Q_{\varepsilon,\tau}} |\nabla\Psi|^{p(x,t)}\,dx\,dt. \tag{4.5}$$

Choosing $\beta = \frac{\alpha p^- - 1}{p^- - 1} > 0$ and plugging these inequalities (4.3)–(4.5) into (4.2), we get

$$\frac{1}{2}\big[1-(3/2)^{1-\alpha}\big]\varepsilon^{2-\alpha-\beta+\frac{(\alpha-1)p^-}{p^--1}}\mu(E) \leqslant (\delta - 3\varepsilon/2)\big[1-(3/2)^{1-\alpha}\big]\varepsilon^{1-\alpha-\beta+\frac{(\alpha-1)p^-}{p^--1}}\mu(E)$$

$$\leqslant \widetilde{C}\big(1+\varepsilon^{\frac{(\alpha-1)p^-}{p^--1}-\beta p^+}\big), \tag{4.6}$$

with a constant $\widetilde{C}$ independent of ε.

Noting that $2 < \sigma < \frac{2(p^+-p^-)}{p^-(p^+-1)} < \frac{2p^+}{p^+-1}$, we have

$$1 < \alpha = \frac{\sigma}{2} < \frac{(p^+-p^-)}{p^-(p^+-1)}, \quad 1 - \beta + \frac{(\alpha-1)p^-}{p^--1} = 0; \tag{4.7}$$

$$\frac{(\alpha-1)p^-}{p^--1} - \beta p^+ = \frac{(p^+-p^-) - \alpha p^-(p^+-1)}{p^--1} > 0. \tag{4.8}$$

Assume that there exists a $\tau_0 \in (0,T)$ such that $\mu(E) \neq 0$. Thus, by virtue of (4.7)–(4.8), letting $\varepsilon \to 0$ in (4.6) yields a contradiction. Hence, for all $\delta \in (0,1)$ and a.e. $\tau \in (0,T)$, we have

$$\mu\big(\big\{x \in \{\Psi = 1\};\ u(x,\tau) > \delta\big\}\big) = 0.$$

Then Theorem 4.1 follows from this and the arbitrariness of $\lambda \in (0,1)$. □

Acknowledgment

The authors would like to thank the referees for their helpful comments and suggestions.

References

[1] M. Tsutsumi, On solutions of some doubly nonlinear degenerate parabolic equations with absorption, J. Math. Anal. Appl. 132 (1988) 187–212.
[2] E. Dibenedetto, Degenerate Parabolic Equations, Springer-Verlag, New York, 1993.
[3] W.S. Zhou, Z.Q. Wu, Existence and nonuniqueness of weak solutions of the initial-boundary value problem for $u_t = u^\sigma \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Northeast. Math. J. 21 (2005) 189–206.
[4] D. Erdem, Blow-up of solutions to quasilinear parabolic equations, Appl. Math. Lett. 12 (1999) 65–69.
[5] M. Ruzicka, Electrorheological Fluids: Modelling and Mathematical Theory, Lecture Notes in Math., vol. 1748, Springer, Berlin, 2000.
[6] S.N. Antontsev, S.I. Shmarev, Anisotropic parabolic equations with variable nonlinearity, CMAF, University of Lisbon, Portugal, preprint 2007-013, 2007, pp. 1–34.
[7] S.N. Antontsev, S.I. Shmarev, Blow up of solutions to parabolic equations with nonstandard growth conditions, CMAF, University of Lisbon, Portugal, preprint 2009-02, 2009, pp. 1–16.

[8] S.N. Antontsev, S.I. Shmarev, Parabolic Equations with Anisotropic Nonstandard Growth Conditions, Internat. Ser. Numer. Math., vol. 154, 2007, pp. 33–44.
[9] Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (2006) 1383–1406.
[10] R. Aboulaich, D. Meskine, A. Souissi, New diffusion models in image processing, Comput. Math. Appl. 56 (2008) 874–882.
[11] F. Andreu-Vaillo, V. Caselles, J.M. Mazón, Parabolic Quasilinear Equations Minimizing Linear Growth Functionals, Progr. Math., vol. 223, Birkhäuser Verlag, Basel, 2004.
[12] J.N. Zhao, Existence and nonexistence of solutions for $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$, J. Math. Anal. Appl. 172 (1993) 130–146.