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Higher Point Derivations on Commutative Banach Algebras, I

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A higher point derivation on a commutative algebra is a finite or infinite sequence of linear functionals connected by the condition that they satisfy the Leibnitz identities. We discuss here higher point derivations on commutative Banach algebras. We study the extent to which point derivations are automatically continuous, and, for certain Banach algebras, we consider the maximum possible order of a higher point derivation. In particular, we obtain a complete description of the order and continuity properties of higher point derivations on the Banach algebra of n-times continuously differentiable functions on the unit interval.

INTRODUCTION

Let A be a commutative algebra with identity 1 over the complex field $\mathbf{C}$, and let d_0 be a character (nonzero multiplicative linear functional) on A. A *point derivation of order* q (respectively, *of infinite order*) *on* A *at* d_0 is a sequence $d_1,..., d_q$ (respectively, $d_1, d_2,...$) of linear functionals on A such that, for f and g in A and $k = 1,..., q$ (respectively, 1, 2,...),

$$d_k(fg) = \sum_{j=0}^{k} d_j(f)\, d_{k-j}(g). \tag{1.1}$$

These equations will be called the *(normalized) Leibnitz identities*.

When it can cause no confusion, we may omit reference to the character d_0 and the algebra A in the terminology. Also, we may use the term "point deriva-

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tion" without specifying the order; in such cases, either the order will be clear from the context or the statement will apply to point derivations of any order. Our "point derivations" are often called "higher point derivations" in the literature, and "point derivations of order 1" are elsewhere termed "point derivations"; they are often studied in the context of a search for analyticity in the character space of a commutative Banach algebra, but this is not our present concern.

We say that a point derivation $d_1, \ldots, d_q$ *belongs to a point derivation of order* p (where $p > q$) if there are linear functionals $d_{q+1}, \ldots, d_p$ such that $d_1, \ldots, d_p$ is a point derivation of order p. If A is a topological algebra, a point derivation on A is *continuous* if d_j is continuous for each $j \geqslant 1$, and it is *totally discontinuous* if d_j is discontinuous for each $j \geqslant 1$.

A point derivation is, of course, a special case of a higher derivation, or system of derivations, into a module, the module being the complex field and the module action being determined by a character. The algebraic theory of the general situation has received a good deal of attention; one fairly recent survey is the paper of Heerema [8]. More recently, Badé and Curtis [2] have investigated continuity properties of derivations (of order 1, in our terminology) in the case that A is a Banach algebra and the module is a Banach A-module. When the module is the algebra itself, or a superalgebra, or an algebra containing a homomorphic image of the algebra, there are a number of results about the continuity of derivations; see, for example, [7, 9], and some of the references in those articles. In contrast, relatively little seems to have been done in the case of point derivations. We shall study these point derivations in a series of articles, concentrating on the determination of the point derivations for particular Banach algebras.

We shall be concerned with the following question (for a Banach algebra A and a character d_0 on A): Is there a function $q \mapsto p(q)$ on the positive integers such that, whenever a point derivation of order q belongs to a point derivation of order $p(q)$, the point derivation of order q is necessarily continuous? We show, for many of the familiar Banach algebras known to have discontinuous point derivations, that the answer is "yes," with $p(q)$ no larger than $2q$. In particular, in the present paper, we examine the point derivations on the algebra $C^{(n)} = C^{(n)}([0, 1])$ of n-times continuously differentiable functions on $[0, 1]$ ($n = 1, 2, \ldots$) and we show that, for these algebras, $p(q) = 2q$ is the exact bound. Note that $p(q)$ is independent of n.

We also consider the question: Given k, what is the maximum order of a point derivation satisfying the condition $d_k \neq 0$? We obtain in Section 2 a general result, which, although found with certain Banach algebras in mind, is entirely algebraic, and in Section 3 we obtain for the algebras $C^{(n)}$ the exact bound, which is $(2n + 1)k - 1$. Thus, our theme is that on certain commutative Banach algebras, the linear functionals in the "first half" of a point derivation are necessarily continuous, and that, if the order of the point derivation is sufficiently large, a certain initial segment will be *trivial* in the sense that it consists entirely of zero functionals.

The theory of point derivations (in our sense) has some connection with the theory of derivations from Banach algebras into finite-dimensional modules; see [3] for a discussion of this topic, particularly in relation to the algebras $C^{(n)}$.

In a subsequent paper, we hope to study in some detail the point derivations on certain other commutative Banach algebras.

Examples of discontinuous point derivations of order 1 are well known [1]; such objects are of interest in connection with the existence of discontinuous homomorphisms from Banach algebras [1, 5, 6, 10]. In a further paper, we shall show that there exist totally discontinuous point derivations of any order, including infinite order. This implies the existence of a discontinuous homomorphism from a Banach algebra into $\mathbf{C}[[X]]$ (in fact, we shall construct an epimorphism), and it also shows that there is not always a function $q \mapsto p(q)$ of the type referred to above.

1. Preliminaries

In this section, we establish some conventions and further terminology, and make some preliminary observations.

All the algebras considered in this paper will be commutative, linear, associative algebras over the complex field $\mathbf{C}$. They will not be assumed to have an identity unless this is explicitly stated; however, we shall usually write A for an algebra with identity and M for an algebra without identity, and we shall write $M \oplus \mathbf{C}1$ for the algebra formed from M by adjoining an identity.

If V is a vector space, if λ is a linear functional on V, and if $S \subset V$, then we write $\lambda \perp S$ to mean that $\lambda(v) = 0$ $(v \in S)$.

If F and G are nonempty subsets of an algebra A, then FG denotes the set of all sums $\sum_{i=1}^{n} \lambda_i f_i g_i$, where $\lambda_i \in \mathbf{C}$, $f_i \in F$, $g_i \in G$, and n is an arbitrary positive integer. If F or G is an ideal in A, then FG is also an ideal in A. In particular, if M is an ideal in A, we shall often consider the descending chain of ideals M, M^2, $M^3, \ldots$.

Suppose that A has an identity and that d_0 is a character on A. Then any point derivation at d_0 is easily seen to satisfy

$$d_j(1) = 0 \qquad (j \geqslant 1). \tag{1.2}$$

Also, if we restrict Eqs. (1.1) to elements in $\ker d_0$, we get

$$\begin{aligned} d_1(fg) &= 0, \\ d_k(fg) &= \sum_{j=1}^{k-1} d_j(f)\, d_{k-j}(g) \qquad (k \geqslant 2), \end{aligned} \tag{1.3}$$

for f and g in $\ker d_0$.

Now let M be an algebra without identity. A *point derivation at* ∞ *on* M is a sequence (finite or infinite, with the terminology concerning the order as before) of linear functionals on M such that Eqs. (1.3) hold for f and g in M.

It is a familiar fact that point derivations of order 1 at d_0 on A (with identity) are characterized by the property $d_1 \perp \mathbf{C}1 + (\ker d_0)^2$, and it is not hard to see that point derivations of all orders are characterized by Eqs. (1.2) and (1.3). Thus, point derivations at ∞ on M could be defined (equivalently) as the restrictions to M of point derivations on $M \oplus \mathbf{C}1$ at the character whose kernel is M.

Another comment about Eqs. (1.3) may be in order. These equations show that the nature of the point derivations at d_0 is strongly related to the ideal $(\ker d_0)^2$ (and, via relations of the form $d_j \perp (\ker d_0)^{j+1}$, to higher powers of $\ker d_0$). Although we do not emphasize this point of view in our exposition, the reader can observe that each of our positive results is based on some properties, usually of a function–theoretic nature, of the ideal $(\ker d_0)^2$ and/or other closely related ideals. For example, Theorem 3.1 is crucial to later results in Section 3, and if we took the point of view that the theory of point derivations was a part of ideal theory in general, we would say that Theorem 3.1 was the main result of Section 3.

Let $\mathscr{F} = \mathbf{C}[[X]]$ denote the algebra of formal power series in one variable over $\mathbf{C}$, and throughout let $p_j\colon \sum \lambda_i X^i \to \lambda_j$ denote the coordinate projections on $\mathscr{F}$. Then $\mathscr{F}$ is a complete Fréchet algebra with respect to the topology determined by the family of seminorms $\{|p_j|\}$. Let $\mathscr{F}_n$ denote the closed ideal $\bigcap_{j=0}^n \ker p_j$ of $\mathscr{F}$. Then the set of point derivations of infinite order at the character d_0 of a Banach algebra A corresponds bijectively to the set $\{\theta \in \mathrm{Hom}(A, \mathscr{F})\colon p_0 \circ \theta = d_0\}$: of course, $\theta(a) = \sum d_i(a)X^i$, $d_i = p_i \circ \theta$. Similarly, point derivations of order n correspond bijectively to homomorphisms into $\mathscr{F}/\mathscr{F}_n$. These homomorphisms are continuous if and only if the corresponding point derivations are continuous. It is noted in [12] that the automorphisms of $\mathscr{F}$ correspond to maps of the form $\lambda \mapsto \lambda \circ \mu$ (formal composition of power series) for those μ in $\mathscr{F}$ such that $p_0(\mu) = 0$, $p_1(\mu) \neq 0$, and that these automorphisms are continuous. Clearly, these automorphisms induce continuous automorphisms on each $\mathscr{F}/\mathscr{F}_n$. This point of view leads to an easy proof of the following useful technical result.

