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Mass formula of division algebras over global function fields

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ABSTRACT

In this paper we give two proofs of the mass formula for definite central division algebras over global function fields, due to Denert and Van Geel. The first proof is based on a calculation of Tamagawa measures. The second proof is based on analytic methods, in which we establish the relationship directly between the mass and the value of the associated zeta function at zero.

1. Introduction

Let K be a global function field with constant field $\mathbb{F}_q$. Fix a place ∞ of K, referred as the place at infinity. Let A be the subring of functions in K regular everywhere outside ∞. Let B be a definite central division algebra of dimension r^2 over K; see Section 2. Let R be a maximal A-order in B and let G' be the multiplicative group of R, regarded as a group scheme over A. Denote by $\widehat{A}$ the pro-finite completion of A, which is the maximal open compact topological subring of the ring $\mathbb{A}_K^\infty$ of finite adeles of K. The mass associated to the double coset space $G'(K)\backslash G'(\mathbb{A}_K^\infty)/G'(\widehat{A})$ is defined as

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$$\mathrm{Mass}\big(G', G'(\widehat{A})\big) := \sum_{i=1}^{h} |\Gamma_i|^{-1}, \quad \Gamma_i := G'(K) \cap c_i G'(\widehat{A}) c_i^{-1}, \tag{1.1}$$

where $c_1, \ldots, c_h$ are complete representatives for the double coset space.

In this paper we prove the following result.

Theorem 1.1. *We have*

$$\mathrm{Mass}\big(G', G'(\widehat{A})\big) = \frac{\#\,\mathrm{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K(-i) \cdot \prod_{v \in S} \lambda_v, \tag{1.2}$$

where $\mathrm{Pic}(A)$ *is the Picard group of* A*,*

$$\zeta_K(s) := \prod_v \big(1 - N(v)^{-s}\big)^{-1}$$

is the zeta function of K*,* S *is the finite subset of ramified places for* B *and*

$$\lambda_v = \prod_{1 \leqslant i \leqslant r-1, d_v \nmid i} \big(N(v)^i - 1\big), \tag{1.3}$$

where d_v *is the index of the central simple algebra* $B_v = B \otimes_K K_v$*.*

We remark that the mass (1.1) is defined only when the central simple algebra is definite. For the complementary cases where the central simple algebra is not definite, one easily shows that its class number is equal to the class number of A (Corollary 2.3). This is the analogue of the classical theorem (due to Eichler) that any central simple algebra over $\mathbb{Q}$ which is not a definite quaternion algebra has class number one.

We say that a central simple algebra B over K is of Drinfeld type if the invariant of B at ∞ is $-1/r$ and B is ramified at one more (finite) place $\mathfrak{p} \subset A$. Recall that a Drinfeld A-module ϕ of rank r over a $\kappa(\mathfrak{p})$-field κ_1 is called *supersingular* if the group of $\bar{\kappa}_1$-valued points of the $\mathfrak{p}$-torsion subgroup $\phi[\mathfrak{p}]$ is trivial, where $\bar{\kappa}_1$ denotes an algebraic closure of κ_1. Let $\Sigma(r, \mathfrak{p})$ denote the set of isomorphism classes of supersingular Drinfeld A-modules of rank r over $\overline{\kappa(\mathfrak{p})}$. The set $\Sigma(r, \mathfrak{p})$ is in bijection with the double space $G'(K)\backslash G'(\mathbb{A}_K^{\infty})/G'(\widehat{A})$ associated to the algebra B of Drinfeld type ramified at $\{\infty, \mathfrak{p}\}$ and each object ϕ in $\Sigma(r, \mathfrak{p})$ has only finitely many automorphisms. One associates the geometric mass $\mathrm{Mass}(\Sigma(r, \mathfrak{p}))$ as

$$\mathrm{Mass}\big(\Sigma(r, \mathfrak{p})\big) := \sum_{[\phi] \in \Sigma(r, \mathfrak{p})} \big|\mathrm{Aut}(\phi)\big|^{-1}. \tag{1.4}$$

As an immediate consequence of Theorem 1.1 applied to Drinfeld type division algebras B, we obtain the following geometric mass formula [14, Theorem 2.1]. This is the function field analogue of the Deuring–Eichler mass formula for supersingular elliptic curves.