1.1 Lemma. *Let A be an algebra with identity, let d_0 be a character on A, and let $d_1, d_2, \ldots$ be a point derivation at d_0 on A, of any order, infinite or finite. Choose a positive integer m and a complex number α, and define $D_0 = d_0$ and*

$$D_r = \sum_{i=0}^{r} \binom{r - i(m-1)}{i} \alpha^i d_{r-i(m-1)} \qquad (r \geq 1).$$

Then $D_1, D_2, \ldots$ is a point derivation at D_0 on A of the same order as $d_1, d_2, \ldots$. If A is a topological algebra and k is a positive integer, then $d_1, \ldots, d_k$ are continuous if and only if $D_1, \ldots, D_k$ are continuous.

Before beginning the proof, we must state the convention about binomial coefficients which applies in the above formula. If r and s are integers, then $\binom{r}{s} = 0$ unless both r and s are nonnegative with $r \geqslant s$, in which case $\binom{r}{s}$ is the usual binomial coefficient.

Proof. Let $\theta: a \to \sum d_i(a)X^i$ be a homomorphism from A into $\mathscr{F}$ or $\mathscr{F}/\mathscr{F}_n$, and let T be the automorphism of $\mathscr{F}$ or $\mathscr{F}/\mathscr{F}_n$ given by $T(\lambda) = \lambda \circ (X + \alpha X^m)$. Then

$$T\left(\sum_r \lambda_r X^r\right) = \sum_r \lambda_r (X + \alpha X^m)^r$$

$$= \sum_r \lambda_r X^r \sum_{i=0}^{r} \binom{r}{i} \alpha^i X^{i(m-1)}$$

$$= \sum_r \left(\sum_i \binom{r - i(m-1)}{i} \alpha^i \lambda_{r-i(m-1)}\right) X^r,$$

and we see that $D_r = p_r \circ T \circ \theta$. Thus, $D_1, D_2, \ldots$ is a point derivation, as required.

Clearly, $T \circ \theta$ is continuous if and only if θ is continuous, and the continuity assertion follows. The lemma is proved.

We conclude this section with a general result on the continuity of point derivations. The result is very simple, and may be known, but we have no reference for it. We need some more terminology, which will be used again later. As we have already indicated, the point derivation defined by $d_j = 0$ for $j \geqslant 1$ is called the *trivial* point derivation. A (nontrivial) point derivation is *degenerate* if $d_1 = 0$, and *nondegenerate* if $d_1 \neq 0$.

1.2 PROPOSITION. *Let A be a Banach algebra with identity, let d_0 be a character on A, and let $d_1, d_2, \ldots$ be a nondegenerate point derivation (of any order) on A at d_0. If $k \geqslant 1$ and d_k is continuous, then d_j is continuous for each $j = 1, 2, \ldots, k$.*

Proof. Let $M = \ker d_0$. It is sufficient to show that the appropriate linear functionals are continuous on M.

If $f \in M$, then $d_k(f^k) = d_1(f)^k$ (see Lemma 2.1), so that, if $f_n \to 0$ in M, $d_1(f_n) \to 0$, and it follows that d_1 is continuous. Since the point derivation is nondegenerate, we can take $f_0 \in M$ with $d_1(f_0) = 1$. Suppose inductively that $d_1, \ldots, d_{r-1}$ are continuous for some $r \in \{2, \ldots, k-1\}$. Now

$$d_k(f_0^{k-r} g) = d_r(g) + \sum_{s=1}^{r-1} d_{k-s}(f_0^{k-r})\, d_s(g) \qquad (g \in M).$$

Apply with $g = g_n$ where $g_n \to 0$ in M to deduce that $d_r(g_n) \to 0$ and that d_r is continuous, continuing the induction.

If the derivation is degenerate, the situation is more complicated, but a simple example shows that the above result need not hold. If $d_1, \ldots, d_q$ is a point derivation and if d is any first-order point derivation, then $0, d_1, 0, d_2, \ldots, 0, d_q, d$ is a point derivation.

2. A General Algebraic Result

Throughout this section, A is an algebra with identity, d_0 is a character on A, and we write M for the maximal ideal $\ker d_0$. The main result is Theorem 2.3; before stating it, we shall isolate as lemmas two of the computations required.

2.1 Lemma. *Let $d_1, \ldots, d_q$ be a point derivation (on A at d_0), let k be a positive integer, and suppose that $d_j = 0$ for $1 \leqslant j < k$ (vacuous if $k = 1$). Then, for any positive integer m such that $mk \leqslant q$ and any $f \in M$,*

(i) $d_{mk}(f^m) = d_k(f)^m$;

(ii) $d_r(f^m) = 0$ *if* $r < mk$.

Proof. The proof is by induction on m. Before beginning, we remark that in the case $k = 1$, the conclusions (i) and (ii) are the familiar (and easily checked) properties of point derivations that $d_m(f^m) = d_1(f)^m$ $(f \in M)$ and that $d_r \perp M^{r+1}$, respectively.

For the proof of the lemma, first observe that if $m = 1$, there is nothing to prove. Suppose that (i) and (ii) hold for some $m \geqslant 1$, and let f belong to M. Then, writing $f^{m+1} = ff^m$ and using Eqs. (1.1) and the information that $d_j = 0$ for $1 \leqslant j < k$,

$$d_{(m+1)k}(f^{m+1}) = d_k(f)\, d_{mk}(f^m) + \Sigma,$$

where Σ is a finite sum, each of whose terms has a factor $d_r(f^m)$ for some $r < mk$. So, using both (i) and (ii),

$$d_{(m+1)k}(f^{m+1}) = d_k(f)^{m+1}.$$

Also, if $r < (m + 1)k$, we have

$$d_r(f^{m+1}) = d_k(f)\, d_{r-k}(f^m) + \cdots,$$

and in this case all the terms vanish by (ii), so that $d_r(f^{m+1}) = 0$, as required.

2.2 Lemma. *Let $d_1, \ldots, d_p$ be a point derivation. Suppose that $g \in M$ and that m is a positive integer such that $d_r(g) = 0$ for $r \leqslant m$. Then, for any positive integer j,*

$$d_r(g^j) = 0 \qquad (r \leqslant \min\{p, jm + j - 1\}).$$

Proof. The proof is by induction on j. First observe that the case $j = 1$ is given. Suppose $k > 1$ and the result holds for all positive integers $j < k$. For any $r \in \{1,..., p\}$,

$$d_r(g^k) = \sum_{j=1}^{r-1} d_j(g)\, d_{r-j}(g^{k-1}).$$

The three conditions $j > m$, $r - j > (k - 1)m + k - 2$, and $r \leqslant km + k - 1$ cannot occur simultaneously, so the inductive hypothesis and the data imply that

$$d_r(g^k) = 0 \qquad (r \leqslant \min\{p, km + k - 1\}),$$

as required.

2.3 THEOREM. *Let A be an algebra with identity, let d_0 be a character on A, and let $M = \ker d_0$. Suppose there are a set $\Lambda \subset M$ and a positive integer n such that*

(i) *Λ generates M in A (i.e., $A\Lambda = M$);*

(ii) *given $f \in \Lambda$ and a positive integer p, there exists g in M such that $g^p = f^{np+1}$.*

Then, for each positive integer q, the only point derivation of order q on A at d_0 which belongs to a point derivation of order $q(2^q n + 1)$ is the trivial point derivation of order q.

Proof. Let $d_1 ,..., d_p$ be a point derivation of order p; we have to show that $p \geqslant q(2^q n + 1)$ implies that $d_1 = \cdots = d_q = 0$. Because of Eqs. (1.2), it is sufficient to show $d_j \perp M$ for $j = 1,..., q$, and for this it is sufficient by hypothesis (i) to show that $d_j(f) = 0$ for $f \in \Lambda$ and $j = 1,..., q$. The proof is by induction on q; hence, suppose that q is a positive integer and that, if $q > 1$, the result holds for the integers $1,..., q - 1$. If $p \geqslant q(2^q n + 1)$, then $d_j \perp M$ for $j < q$, and we must show that $d_q(f) = 0$ $(f \in \Lambda)$.

To obtain a contradiction, suppose there exists $f \in \Lambda$ such that $d_q(f) = \alpha \neq 0$. Take $g \in M$ such that $g^{2^q} = f^{2^q n+1}$. By Lemma 2.1(ii), $r \leqslant 2^q qn$ $(<q(2^q n + 1))$ implies that

$$d_r(f^{2^q n+1}) = 0,$$

and so

$$d_r(g^{2^q}) = 0.$$

Now an induction on j shows that

$$d_r(g^{2^{q-j}}) = 0 \qquad (r \leqslant 2^{q-j} qn) \tag{2.1}$$

for $j = 0, 1, \ldots, q$: If (2.1) holds for $j < k \leqslant q$, if $s \leqslant 2^{q-k}qn$, and if $d_r(g^{2^{q-k}}) = 0$ for all $r < s$, then

$$0 = d_{2s}(g^{2^{q-k+1}}) = d_s(g^{2^{q-k}})^2.$$

The case $j = q$ of (2.1) gives

$$d_r(g) = 0 \qquad (r \leqslant qn).$$

Therefore, $d_{q(2^qn+1)}(g^{2^q}) = 0$ by Lemma 2.2, since $q(2^qn + 1) \leqslant 2^qqn + 2^q - 1$. On the other hand,

$$d_{q(2^qn+1)}(f^{2^qn+1}) = \alpha^{2^qn+1} \neq 0,$$

by Lemma 2.1(i). This is the required contradiction.

2.4 COROLLARY. *A nondegenerate point derivation on A at d_0 has order at most $2n$. The only point derivation of infinite order on A at d_0 is the trivial point derivation of infinite order.*

Before pointing out some of the algebras to which Theorem 2.3 can be applied, we remark that for any particular value of n, it may be possible to improve on the number $q(2^qn + 1)$ of Theorem 2.3. For example, using rather more complicated calculations, it can be shown that in the case $n = 1$ a point derivation of order $p \geqslant q(q + 2)$ has $d_1 = \cdots = d_q = 0$. We do not know if this is best-possible.

We conclude this section with some applications of Theorem 2.3.

2.5 EXAMPLE. *Algebras of differentiable functions on an interval.*

For each positive integer n, let $C^{(n)} \equiv C^{(n)}([0, 1])$ denote the set of functions having at least n continuous derivatives on $[0, 1]$ (one-sided derivatives at 0 and 1). With pointwise operations and the norm

$$\|f\| = \sum_{j=0}^{n} \frac{1}{j!} \sup\{|f^{(j)}(t)|: t \in [0, 1]\} \qquad (f \in C^{(n)}),$$

$C^{(n)}$ is a Banach algebra, singly generated by the coordinate function x (where $x(t) = t$ for $t \in [0, 1]$), and natural on $[0, 1]$. The theory of derivations from this algebra into modules is discussed in [2, 3].

Fix a point $t_0 \in [0, 1]$ and let $M = \{f \in C^{(n)}: f(t_0) = 0\}$ be the corresponding maximal ideal. Then Theorem 2.3 applies with the n from $C^{(n)}$ and with $\Lambda = \{f \in M: f \text{ is real-valued}\}$. It is certainly the case that Λ generates M. Also, for any positive integer p, one can define $t^{n+(1/p)}$ ($t \in \mathbf{R}$) so that the resulting function has n continuous derivatives and vanishes at 0 (but is not, of course, necessarily real-valued). Therefore, for each $f \in \Lambda$, one can produce $f^{n+(1/p)} \equiv$

$g \in M$, and then $g^p = f^{np-1}$, as required. Thus, any nondegenerate point derivation on $C^{(n)}$ has order at most $2n$, and the only homomorphisms $C^{(n)} \to \mathscr{F}$ are the obvious ones of the form $f \mapsto f(t_0)1$, for some fixed $t_0 \in [0, 1]$.

We had the algebras $C^{(n)}$ in mind when we found Theorem 2.3; we shall return to them in Section 3 and study their point derivations more fully.

2.6 Example. *Algebras of differentiable functions on a disc.*

Let U be the open unit disc in $\mathbf{C}$, let n be a positive integer, and let $D^{(n)} = D^{(n)}(\overline{U})$ be the set of functions which are analytic on U and whose derivatives up to and including order n are uniformly continuous on U. With pointwise operations and norm analogous to that defined on $C^{(n)}$, $D^{(n)}$ is a Banach algebra, singly generated by the coordinate function z, and natural on $\overline{U}$.

Now Theorem 2.3 certainly does not apply at points of U. At such points there are nondegenerate point derivations of all orders, including infinite order, because of the analyticity of functions in $D^{(n)}$ at points of U. Clearly, every point derivation at a point of U is continuous.

Fix a point ξ_0 in $T = \{\xi \in \mathbf{C}: |\xi| = 1\}$, and let M be the corresponding maximal ideal $\{f \in D^{(n)}: f(\xi_0) = 0\}$. We claim that now Theorem 2.3 applies, with the n from $D^{(n)}$ and with

$$\Lambda = \{f \in M: f(\xi) \neq 0\ (\xi \in \overline{U}\backslash\{\xi_0\})\}.$$

It is fairly straightforward to see that condition (ii) of Theorem 2.3 is satisfied. For, given $f \in \Lambda$ and a positive integer p, the facts that U is simply connected and that f does not vanish on U allow us to produce $f^{1/p}$, analytic on U and uniformly continuous there (since f was). Then $g = f^n f^{1/p}$ will satisfy the requirements of condition (ii). The following theorem, which we state for reference in a later paper, leads to the fact that Λ generates M.

Theorem. *For each $f \in M$ with $f'(\xi_0) \neq 0$, there are $g \in D^{(n)}$ and $h \in M$ such that $g(\xi_0) \neq 0$, $h(\xi) \neq 0$ if $\xi \neq \xi_0$, and $f = gh$.*

Proof. Since $f'(\xi_0) \neq 0$, we can choose a relative neighborhood V of ξ_0 so that f does not vanish in $V\backslash\{\xi_0\}$. Therefore, we can choose a relatively open cover $\{V_1, V_2\}$ of $\overline{U}$, with $\xi_0 \in V_1$, say, and with ∂V_i and $\partial(V_1 \cap V_2)$ as smooth as we like, and so that $f \mid (V_1 \cap V_2)$ is an invertible element of $D^{(n)}(\overline{V_1 \cap V_2})$ (obvious definition). By Theorem 20 of [11], there are invertible elements F_i in $D^{(n)}(\overline{V}_i)$ such that $f = F_1F_2$ in $V_1 \cap V_2$. Therefore, $fF_1^{-1} = F_2$ in $V_1 \cap V_2$, and defining

$$\begin{aligned} h(\xi) &= f(\xi)F_1(\xi)^{-1} \qquad (\xi \in V_1), \\ &= F_2(\xi) \qquad (\xi \in V_2), \end{aligned}$$

and $g = f/h$ gives the required factorization.

To see that A generates M, observe that both $z - \xi_0$ and the function h of the theorem belong to A, and that any $f \in M$ can be written as $\alpha(z - \xi_0) + f_1$ for some $\alpha \in \mathbf{C}$ and some $f_1 \in M$ such that $f_1'(\xi_0) \neq 0$ (any $\alpha \neq f'(\xi_0)$ will do).

Hence, again, we can conclude that a nondegenerate point derivation on $D^{(n)}$ at a point of T has order at most $2n$.

We are grateful to Professor E. L. Stout, who referred us to Nagel's paper and who also provided an ingenious direct proof of the factorization theorem given above. The algebras $D^{(n)}$ will be studied later. Note that as far as the above argument is concerned, the unit disc could be replaced by any simply connected domain with a sufficiently smooth boundary, the degree of smoothness depending on n; see [11] for details.

2.7 EXAMPLE. *Point derivations at infinity on l^p.*

Let p be a real number, $1 \leqslant p < \infty$, and let l^p denote the Banach space of pth power summable sequences. With coordinate-wise multiplication, l^p is a Banach algebra without identity. Theorem 2.3 applies to l^p, with $n = 1$ and $A = \{s = (s_j) \in l^p \colon s_j \in \mathbf{R} \text{ for all } j\}$. Thus, for example, a nondegenerate point derivation at ∞ on l^p has order at most 2. In fact, we can show that more is true, and we can determine the nature of all point derivations for this algebra. The point derivations of order 1 are well known [1]: since $(l^p)^2 = l^{p/2}$, which is dense in l^p, the space of such point derivations is infinite-dimensional, and all but the zero functional are discontinuous. So, given a point derivation d_1, d_2, a proof that d_1 is continuous entails a proof that $d_1 = 0$. This is a consequence of the next theorem.

THEOREM. *Every nondegenerate point derivation at ∞ on l^p ($1 \leqslant p < \infty$) has order* 1.

Proof. Suppose, if possible, that d, d_2 is a point derivation of order 2 and that there exists $f_0 \in l^p$ with $d(f_0) = 1$.