Theorem 1.2. *We have*

$$\mathrm{Mass}\big(\Sigma(r, \mathfrak{p})\big) := \frac{\#\,\mathrm{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K^{\infty, \mathfrak{p}}(-i), \tag{1.5}$$

where $\zeta_K^{\infty,\mathfrak{p}}(s) = \prod_{v \neq \infty, \mathfrak{p}} (1 - N(v)^{-s})^{-1}$ *is the zeta function of* K *with the local factors at* ∞ *and* $\mathfrak{p}$ *removed.*

Theorem 1.2 was proved by Gekeler when $r = 2$ or K is the rational function field ([4, Theorem 1, p. 144], [5, Theorem 2.5, p. 321 and 5.1, p. 328]), and by J. Yu and the second named author [14] for both arbitrary r and global function fields K. The proof in [14] consists of two parts: The first one deduces the mass, through manipulating Tamagawa measures, as a product of zeta values up to explicit local indices (the ratio of the volumes of two local open compact groups at each ramified place); see also (3.4). Then one uses Gekeler's result of geometric mass formula for the rational global function field case to determine the local index. This argument yields the local index where the local invariant of B is $\pm 1/r$ for free, however, its proof roots in the result of counting supersingular Drinfeld modules in the Drinfeld moduli scheme modulo the finite prime $\mathfrak{p}$.

Since central division algebras considered in Theorem 1.1 may not arise from geometry, that is, as endomorphism algebras of certain Drinfeld modules, the question of determining the mass formula goes beyond the reach of geometric methods and hence a different approach is needed. In this paper we give two proofs of Theorem 1.1. As a consequence we obtain two different proofs of the geometric mass formula (1.5). For the first proof we calculate the remaining the ratio of local volumes directly. The proof is given in Section 5; some basic results in central simple algebras over local fields are recalled in Section 4.

The second proof is analytic. In this proof we directly show that the associated mass is equal to the value of the associated zeta function at zero; see Section 6.1.

After the present manuscript was completed, the authors learned that Theorem 1.1 was first obtained in Denert and Van Geel [1] and that it is also a consequence of deep results of Gopal Prasad [7]. Therefore, the main result presented in this paper is not new. However, we hope that the detailed calculations presented in this paper may be helpful to some readers who wish to know more elementary steps. For more computations of Tamagawa measures and mass formulas we refer to the references [2,3,11–13,15].

2. Preliminaries

2.1. Notation

Let K be a global function field with constant field $\mathbb{F}_q$. Fix a place ∞ of K, referred as the place at infinity. All other places of K are referred as finite places. Let A be the subring of functions in K regular everywhere outside ∞. For each place v of K, denote by K_v the completion of K at v, and denote by O_v the ring of integers in K_v. When v is finite, the ring O_v equals the completion A_v of A at v. We also write $\kappa(v)$ for the residue field O_v/π_v at v, where π_v is a uniformizer of O_v, and put $N(v) := \#\kappa(v)$.

For any A-module or K-module M, we write M_v for $M \otimes_A A_v$ if v is finite, and M_v for $M \otimes_K K_v$ for any place v. Let $\mathbb{A}_K^\infty$ denote the ring of finite adeles of K (with respect to ∞) and put

$$\widehat{A} := \prod_{v:\ \text{finite}} A_v,$$

the pro-finite completion of A.