First note that there exists a function h on $\mathbf{N}$ such that $h(n) \neq 0$ for $n \in \mathbf{N}$, $h(n) \to \infty$ as $n \to \infty$, and $hf_0 \in l^p$. For, set $n_0 = 1$ and define inductively a sequence $(n_k \colon k = 1, 2,\ldots)$ such that $n_{k+1} > n_k$ and $\sum_{n_k}^{\infty} | f_0(n)|^p < 2^{-(k+1)}$. Let

$$h(n) = k \qquad (n_{k-1} \leqslant n < n_k\,, k = 1, 2,\ldots).$$

Then $\sum_1^{\infty} |(hf_0)(n)|^p \leqslant \sum_{k=1}^{\infty} (k^p/2^k) < \infty$, so that h is the required function.

Let $f_1 = hf_0$ and let $g = h^{-1}$, so that $gf_0 \in l^p$. Now $f_1 g f_0 = f_0^2$, so

$$d(f_1)\, d(gf_0) = d_2(f_1 g f_0) = d_2(f_0^2) = d(f_0)^2 = 1,$$

whence $d(f_1) \neq 0$. By replacing h by a scalar multiple, we can suppose that $d(f_1) = 1$.

If $F \in C \equiv C(\mathbf{N} \cup \{\infty\})$, then $Ff_1 \in l^p$ (identifying F with its restriction to $\mathbf{N}$). Define $\lambda \colon C \to \mathbf{C}$ by $\lambda(F) = d(Ff_1)$. Then λ is a linear functional and $\lambda(FG) =$

$d(FGf_1)\,d(f_1) = d(Ff_1)\,d(Gf_1) = \lambda(F)\,\lambda(G)$ for $F, G \in C$, so that λ is a character on C, say $\lambda(F) = F(x)$ $(F \in C)$ for some $x \in \mathbf{N} \cup \{\infty\}$. If $x \in \mathbf{N}$, $\delta_x f_1 \in (l^p)^2$, so that $1 = \lambda(\delta_x) = d(\delta_x f_1) = 0$, a contradiction. If $x = \infty$, set $g(\infty) = 0$ so that $g \in C$. Then $d(f_0) = d(gf_1) = \lambda(g) = g(\infty) = 0$, again a contradiction. The theorem is proved.

2.8 Example. *Algebras of Lipschitz functions.*

Let (X, m) be a metric space and write $\mathrm{Lip}(X, m)$ for the set of bounded functions f on X such that $|f|_m < \infty$, where

$$|f|_m = \sup\{|f(x) - f(y)|/m(x, y): x, y \in X, x \neq y\}.$$

Clearly, such functions are uniformly continuous on the metric space (X, m). The system $\mathrm{Lip}(X, m)$ has been studied by Sherbert [13] who showed that with pointwise operations and the norm

$$\|f\| = |f|_X + |f|_m$$

(where $|f|_X = \sup\{|f(x)|: x \in X\}$), $\mathrm{Lip}(X, m)$ is a regular Banach algebra. Sherbert studied point derivations of order 1 on $\mathrm{Lip}(X, m)$, and in particular obtained characterizations of the space of continuous, first-order point derivations associated with a point $x_0 \in X$. In the present context, we observe that Theorem 2.3 applies to $\mathrm{Lip}(X, m)$, again with $n = 1$ and

$$A = \{f \in \mathrm{Lip}(X, m): f(x_0) = 0 \text{ and } f \text{ is real-valued}\}.$$

Thus, as in the last example, a nondegenerate point derivation at x_0 has order at most 2. An analysis of the point derivations of order 2 on this algebra will be given in a later paper.

3. The Algebras $C^{(n)}([0, 1])$

In this section, we continue the study of the point derivations on the algebras $C^{(n)}$ $(=C^{(n)}([0, 1]))$ for $n = 1, 2,\ldots$, introduced in Example 2.5. We shall give what amounts to a complete description of the point derivations on these algebras, and in particular we shall answer the question raised in the Introduction: Find $p(q)$ such that whenever a point derivation of order q belongs to a point derivation of order $p(q)$, the point derivation of order q is necessarily continuous. Also, we shall improve the upper bound implied by Theorem 2.3 for the order of a point derivation which begins with a specified number of zeros, giving a sharp bound.

First, we point out that the point derivations on $C^{(n)}$ at one point of $[0, 1]$ are formally equivalent to those at any other point. This is not immediately obvious;

e.g., it is probably natural to wonder whether there might be a difference between endpoints and interior points of [0, 1]. However, the reader will observe that for any $t_0 \in [0, 1]$, the function $x - t_0$ plays a (crucial) role in the investigation of the point derivations at t_0 entirely parallel to the role played by the function x with regard to the point derivations at 0. Therefore, we shall fix our attention on the point derivations at 0.

For any nonnegative integer k and any function f which has a kth derivative (in the usual sense) at 0, we write

$$\delta_k(f) = f^{(k)}(0)/k!.$$

We use the usual convention about the 0th derivative. For any $n \geqslant 1$ and each $k = 1,..., n$, it is clear that δ_k is a continuous linear functional on $C^{(n)}$, and also that $\delta_1 ,..., \delta_k$ is a point derivation on $C^{(n)}$ at δ_0. We may refer to these as the *obvious* point derivations on $C^{(n)}$; one of the aims of this section will be to show the relationship of the obvious point derivations to point derivations in general (cf. [6], particularly Corollary 5.1.2).

As might be expected, certain ideals in $C^{(n)}$ associated with the point 0 play an important role in the study of the point derivations. We will fix some notation, setting, for $k = 0, 1,..., n$,

$$M_{n,k} = \{f \in C^{(n)}: f^{(j)}(0) = 0 \ (j = 0, 1,..., k)\}.$$

The $M_{n,k}$ form a descending (finite) sequence of closed ideals in $C^{(n)}$. The first result of this section gives the properties of these ideals which we shall need.

3.1 THEOREM. *Let n be a positive integer.*

(i) $(M_{n,n})^2 = x^n M_{n,n}$.

(ii) *For* $0 \leqslant k < n$, $(M_{n,k})^2 = x^{k+1} M_{n,k}$.

Proof. (i) We need the following technical fact. If $f \in M_{n,n}$, then there is a function ϕ such that: ϕ is infinitely differentiable on (0, 1] and ϕ is continuous on [0, 1];

$$\phi(0) = 0, \qquad \text{and} \qquad \phi(t) > 0 \qquad \text{for} \quad t > 0;$$

$$\phi(t) \geqslant \sup\{|f^{(n)}(s)|: 0 \leqslant s \leqslant t\};$$

$$\lim_{t\to 0+} f^{(n)}(t)/\phi(t) = 0.$$

To prove this, suppose (without loss of generality) that $|f^{(n)}(t)| < 1$ for $t \in [0, 1]$, and define $g(t) = \sup\{|f^{(n)}(s)|: 0 \leqslant s \leqslant t\}$ for $t \in [0, 1]$. Then g is nonnegative, continuous, and nondecreasing on [0, 1], $g(0) = 0$, and $g(1) < 1$.

We can choose a sequence of points $(t_i) \subset (0, 1)$, decreasing to 0, and for each i a step function h_i on $[t_i, 1]$ such that

$$\begin{array}{ll} 1 > h_i > g & \text{on } [t_i, 1]; \\ h_i & \text{is nondecreasing;} \\ \text{if } j > i, & h_j = h_i \text{ on } [t_i, 1]; \\ h_i(t_i) \to 0 & \text{as } i \to \infty. \end{array}$$

Now define h on $(0, 1]$ by $h(t) = h_i(t)$ if $t \geqslant t_i$. From h produce a "polygonal" function k by taking the line segments which join the left-hand ends of adjacent "steps" of h; observe that we have $1 > k > g$ on $(0, 1]$, and $\lim_{t\to 0+} k(t) = 0$. Round off the corners of k to produce ψ, infinitely differentiable on $(0, 1]$; clearly, we can do this so that $1 > \psi > g$ on $(0, 1]$ and $\lim_{t\to 0+} \psi(t) = 0$. Define $\psi(0) = 0$, so that ψ is continuous on $[0, 1]$. We have $|f^{(n)}/\psi| < 1$ on $(0, 1]$, and it is now easy to see that $\phi = \psi^{1/2}$ will satisfy all the requirements.

To prove (i), fix $f \in M_{n,n}$. We shall find $h \in M_{n,n}$ and $k \in M_{n,n}$ such that $x^n f = hk$. Take ϕ related to f as above, and define k on $[0, 1]$ by

$$k(t) = \int_0^t (t - s)^{n-1} \phi(s)\, ds.$$

(The motivation for this is that the integral used to define k is a multiple of $\int_0^t \int_0^u \cdots \int_0^w \int_0^r \phi(s)\, ds\, dr \cdots dv\, du$ (n integrations), as can be seen by changing the order of the integrations.) Thus,

$$k^{(j)}(t) = (n-1) \cdots (n-j) \int_0^t (t-s)^{n-j-1} \phi(s)\, ds \qquad (j = 1, \ldots, n-1)$$

and

$$k^{(n)}(t) = (n-1)!\, \phi(t),$$

so that, clearly, $k \in M_{n,n}$. Now define h on $[0, 1]$ by

$$h(t) = t^n f(t)/k(t) \qquad (t \in (0, 1]),$$
$$h(0) = 0.$$

We write $h = x^n f/k$ for short. Note that k is positive on $(0, 1]$. Since $x^n f = hk$, we will have the required factorization of $x^n f$ once we have shown that $h \in M_{n,n}$. It is clear that h has n continuous derivatives on $(0, 1]$, so only the behavior at 0 has to be determined. This is done by direct calculation, and we omit most of the details, which are straightforward. The crucial fact in that $h(t)/t^n \to 0$ as $t \to 0+$ (and hence $h(t)/t^j \to 0$ as $t \to 0+$ for $j = 0, 1, \ldots, n$), which follows from the equation $h(t)/t^n = f(t)/k(t)$ for $t \neq 0$ and l'Hôpital's rule:

$$\lim_{t\to 0+} f(t)/k(t) = \lim_{t\to 0+} f^{(n)}(t)/k^{(n)}(t) = [(n-1)!]^{-1} \lim_{t\to 0+} f^{(n)}(t)/\phi(t) = 0.$$

We have shown that $x^n M_{n,n} \subset (M_{n,n})^2$, and this is the difficult part of the proof. To see the reverse inclusion, suppose $h \in M_{n,n}$ and $k \in M_{n,n}$, and take $f = hk/x^n$. As before, it is clear that f has n continuous derivatives on (0, 1], and to complete the proof that $f \in M_{n,n}$, it suffices as above to show that $f(t)/t^n \to 0$ as $t \to 0+$. But $f(t)/t^n = (h(t)/t^n)(k(t)/t^n)$, which tends to 0 as $t \to 0+$ by l'Hôpital's rule, since both h and k belong to $M_{n,n}$. This completes the proof of (i).

(ii) Since $x^{k+1} \in M_{n,k}$, the inclusion $x^{k+1}M_{n,k} \subset (M_{n,k})^2$ is clear. For the reverse inclusion, suppose that $f_i \in M_{n,k}$ for $i = 1, 2$. Then there are Taylor expansions

$$f_i = \sum_{j=k+1}^{n} \delta_j(f_i)\, x^j + R_n f_i ,$$

where $R_n f_i$ belong to $M_{n,n}$. To see that $f_1 f_2 / x^{k+1}$ belongs to $M_{n,k}$, simply multiply the two Taylor expansions and divide by x^{k+1}, using (i) to see that $(R_n f_1)(R_n f_2)/x^{k+1}$ belongs to $M_{n,n}$, which is contained in $M_{n,k}$. This proves (ii).

The first part of the above theorem was found by A. Browder and P. C. Curtis. We have given Browder's proof, in a form shown to us by W. G. Badé. It is entirely elementary. Another elegant proof, due to Curtis, depends on the fact that

$$\begin{aligned} A_n &= \{f/x^n : f \in M_{n,n}\} \\ &= \{g \in C^{(n)}((0, 1]) : t^j g^{(j)}(t) \to 0 \text{ as } t \to 0+ \ (j = 0, 1, \ldots, n)\} \end{aligned}$$

is a Banach algebra (without identity) with respect to the norm

$$\| g \| = \sum_{j=0}^{n} \left(\frac{1}{j!}\right) \sup\{t^j \mid g^{(j)}(t)| : t \in (0, 1]\},$$

and has a bounded approximate identity. Thus, by Cohen's theorem [4, 11.10], if $f \in M_{n,n}$, there are g and h in $M_{n,n}$ such that $f/x^n = (g/x^n)(h/x^n)$, and hence $x^n f = gh$.

Part (ii) of Theorem 3.1, in the particular case $k = 0$ must be regarded as the ancestor of the results in this section. It is found in [2], Example 3, with the additional information that

$$(M_{n,0})^2 = xM_{n,0} = \{f \in M_{n,0} : f'(0) = 0 \text{ and } f^{(n+1)}(0) \text{ exists}\}.$$

From the last equality, it follows, first, that $\overline{(M_{n,0})^2} = M_{n,1}$. Since continuous point derivations of order 1 on $M_{n,0}$ are just those linear functionals which annihilate $\overline{(M_{n,0})^2}$, the space of such derivations is one-dimensional and each continuous point derivation is a multiple of δ_1. Second, $(M_{n,0})^2$ has infinite codimension in $M_{n,0}$, and so the space of discontinuous point derivations of order 1 is infinite-dimensional. Third, we see that $C^{(n)}$ has a nondegenerate point derivation of order $n+1$. For certainly δ_{n+1} exists for f in $(M_{n,0})^2$, and, if d_{n+1} is any linear functional on $C^{(n)}$ such that $d_{n+1}(1) = 0$ and $d_{n+1} = \delta_{n+1}$ on $(M_{n,0})^2$, then $\delta_1, \ldots, \delta_n, d_{n+1}$ is a point derivation. Since we can easily find a set in $(M_{n,0})^2$ which is bounded with respect to the norm of $C^{(n)}$, but on which δ_{n+1} is unbounded, such a d_{n+1} is necessarily discontinuous. It is less clear whether or not there exist nondegenerate point derivations on $C^{(n)}$ of order q with $n+1 < q \leqslant 2n$; for this, see Theorem 3.4(iii).

Next we give a technical lemma to be used in the proofs of the main theorems of this section.

3.2 LEMMA. *Suppose* $f \in M_{n,n-1}$, $\delta_n(f) \neq 0$, *and* $f(t) \neq 0$ *if* $t \neq 0$. *Then* $g = x^{2n}/f \in M_{n,n-1}$ *and* $\delta_n(g) = 1/\delta_n(f)$.

Proof. Induction (on k) shows that, for $k = 0, 1, \ldots, n$, $(1/f)^{(k)}$ is a sum of terms of the form $c_\sigma f^{(\sigma)}/f^{|\sigma|+1}$, where $\sigma = \{r(j): j = 1, 2, \ldots\}$ is a finite sequence of nonnegative integers such that $\Sigma j r(j) = k$, c_σ is a numerical coefficient, $|\sigma| = \Sigma r(j)$, and $f^{(\sigma)} = (f^{(1)})^{r_1} \cdots (f^{(n)})^{r_n}$. Since

$$(x^{2n}/f)^{(p)} = \sum_{k=0}^{p} \binom{p}{k} (1/f)^{(k)} (x^{2n})^{(p-k)},$$

we see that $(x^{2n}/f)^{(p)}$ is a sum of terms of the form

$$c_{p,\sigma} x^{2n-p+k} f^{(\sigma)}/f^{|\sigma|+1}.$$

Ignoring the numerical coefficient $c_{p,\sigma}$, the last expression can be written

$$\left(\frac{f^{(1)}}{x^{n-1}}\right)^{r(1)} \cdots \left(\frac{f^{(k)}}{x^{n-k}}\right)^{r(k)} \frac{x^{2n-p+k+\Sigma r(j)(n-j)}}{f^{1+\Sigma r(j)}}$$

$$= \left(\frac{f^{(1)}}{x^{n-1}}\right)^{r(1)} \cdots \left(\frac{f^{(k)}}{x^{n-k}}\right)^{r(k)} \left(\frac{x^n}{f}\right)^{1+\Sigma r(j)} x^{n-p}.$$

Now, l'Hôpital's rule used repeatedly and the assumption that $\delta_n(f) \neq 0$ imply that

(a) $g = x^{2n}/f$ has $n-1$ continuous derivatives on $[0, 1]$, and $\delta_j(g) = 0$ for $j = 0, 1, \ldots, n-1$;

(b) $g^{(n)}$ exists and is continuous on $(0, 1]$, and $\lim_{t \to 0+} g^{(n)}(t)$ exists.

Therefore, $g \in M_{n,n-1}$. (It is a fact that if a function ϕ is differentiable in a deleted neighborhood of a point a and if $\lim_{t \to a} \phi'(t) = L$ exists, then $\phi'(a)$ exists and equals L.)

The fact that $\delta_n(g) = 1/\delta_n(f)$ can be seen by looking at the equation $fg/x^n = x^n$ and using Taylor expansions for f and g as in the proof of Theorem 3.1(ii).

This brings us to the first main result of this section, which gives the function $p(q)$ referred to in the Introduction for *nondegenerate* point derivations on $C^{(n)}$.

3.3 THEOREM. *Let n be a positive integer, and let $d_1, \ldots, d_p$ be a nondegenerate point derivation at 0 on $C^{(n)}$ of order p. If q is a positive integer and $p \geqslant 2q$, then the point derivation $d_1, \ldots, d_q$ is continuous.*

Proof. First, observe that the hypothesis of nondegeneracy forces $p \leqslant 2n$, as in Example 2.5. Also, if $p = 1$, there is nothing to prove, so we shall suppose that $p > 1$.