For a linear algebraic group G over K and an open compact subgroup U of $G(\mathbb{A}_K^\infty)$, denote by $\mathrm{DS}(G, U)$ the double coset space $G(K)\backslash G(\mathbb{A}_K^\infty)/U$. If the arithmetic subgroup $G(K) \cap U$ is finite, or equivalently that any (∞-)arithmetic subgroup Γ of $G(K)$ (i.e. Γ is a subgroup commensurable to $G(K) \cap U$) is finite, define

$$\mathrm{Mass}(G, U) := \sum_{i=1}^{h} |\Gamma_i|^{-1}, \quad \Gamma_i := G(K) \cap c_i U c_i^{-1},$$

where $c_1, \dots, c_h$ are complete representatives for $\mathrm{DS}(G, U)$. It is easy to show that $\mathrm{Mass}(G, U)$ does not depend on the choice of representatives c_i.

2.2. Class numbers of indefinite central simple algebras

Let B be a central simple algebra over K. An *A-order* in B is an A-subring of B which is finite as an A-module and spans B over K. An A-order in B is called *maximal* if it is not properly contained in another A-order in B. Let Λ be a maximal A-order in B. By a right fractional ideal of Λ we mean a non-zero finite right Λ-submodule I in B; I is called *full* if it spans B over K. When B is a division algebra, any fractional ideal of Λ is full. Let $\mathcal{L}$ be the set of all full right fractional ideals of Λ in B. Two right fractional ideals I and I' are said to be *locally equivalent at a finite place v* if $I'_v = g_v I_v$ for some element $g_v \in B_v^\times$; they are said to be *globally equivalent* if there is an element $g \in B^\times$ such that $I' = gI$. This is equivalent to that I_v and I'_v (resp. I and I') are isomorphic as Λ_v-modules (resp. as Λ-modules).

Since Λ_v is a maximal A_v-order, any one-sided ideal of Λ_v is principal [8, Theorem 18.7, p. 179]. It follows that any two full right fractional ideals are locally equivalent everywhere, that is, the set $\mathcal{L}$ of ideals forms a single genus. Let $\mathcal{L}/\sim$ denote the set of global equivalence classes of right ideals in $\mathcal{L}$. Let G' be the group scheme over A associated to the multiplicative group of Λ. For each commutative A-algebra L, the group of L-valued points of G' is

$$G'(L) = (\Lambda \otimes_A L)^\times.$$

The above argument establishes the following well-known basic fact:

Lemma 2.1. *There is a natural bijection*

$$\varphi : G'(K)\backslash G'\big(\mathbb{A}_K^\infty\big)/G'(\widehat{A}) \to \mathcal{L}/\sim$$

which maps the identity class to the trivial class $[\Lambda]$.

The cardinality of $\mathcal{L}/\sim$ is independent of the choice of the maximal order Λ; this follows from the basic fact that any two maximal orders are locally conjugate. The number $\#\mathcal{L}/\sim$ is called the class number of B (relative to ∞), which we denote by $h^\infty(B)$ or simply by $h(B)$ as the place ∞ has been fixed. We shall call B definite (at ∞) if $B_\infty := B \otimes_K K_\infty$ is a division algebra, and B indefinite (at ∞) otherwise.

Lemma 2.2. *Assume that B is indefinite. Let U be an open compact subgroup of $G'(\mathbb{A}_K^\infty)$. Then the reduced norm map $N_{B/K}$ induces a bijection of double coset spaces*

$$N_{B/K} : G'(K)\backslash G'\big(\mathbb{A}_K^\infty\big)/U \simeq K^\times\backslash \mathbb{A}_K^{\infty,\times}/N_{B/K}(U).$$

Proof. This follows from the strong approximation; we provide the proof for the reader's convenience. We may assume that $B \neq K$. Clearly the induced map is surjective. We show the injectivity. Let $[a]$ be an element in the target space. Fix a section $s : \mathbb{A}_K^{\infty,\times} \to G'(\mathbb{A}_K^\infty)$ of the map $N_{B/K}$. Then the inverse image $T_{[a]}$ of the class $[a]$ consists of elements ${}_{G'(K)}[gs(a)]_U$ for all $g \in G'_1(\mathbb{A}_K^\infty)$, where $G'_1 \subset G'$ is the reduced norm one algebraic subgroup. The surjective map $g \mapsto {}_{G'(K)}[gs(a)]_U$ induces a surjective map

$$\alpha : G'_1(K)\backslash G'_1\big(\mathbb{A}_K^\infty\big)/U' \to T_{[a]},$$

where $U' := s(a)Us(a)^{-1} \cap G'_1(\mathbb{A}_K^\infty)$. Since the group G'_1 is semi-simple, simply connected and $G'_1(K_\infty)$ is not compact, the strong approximation holds for the algebraic group G'_1. Therefore, $T_{[a]}$ consists of a single element and this proves the lemma. $\square$