Since the point derivation is nondegenerate, $d_1(x) \neq 0$, for, given $f \in M_{n,0}$, there exists $g \in M_{n,0}$ with $f^2 = xg$, and, if $d_1(x) = 0$, $d_1(f)^2 = d_2(f^2) = d_2(xg) = d_1(x)\, d_1(g) = 0$, so that $d_1 = 0$, a contradiction.

We now transform the derivation, using Lemma 1.1. When applying this lemma, we shall always use the same symbols for the members of the point derivation both before and after the transformation. Thus, first we can suppose $d_1(x) = 1$ (Lemma 1.1, $m = 1$, $\alpha = d_1(x)^{-1} - 1$), and this implies that $d_i(x^j) = \delta_{ij}$ for $i = 1, \ldots, p$ and $j \geqslant i$ (Lemma 2.1). Apply Lemma 1.1 with $m = 2$ and $\alpha = -d_2(x)$. Then we have $d_2(x) = 0$, and it is easily checked that $d_i(x^{i-1}) = 0$ for $i = 2, 3, \ldots, p$. Apply Lemma 1.1 with $m = 3$ and $\alpha = -d_3(x)$. Then $d_3(x) = 0$ and hence $d_i(x^{i-2}) = 0$ for $i = 3, \ldots, p$. Continuing, we see that because of the continuity assertion in Lemma 1.1, we can suppose without loss of generality that

$$d_i(x^j) = \delta_{ij} \qquad (i = 1, \ldots, p, j = 1, 2, \ldots). \tag{3.1}$$

The proof of the theorem is by induction on q. In fact, the case $q = 1$ includes in simple form the method for the general inductive step. Note that the arguments in the cases $n = 1$ and $n > 1$ are different.

Thus, take $q = 1$ and $n = 1$. If $f \in M_{1,0}$ and $\delta_1(f) \neq 0$, then f does not vanish in some deleted neighborhood of 0. Changing f outside a neighborhood of 0 does not affect either $d_1(f)$ or $\delta_1(f)$, so that we can suppose $f(t) \neq 0$ if $t \neq 0$. Then $g = x^2/f \in M_{1,0}$, by Lemma 3.2, and therefore $1 = d_2(x^2) = d_2(fg) = d_1(f)\, d_1(g)$, so that $d_1(f) \neq 0$. This implies $\ker d_1 \subset \ker \delta_1$, and therefore d_1 is a multiple of δ_1, which shows that d_1 is continuous. (Because of the normalization carried out above, $d_1 = \delta_1$.) On the other hand, if $n > 1$ and $f \in M_{n,1}$, then, by Theorem 3.1(ii), there is $g \in M_{n,1}$ such that $f^2 = x^2 g$. In particular, $f^2 \in (M_{n,0})^3$, and therefore $d_1(f)^2 = d_2(f^2) = 0$, so $d_1(f) = 0$. This shows that $\ker \delta_1 \subset \ker d_1$, and again it follows that d_1 equals δ_1 and is continuous.

Now suppose that $2 < 2k + 2 \leqslant p$ and that $d_1, \ldots, d_k$ are continuous. Since $d_i(x^j) = \delta_{ij} = \delta_i(x^j)$, d_i and δ_i agree on polynomials, and so $d_i = \delta_i$ $(i = 1, \ldots, k)$. It follows that $d_{k+1} = \delta_{k+1} + d$ for some first-order point derivation d. (Note that $k + 1 \leqslant n$ because $2k + 2 \leqslant p \leqslant 2n$.) Clearly, d_{k+1} is continuous if and only if d is continuous. Since $d(x) = d_{k+1}(x) = 0$, d is continuous if and only if $d = 0$, and to show that $d = 0$ it suffices, by (3.1), to show that $d \perp M_{n,n}$.

Let f and g belong to $M_{n,k}$, where $1 \leq k \leq n - 1$. By Theorem 3.1, $fg/x^{k+1} \in M_{n,k}$, and so, using (3.1),

$$d_{2k+2}(fg) = d_{2k+2}\left(\frac{fg}{x^{k+1}} \cdot x^{k+1}\right) = d_{k+1}\left(\frac{fg}{x^{k+1}}\right).$$

Also,

$$d_{2k+2}(fg) = d_{k+1}(f)\, d_{k+1}(g),$$

and so

$$d_{k+1}\left(\frac{fg}{x^{k+1}}\right) = d_{k+1}(f)\, d_{k+1}(g) \qquad (f, g \in M_{n,k}). \tag{3.2}$$

In particular, if $k + 1 < n$ and $f \in M_{n,k+1}$, then we know that f^2/x^{k+1} belongs to $xM_{n,k+1}$ which is contained in $(M_{n,0})^2$, and so (3.2) implies that $d(f)^2 = d_{k+1}(f)^2 = d_{k+1}(f^2/x^{k+1}) = d(f^2/x^{k+1}) = 0$, whence $d(f) = 0$ for each f in $M_{n,n}$, as required.

Now consider the case when $k + 1 = n$. It is easy to check that $d_n \perp (M_{n,n-1})^2$. If λ is a linear functional on $M_{n,n-1}$ such that $\lambda(f) = d_n(f/x^n)$ whenever $f \in (M_{n,n-1})^2$, then (3.2) (with $k = n - 1$) implies that the pair d_n, λ is a point derivation of order 2 at ∞ on $M_{n,n-1}$. If $f \in M_{n,n-1}$ and $\delta_n(f) \neq 0$, then (as in the case $n = 1$) f is zero-free in some deleted neighborhood of 0, and since changing f outside a neighborhood of 0 affects neither $\delta_n(f)$ nor $d(f)$, we can assume that $f(t) \neq 0$ for $t \neq 0$. By Lemma 3.2, $g = x^{2n}/f$ belongs to $M_{n,n-1}$. Then $fg = x^{2n} = (x^n)^2$, so that $d_n(f)\, d_n(g) = \lambda(fg) = d_n(x^n)^2 = 1$, and therefore $d_n(f) \neq 0$. Thus $f \in M_{n,n-1}$ and $d_n(f) = 0$ together imply $\delta_n(f) = 0$, and so d_n and δ_n, as linear functionals on $M_{n,n-1}$, have the same kernel, namely $M_{n,n}$. Thus $d = d_n - \delta_n$ vanishes on $M_{n,n}$, as required.

This completes the proof.

The next result shows that all the situations not excluded by Theorem 3.3 (and Theorem 2.3) for nondegenerate point derivations on $C^{(n)}$ do actually occur. Also, the proof shows how the obvious point derivations are related to nondegenerate point derivations in general.

3.4 THEOREM. (i) *A continuous, nondegenerate point derivation on $C^{(n)}$ has order at most n.*

(ii) *A continuous, nondegenerate point derivation $d_1, \ldots, d_q$ on $C^{(n)}$ of order $q < n$ belongs to a point derivation of order $2q + 1$ with d_j discontinuous for $j = q + 1, \ldots, 2q + 1$.*

(iii) *A continuous, nondegenerate point derivation $d_1, \ldots, d_n$ on $C^{(n)}$ of order n belongs to a point derivation of order $2n$ with d_j discontinuous for $j = n + 1, \ldots, 2n$.*

Proof. (i) Assume that $d_1, \ldots, d_q$ is a continuous, nondegenerate point derivation with $q > n \geqslant 1$. As in the proof of Theorem 3.3, we can apply Lemma 3.3 and suppose that $d_i(x^j) = \delta_{ij}$ for $i = 1, \ldots, q$ and $j = 1, 2, \ldots$. Then the continuity of the d_i's and the density of the polynomials in $C^{(n)}$ imply $d_i = \delta_i$ for $i = 1, \ldots, n$. By a remark following Theorem 3.1, δ_{n+1} is defined on $(M_{n,0})^2$. Since both d_{n+1} and δ_{n+1} obey the Leibnitz identity with $\delta_1, \ldots, \delta_n$ for products of elements in $M_{n,0}$, we have $d_{n+1} = \delta_{n+1}$ on $(M_{n,0})^2$. But δ_{n+1} is discontinuous on $(M_{n,0})^2$ with respect to the norm of $C^{(n)}$, a contradiction.

(ii) Note that the case $n = 1$ is implicitly excluded here. Also note that, by Proposition 1.2, we need only show that d_{q+1} can be discontinuous in order to ensure that each d_j ($j = q + 1, \ldots, 2q + 1$) can be chosen to be discontinuous.

First, suppose that $d_i = \delta_i$ for $i = 1, \ldots, q$. Let d_{q+1} be a linear functional on $C^{(n)}$ such that

(a) if $f \in (M_{n,0})^2$, then $d_{q+1}(f) = d_q(f/x)$;

(b) $d_{q+1}(1) = d_{q+1}(x) = 0$;

(c) d_{q+1} is discontinuous.

Observe that the case $k = 0$ of Theorem 3.1(ii) implies that (a) can be satisfied. Also, (b) can be satisfied since neither 1 nor x belongs to $(M_{n,0})^2$. Finally, (c) can also be satisfied because $(M_{n,0})^2 = \{f \in M_{n,0}: f'(0) = 0 \text{ and } f^{(n+1)}(0) \text{ exists}\}$ ([2], Example 3), and so $(M_{n,0})^2$ has infinite codimension in $M_{n,0}$.

Now we must show that $d_1, \ldots, d_{q+1}$ is a point derivation, and it will be enough to show that, if f and g belong to $M_{n,0}$, then $d_{q+1}(fg) = \sum_{i=1}^{q} d_i(f)\, d_{q+1-i}(g)$. Take the Taylor expansion of f in the form

$$f = \sum_{i=1}^{q} \delta_i(f)\, x^i + R_q f, \tag{3.3}$$

where $R_q f \in M_{n,q}$, and take a similar expansion for g. Multiplying the two Taylor expansions gives

$$fg = \sum_{i=2}^{q+1} \left(\sum_{j=1}^{i-1} \delta_j(f)\, \delta_{i-j}(g) \right) x^i + \sum_{i=q+2}^{2q} \left(\sum_{j=i-q}^{q} \delta_j(f)\, \delta_{i-j}(g) \right) x^i$$
$$+ \sum_{i=1}^{q} (\delta_i(f)(R_q g) + \delta_i(g)(R_q f))\, x^i + (R_q f)(R_q g). \tag{3.4}$$

Because $d_i = \delta_i$ for $i = 1, \ldots, q$ and because of the properties of d_{q+1}, it is easy to see that $d_i(x^j) = \delta_{ij}$ for $i = 1, \ldots, q + 1$ and $j = 1, 2, \ldots$, and hence that

$$d_i(x^j F) = d_{i-j}(F) \qquad (F \in M_{n,0},\ i = 1, \ldots, q + 1, j < i).$$

(In fact, these relations motivate the definition of d_{q+1}.) Because of Theorem 3.1(ii), there exists h in $M_{n,q}$ such that

$$(R_q f)(R_q g) = x^{q+1}h. \tag{3.5}$$

With these facts in mind, we apply d_{q+1} to fg, using (3.4) and (3.5), and get

$$\begin{aligned} d_{q+1}(fg) &= \sum_{j=1}^{q} \delta_j(f)\,\delta_{q+1-j}(g) \\ &\quad + \sum_{i=1}^{q} (\delta_i(f)\,d_{q+1-i}(R_q g) + \delta_i(g)\,d_{q+1-i}(R_q f)) + d_1(xh) \\ &= \sum_{j=1}^{q} d_j(f)\,d_{q+1-j}(g), \end{aligned}$$

as required; the last equality is obtained from the facts that $d_i = \delta_i$ for $i = 1,\dots, q$ and that $R_q f$ and $R_q g$ belong to $M_{n,q}$.

We now suppose inductively that $q + 1 \leqslant k < 2q + 1$ and that we have a point derivation $d_1, \dots, d_k$ with $d_i = \delta_i$ for $i = 1,\dots, q$ and with $d_i(x^j) = \delta_{ij}$ for $i = 1,\dots, k$ and $j = 1, 2,\dots$. As before, there is a linear functional d_{k+1} such that $d_{k+1}(F) = d_k(F/x)$ for $F \in (M_{n,0})^2$ and $d_{k+1}(x) = d_{k+1}(1) = 0$. Again, we have $d_i(x^j) = \delta_{ij}$ for $i = 1,\dots, k + 1$ and $j = 1, 2,\dots,$ and

$$d_i(x^j F) = d_{i-j}(F) \qquad (F \in M_{n,0},\ i = 1,\dots, k + 1, j < i).$$

Also, using (3.3), we see that $d_i(f) = d_i(R_q f)$ for $i = q + 1,\dots, k$, and similarly for g. Applying d_{k+1} to fg, using (3.4) and (3.5), and taking account of the immediately preceeding remarks, gives

$$\begin{aligned} d_{k+1}(fg) &= \sum_{j=k+1-q}^{q} \delta_j(f)\,\delta_{k+1-j}(g) \\ &\quad + \sum_{i=1}^{q} (\delta_i(f)\,d_{k+1-i}(R_q g) + \delta_i(g)\,d_{k+1-i}(R_q f)) + d_{k-q}(h) \\ &= \sum_{j=k+1-q}^{q} \delta_j(f)\,\delta_{k+1-j}(g) + \sum_{i=1}^{k-q} (\delta_i(f)\,d_{k+1-i}(g) + \delta_i(g)\,d_{k+1-i}(f)) \\ &= \sum_{i=1}^{k} d_i(f)\,d_{k+1-i}(g). \end{aligned}$$

This proves that $d_1, \dots, d_{k+1}$ is a point derivation. Therefore, by induction, the required derivation of order $2q + 1$ exists.

Finally, if $d_1, \ldots, d_q$ is any nondegenerate, continuous point derivation of order q, where q is less than n, we can use Lemma 1.1 as in the proof of Theorem 3.3 to transform $d_1, \ldots, d_q$ into a continuous point derivation for which $d_i(x^j) = \delta_{ij}$ for $i = 1, \ldots, q$ and $j = 1, 2, \ldots$, and so we can transform $d_1, \ldots, d_q$ into $\delta_1, \ldots, \delta_q$. We carry out the above construction and then use the inverse of the transformation to go back from $\delta_1, \ldots, \delta_q$ to $d_1, \ldots, d_q$ modifying $d_{q+1}, \ldots, d_{2q+1}$ so that, in the end, we obtain a point derivation $d_1, \ldots, d_q, d_{q+1}, \ldots, d_{2q+1}$, as required.

(iii) This is very similar to (ii), and we omit the details. Observe that when the inductive construction of (ii) is used to extend $\delta_1, \ldots, \delta_n$, the process must stop when d_{2n} is reached. This is because of the difference between $(M_{n,n})^2$ and $(M_{n,k})^2$ for $k < n$, as expressed in Theorem 3.1; specifically, (3.5) must be replaced by the equation

$$(R_n f)(R_n g) = x^n h$$

for some h in $M_{n,n}$. However, we know by Theorem 2.3 that $C^{(n)}$ has no nondegenerate point derivation of order greater than $2n$. Note also that d_{n+1} is necessarily discontinuous by (i).

This completes the proof of the theorem.

It is a consequence of Theorem 3.4(iii) that the estimate given by Theorem 2.3 for the maximum order of a point derivation is best-possible for nondegenerate point derivations on $C^{(n)}$. The final theorem concerning $C^{(n)}$ of this section improves the estimate given by Theorem 2.3 for these algebras, giving the best-possible bound. It also shows that the hypothesis of nondegeneracy was unnecessary in Theorem 3.3. We need one more technical lemma; the case $n = 1$ was shown to us by P. C. Curtis.

3.5 Lemma. *Let $d_1, \ldots, d_{n+1}$ be linear functionals on $M_{n,0}$ such that d_1 is a point derivation of order* 1, *and*

$$d_{n+1}(fg) = \sum_{i=1}^{n} (\delta_i(f)\, d_{n+1-i}(g) + d_{n+1-i}(f)\, \delta_i(g)) \qquad (f, g \in M_{n,0}).$$

Then d_1 is a multiple of δ_1; in particular, d_1 is continuous.

Proof. If f belongs to $M_{n,n}$, then the identity given for d_{n+1} implies that $d_{n+1}(x^n f) = d_1(f)$. If $\delta_1(f) = 0$, then $f = \sum_{i=2}^{n} \delta_i(f) x^i + R_n f$, and so $d_1(f) = d_1(R_n f)$ because d_1 is a point derivation. Thus, given f such that $\delta_1(f) = 0$, we can suppose that f belongs to $M_{n,n}$ without affecting $d_1(f)$. Then, by Theorem 3.1(i), there are g and h in $M_{n,n}$ such that $x^n f = gh$. So $d_1(f) = d_{n+1}(x^n f) = d_{n+1}(gh) = 0$. This shows that $\ker \delta_1 \subset \ker d_1$, and proves the lemma.

3.6 Theorem. *Let n be a positive integer, and let $d_1, \ldots, d_p$ be a point derivation* *at 0 on $C^{(n)}$ of order p. Let k be a positive integer, and suppose that $d_j = 0$ for $j < k$ and that $d_k \neq 0$.*

(i) *If q is a positive integer and $p \geqslant 2q$, then the point derivation $d_1, \ldots, d_q$ is continuous.*

(ii) $p < (2n + 1)k$.

Proof. If $k = 1$, assertion (i) is Theorem 3.3 and assertion (ii) comes from Theorem 2.3, so we suppose henceforth that $k > 1$.