Corollary 2.3. *Assume that B is indefinite.*

(1) *We have $h(B) = \#\operatorname{Pic}(A) =: h(A)$, where $\operatorname{Pic}(A)$ is the Picard group of A.*
(2) *If A is a principal ideal domain, then any full one-sided ideal of Λ is principal.*

Proof. These easily follow from Lemmas 2.1 and 2.2. □

Lemma 2.4. *Notation as above. The algebra B is definite if and only if any ∞-arithmetic subgroup Γ of $G'(K)$ is finite.*

Proof. This is clear. □

From discussion above, the mass $\operatorname{Mass}(G', U)$ is defined only when the central simple algebra B is definite. When B is indefinite, the class number $h(B)$ is equal to $h(A)$. One can calculate the class number $h(A)$ of A by the following formula [4, p. 143, (1.5)]:

$$h(A) = \deg \infty P(1), \tag{2.1}$$

where $\deg\infty$ is the degree of ∞, and $P(T) \in \mathbb{Z}[T]$ is the polynomial so that

$$\zeta_K(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

In the sequel we shall only consider the case where B is a definite central division algebra over K.

3. Proof of Theorem 1.1

Keep the notation as in Section 1; B and R are as before. Let B_0 be the matrix algebra $\operatorname{Mat}_r(K)$ and let $R_0 := \operatorname{Mat}_r(A)$ be the standard maximal A-order in B_0. Let G and G' be the group schemes over A associated to the multiplicative groups of R_0 and R, respectively. Let G_1 (resp. G'_1) denote the reduced norm one subgroup schemes of G (resp. G').

First we have

$$\operatorname{Mass}\big(G'(\widehat{A})\big) := \operatorname{Mass}\big(G', G'(\widehat{A})\big) = \frac{\operatorname{vol}(G'(K)\backslash G'(\mathbb{A}_K^\infty))}{\operatorname{vol}(G'(\widehat{A}))}, \tag{3.1}$$

for any Haar measure dg' on $G'(\mathbb{A}_K^\infty)$. A simple computation (cf. [14]) shows that

$$\begin{aligned}\operatorname{Mass}\big(G'(\widehat{A})\big) &= \frac{\#\operatorname{Pic}(A)}{q-1}\cdot \tau\big(G'_1\big)\cdot \omega'_{\mathbb{A}}\big(P'\big)^{-1}\\ &= \frac{\#\operatorname{Pic}(A)}{q-1}\cdot \omega'_{\mathbb{A}}\big(P'\big)^{-1} \quad \big(\tau\big(G'_1\big)=1,\ \text{Weil's Theorem [10]}\big),\end{aligned} \tag{3.2}$$

where $P' := \prod_v P'_v$ with $P'_v := G'_1(O_v)$, $\omega'_{\mathbb{A}}$ is the Tamagawa measure on $G'_1(\mathbb{A}_K)$ and $\tau(G'_1)$ is the Tamagawa number of G'_1.

Let ω be an invariant K-rational differential form of top degree on the group G_1, and let ω' be the pull back of ω via an inner twist $\alpha : G'_1 \xrightarrow{\sim} G_1$ (over a finite extension of K). They give rise to the Tamagawa measures $\omega_{\mathbb{A}}$ and $\omega'_{\mathbb{A}}$ on G_1 and G'_1, respectively. Then

$$\omega'_{\mathbb{A}}\big(P'\big) = \omega_{\mathbb{A}}(P)\cdot \prod_{v\in S}\frac{\omega'_v(P'_v)}{\omega_v(P_v)}, \tag{3.3}$$

where $P = \prod_v P_v$, $P_v := G_1(O_v)$ and S is the finite set of ramified places for B. From the well-known fact that $\omega_{\mathbb{A}}(P)^{-1} = \prod_{i=1}^{r-1} \zeta_K(-i)$, we get

$$\mathrm{Mass}\big(G'(\widehat{A})\big) = \frac{\#\,\mathrm{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K(-i) \cdot \prod_{v \in S} \lambda_v, \quad \lambda_v := \frac{\omega_v(P_v)}{\omega'_v(P'_v)}. \tag{3.4}$$

Proposition 3.1. *Suppose that $B_v \simeq \mathrm{Mat}_{m_v}(\Delta_v)$, where Δ_v is the division part of B_v, and let d_v be the index of Δ_v. Then*

$$\lambda_v = \prod_{1 \leqslant i \leqslant r-1, d_v \nmid i} \big(N(v)^i - 1\big). \tag{3.5}$$

The proof of Proposition 3.1 will be given in Section 5. By (3.4) and Proposition 3.1, Theorem 1.1 is proved.

4. Division algebras over local fields

In this section we make preparation on central division algebras over non-Archimedean local fields. This will be used in the next section. Let K_v, O_v, π_v, $\kappa(v)$, $N(v)$ be as before.

4.1. Maximal orders

Let Δ be a central division algebra of dimension d^2 over K_v. Let L be the unramified field extension of K_v of degree d, and O_L its ring of integers. Let σ be the (arithmetic) Frobenius automorphism of L over K_v. Suppose $\mathrm{inv}(\Delta) = b/d$, where b is a positive integer with $(b,d) = 1$ and $b < d$. We use the normalization of invariant of Δ in Pierce [6]; see p. 338 and p. 277. We can write

$$\Delta = L\big[\Pi'\big], \qquad \big(\Pi'\big)^d = \pi_v^b, \qquad \big(\Pi'\big)^{-1} c \Pi' = \sigma(c), \quad \forall c \in L. \tag{4.1}$$

Note that the normalization in Reiner [8] is different; the invariant of Δ in (4.1) is defined to be $-b/d$ there; see [8, (31.7), p. 266 and p. 264].

Choose integers m and m' such that $bm + dm' = 1$. We may take $1 \leqslant m \leqslant d$. Put $\Pi := (\Pi')^m \pi_v^{m'}$. It is easy to check that

$$\Pi^d = (\pi_v)^{bm} \pi_v^{dm'} = \pi_v, \qquad \Pi^{-1} c \Pi = \sigma^m(c), \quad \forall c \in L. \tag{4.2}$$

Put $\tau := \sigma^m$; we have $\mathrm{Gal}(L/K_v) = \langle \tau \rangle$. The subring

$$O_\Delta := O_L[\Pi] \subset \Delta$$

is the unique maximal order; see [8, Theorem 13.3, p. 140 and p. 146].

We regard Δ as a right vector space over L, with basis $1, \Pi, \ldots, \Pi^{d-1}$. The left translation of Δ on Δ gives an embedding

$$\Phi : \Delta \to \mathrm{Mat}_d(L)$$

as K_v-algebras. From the relation $a_0 \Pi^i = \Pi^i \tau^i(a_0)$ for $a_0 \in L$, we have

$$\Phi(a_0)=\begin{pmatrix} a_0 & 0 & \cdots & 0\\ 0 & \tau(a_0) & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & & \tau^{d-1}(a_0)\end{pmatrix},\qquad \Phi(\Pi)=\begin{pmatrix} 0 & 0 & \cdots & 0 & \pi_v\\ 1 & 0 & & 0 & 0\\ 0 & 1 & \ddots & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}. \tag{4.3}$$

For example, when $d=3$, we have

$$\Phi\big(a_0+\Pi a_1+\Pi^2 a_2\big)=\begin{pmatrix} a_0 & \pi_v\tau(a_2) & \pi_v\tau^2(a_1)\\ a_1 & \tau(a_0) & \pi_v\tau^2(a_2)\\ a_2 & \tau(a_1) & \tau^2(a_0)\end{pmatrix}. \tag{4.4}$$