To begin, consider, for $r \geqslant 1$, the hypothesis

$$(H_r) \begin{cases} p \geqslant 2rk \text{ and the derivation has the form } 0,\ldots, 0, d_k, 0,\ldots, 0, d_{2k}, \\ 0,\ldots, 0, d_{rk}, d_{rk+1}, \ldots, d_p, \text{ with } d_k \neq 0 \text{ and where each sequence} \\ \text{of zeros has length } k - 1. \end{cases}$$

By the hypotheses of the theorem, either $p < 2k$, in which case both assertions (i) and (ii) are clear, or (H_1) is true.

Suppose now that (H_r) holds for some $r < n$. An easy calculation shows that $d_k, d_{2k}, \ldots, d_{2rk}$ is a ***nondegenerate*** point derivation of order $2r$. If $q \leqslant 2rk$, the continuity assertion (i) follows from Theorem 3.3, so that (H_r) entails (i) unless $p \geqslant 2rk + 2$. If $p \geqslant 2rk + 2$, then, using Lemma 1.1 as in the proof of Theorem 3.3, we see that we can suppose that

$$d_{jk} = \delta_j \qquad (j = 1, \ldots, r)$$

and that

$$d_j(x) = 0 \qquad (j > rk)$$

without changing the sequences of zeros. (For example, supposing that $d_k = \delta_1$, to ensure that $d_{2k} = \delta_2$, apply Lemma 1.1 with $\alpha = -d_{2k}(x)$ and $m - 1 = k$, and observe that the element k, or any multiple of k, places before a zero is a zero, so that the zeros are unchanged.) Also, another easy calculation shows that

$$d_{rk+j} \perp (M_{n,0})^2 \qquad (j = 1, \ldots, k - 1).$$

We now wish to show that, in certain circumstances, (H_{r+1}) is true. Let m be a positive integer with $m < k$, and suppose that

$$d_{rk+j} = 0 \qquad (j = 1, \ldots, m - 1) \tag{3.6}$$

(vacuous if $m = 1$) and that $p \geqslant 2rk + 2m = 2(rk + m)$; we shall show that $d_{rk+m} = 0$. We have already pointed out that $d_k, d_{2k}, \ldots, d_{2rk}$ is a point derivation of order $2r$. Provided that $2m \neq k$, a further calculation shows that

$$\begin{aligned} &d_k, d_{2k}, \ldots, d_{rk}, \\ &\quad d_{(r+1)k} + d_{rk+2m}, \\ &\quad d_{(r+2)k} + d_{(r+1)k+2m}, \\ &\quad \ldots \\ &\quad d_{2rk} + d_{(2r-1)k+2m} \end{aligned}$$

is also a point derivation of order $2r$. Because $r < n$ and because of the normalization carried out above, the construction in the proof of Theorem 3.4(ii) can be used to extend each of these to point derivations of order $2r + 1$. Taking the difference of the $(2r + 1)$th members of these derivations shows that there is a well-defined linear functional λ on $M_{n,0}$ such that, for f and g in $M_{n,0}$,

$$\lambda(fg) = \sum_{j=1}^{r} [d_{jk}(f)\, d_{(2r-j)k+2m}(g) + d_{(2r-j)k+2m}(f)\, d_{jk}(g)],$$

and comparison of this equation with the Leibnitz identity for d_{2rk+2m} shows that

$$(d_{2kr+2m} - \lambda)(fg) = d_{rk+m}(f)\, d_{rk+m}(g) \qquad (f, g \in M_{n,0}),$$

where we recall Eq. (3.6). That is, we have shown that d_{rk+m} belongs to a point derivation of order 2. By Theorem 3.3, d_{rk+m} is either 0 or a multiple of δ_1, so the fact that $d_{rk+m}(x) = 0$ implies that $d_{rk+m} = 0$, as required.

On the other hand, if $2m = k$, one needs only the observation that $d_k, \ldots, d_{2rk}$ belongs to a point derivation of order $2r + 1$ to conclude that d_{rk+m} belongs to a point derivation of order 2, and hence that $d_{rk+m} = 0$.

For each $m < k$ such that (3.6) holds, either $p < 2rk + 2m$ or $p \geqslant 2rk + 2m$. In the former case, the condition $2q \leqslant p$ implies that $q \leqslant rk + m - 1$ and so d_q is continuous, being either 0 or $d_{jk} = \delta_j$ for some $j \in \{1, \ldots, r\}$. In the latter case, $d_{rk+m} = 0$. Induction on m shows that either the continuity assertion (i) holds or that we can deduce that (H_{r+1}) holds, and then induction on r (for $r < n$) shows that either (i) holds or that we can deduce that (H_n) holds.

Suppose then that (H_n) holds. As before, $d_{jk} = \delta_j$ for $j = 1, \ldots, n$ and $d_{nk+j} \perp (M_{n,0})^2$ for $j = 1, \ldots, k - 1$. Again, suppose that $1 \leqslant m < k$, and that $d_{nk+j} = 0$ for $j = 1, \ldots, m - 1$. If $p \geqslant 2nk + m$ (NOT "$p \geqslant 2nk + 2m$") and if $f, g \in M_{n,0}$, then

$$d_{2nk+m}(fg) = \sum_{j=1}^{n} [d_{jk}(f)\, d_{(2n-j)k+m}(g) + d_{(2n-j)k+m}(f)\, d_{jk}(g)],$$

so that d_{nk+m} is a multiple of δ_1 by Lemma 3.5 (with $d_{nk+m}, d_{(n+1)k+m}, \ldots, d_{2nk+m}$

replacing $d_1, d_2, \ldots, d_{n+1}$). Since $d_{nk+m}(x) = 0$, we have $d_{nk+m} = 0$. Thus, we either prove (i) for q in the range $nk + 1 \leqslant q < (n + 1)k$ (something a little better, in fact), or we can deduce that $p \geqslant (2n + 1)k$ and that the derivation has the form

$$0, \ldots, 0, d_k, 0, \ldots, 0, d_{2k}, 0, \ldots, 0, d_{nk}, 0, \ldots, 0, d_{(n+1)k}, d_{(n+1)k+1}, \ldots, d_{(2n+1)k}, \ldots, d_p .$$

This latter implies, by a method similar to that used above, that d_k, $d_{2k}, \ldots, d_{(2n+1)k}$ is a *nondegenerate* point derivation of order $2n + 1$ on $C^{(n)}$. But we know that this is not possible, so that we have proved both (i) and (ii), as required.

This completes the proof of the theorem.

The estimate (ii) in the above result is best-possible: Take a point derivation of order $2n$ (given by Theorem 3.4(iii)), put $k - 1$ zeros before each term, and add any $k - 1$ first-order point derivations at the end. We could also show that the estimate (i) (modified according to the last part of the proof for $nk < q < (n + 1)k$) is sharp. This would amount to something like Theorem 3.4, with complications brought on by the degeneracy, and we omit it.

As an addendum to this section, we examine an algebra which is closely related to the algebras $C^{(n)}([0, 1])$. This is the algebra $C^\infty = C^\infty([0, 1])$ of infinitely differentiable functions on $[0, 1]$. It is well known [4, 18.22] that C^∞ cannot be normed so that it becomes a Banach algebra, but C^∞ is a complete Fréchet algebra on $[0, 1]$ with respect to the seminorms

$$\|f\|_n = \sum_{j=0}^{n} \frac{1}{j!} \sup\{|f^{(j)}(t)| : t \in [0, 1]\} \qquad (f \in C^\infty)$$

(for $n = 0, 1, 2, \ldots$), and every character is given by evaluation at a point of $[0, 1]$.

3.7 Theorem. *Every point derivation on C^∞ is continuous.*

Proof. Let $M_{\infty,k} = \{f \in C^\infty : f^{(j)}(0) = 0 \ (j = 0, \ldots, k)\}$. By a standard argument, $M_{\infty,0} = xC^\infty$: if $f \in M_{\infty,0}$ and $t \in [0, 1]$, let $F(s) = f(st)$ $(s \in [0, 1])$, so that $f(t) = F(1) - F(0) = \int_0^1 F'(s)\, ds = tg(t)$ and $f = xg$ for some $g \in C^\infty$. Thus, $M_{\infty,1} \subset xM_{\infty,0}$ and so

$$(M_{\infty,0})^{k+1} = x^k M_{\infty,0} = M_{\infty,k} \qquad (k = 1, 2, \ldots).$$

In particular, $(M_{\infty,0})^{k+1}$ is closed in C^∞ and of finite codimension in $M_{\infty,0}$. Thus, if λ is any linear functional such that $\lambda \perp (M_{\infty,0})^{k+1}$ for some k, then λ is continuous. The theorem follows.

In fact, for each nondegenerate point derivation on C^∞ at 0 of infinite order, the map $f \mapsto \Sigma\, d_i(f) X^i$, $C^\infty \to \mathscr{F}$ is a continuous epimorphism.

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