The map $\Phi:\Delta\to\mathrm{Mat}_d(L)$ induces an isomorphism

$$\Phi_L:\Delta\otimes_{K_v}L\to\mathrm{Mat}_d(L),\quad x\otimes a\mapsto\Phi(x)a. \tag{4.5}$$

Let $Iw\subset\mathrm{Mat}_d(O_L)$ be the hereditary order

$$Iw:=\big\{(a_{ij})\in\mathrm{Mat}_d(O_L)\ \big|\ a_{ij}\in\pi_v O_L,\ \forall i<j\big\}. \tag{4.6}$$

It is not hard to see that

$$O_\Delta=\big\{a\in\Delta\ \big|\ \Phi(a)\in Iw\big\},\quad\text{and}\quad \Phi_L(O_\Delta\otimes_{O_v}O_L)=Iw. \tag{4.7}$$

In other words, the map Φ is an optimal embedding (also called a maximal embedding) of O_Δ into the order $Iw\subset\mathrm{Mat}_d(L)$.

4.2. Haar measures and the base change formula

Let $\{e'_{ij}\}_{1\leqslant i,j\leqslant d}$ be a K_v-basis for the vector space Δ. For any element $x'=\sum_{i,j}x'_{ij}e'_{ij}$ in Δ, write $x'=(x'_{ij})$ and x'_{ij}s are global linear coordinates for Δ, regarded as a commutative algebraic group over K_v. The invariant differential form $dx'=\prod_{i,j}dx'_{ij}$ of top degree naturally gives rise to an additive Haar measure on Δ, which we also denote by dx', by setting

$$\mathrm{vol}\big(B(1),dx'\big)=1,$$

where $B(1):=\{(x'_{ij})\mid x'_{ij}\in O_v,\ \forall i,j\}$ is the unit ball. Let

$$d^\times x':=\frac{dx'}{|N_{\Delta/K_v}(x')|_v^d}$$

be the induced Haar measure on $\Delta^\times$, where N_{Δ/K_v} is the reduced norm map and $|\pi_v|_v=N(v)^{-1}$. Regarding $G'=\Delta^\times$ as an algebraic group over K_v, $d^\times x'$ is also an invariant different form on G' of top degree.

The differential form dx' is a K_v-rational differential form on $\Delta\otimes L$, regarded as a commutative algebraic group over L. The induced Haar measure on $\Delta\otimes L$ will be denoted by $dx'\otimes L$.

Proposition 4.1.

(1) *For any full O_v-lattice M in Δ, we have the base change formula*

$$\mathrm{vol}\big(M, dx'\big)^d = \mathrm{vol}\big(M \otimes_{O_v} O_L, dx' \otimes L\big). \tag{4.8}$$

(2) *We have*

$$\mathrm{vol}\big(O_\Delta, dx'\big) = N(v)^{-d(d-1)/2}\big[\mathrm{vol}\big(\Phi_L^{-1}\big(\mathrm{Mat}_d(O_L)\big), dx' \otimes L\big)\big]^{1/d}. \tag{4.9}$$

Proof. (1) This is clear. (2) It follows from (1) and (4.7) that

$$\begin{aligned}\mathrm{vol}\big(O_\Delta, dx'\big) &= \big[\mathrm{vol}\big(\Phi_L^{-1}(Iw), dx' \otimes L\big)\big]^{1/d} \\ &= \big[\mathrm{Mat}_d(O_L) : Iw\big]^{-1/d}\big[\mathrm{vol}\big(\Phi_L^{-1}\big(\mathrm{Mat}_d(O_L)\big), dx' \otimes L\big)\big]^{1/d}.\end{aligned}$$

Then (4.9) follows from

$$\big[\mathrm{Mat}_d(O_L) : Iw\big] = N(v)^{d^2(d-1)/2}. \qquad \square$$

5. Computation of local indices

In this section we shall give a proof of Proposition 3.1. Suppose $B_v \simeq \mathrm{Mat}_{m_v}(\Delta_v)$, where Δ_v is a central division algebra of dimension d_v^2 over K_v. We have $r = m_v d_v$.

Choose the standard coordinates x_{ij} for $\mathrm{Mat}_r(K_v)$ and form an invariant differential form $dx := \prod_{i,j} dx_{ij}$ of top degree on the commutative algebraic group Mat_r over K_v. Let L be the unramified extension of K_v of degree d_v. The L-algebra isomorphism $\Phi_L : B_v \otimes_{K_v} L \to \mathrm{Mat}_r(L)$ constructed in Section 4 gives an isomorphism $\alpha : B_v \to \mathrm{Mat}_r$ of ring schemes over L, and defines also an isomorphism $\alpha : G' \to G$ over L. The pull-back differential form $\alpha^* dx$ is K_v-rational and there is an invariant differential form dx' of top degree on B_v such that $dx' \otimes L = \alpha^* dx$. Then the invariant differential forms

$$dg := dx/\big|\det(x)\big|_v^r, \qquad dg' := dx'/\big|N_{B_v/K_v}\big(x'\big)\big|_v^r$$

induce the Haar measures on $G'(K_v)$ and $G(K_v)$ which are transferred to each other via the map α. Choose a Haar measure dt of $K_v^\times$; it defines Haar measures dg_1 on $G_1(K_v)$ and dg_1' on $G_1'(K_v)$ such that $dg = dg_1 dt$ and $dg' = dg_1' dt$. Also dg_1' is the transfer of dg_1. It follows that

$$\lambda_v = \frac{\mathrm{vol}(G_1(O_v), dg_1)}{\mathrm{vol}(G_1'(O_v), dg_1')} = \frac{\mathrm{vol}(G(O_v), dg)}{\mathrm{vol}(G'(O_v), dg')}.$$

We shall calculate the volumes $\mathrm{vol}(G(O_v), dg)$ and $\mathrm{vol}(G'(O_v), dg')$. From our choice of the Haar measure, we have $\mathrm{vol}(\mathrm{Mat}_r(O_v), dx) = 1$. Therefore,

$$\mathrm{vol}\big(G(O_v)\big) = \int\limits_{G(O_v)} dg = \int\limits_{G(O_v)} dx = \mathrm{vol}\big(\mathrm{Mat}_r(O_v), dx\big)\frac{\#\,\mathrm{GL}_r(\kappa(v))}{\#\,\mathrm{Mat}_r(\kappa(v))}. \tag{5.1}$$

It is known that $\#\,\mathrm{GL}_r(\kappa(v)) = N(v)^{r(r-1)/2}\prod_{i=1}^r (N(v)^i - 1)$, and we get

$$\operatorname{vol}\bigl(G(O_v)\bigr) = \frac{\#\,\mathrm{GL}_r(\kappa(v))}{\#\,\mathrm{Mat}_r(\kappa(v))} = \frac{\prod_{i=1}^{r}(N(v)^i - 1)}{N(v)^{r(r+1)/2}}. \tag{5.2}$$

On the other hand, we have

$$\operatorname{vol}\bigl(G'(O_v)\bigr) = \operatorname{vol}\bigl(\mathrm{Mat}_{m_v}(O_{\Delta_v}), dx'\bigr)\frac{\#\,\mathrm{GL}_{m_v}(O_{\Delta_v}/\Pi_v)}{\#\,\mathrm{Mat}_{m_v}(O_{\Delta_v}/\Pi_v)}. \tag{5.3}$$

It follows from Proposition 4.1 that

$$\operatorname{vol}\bigl(\mathrm{Mat}_{m_v}(O_{\Delta_v}), dx'\bigr) = N(v)^{-m_v^2 d_v(d_v-1)/2}.$$

Similar to (5.2), we have

$$\frac{\#\,\mathrm{GL}_{m_v}(O_{\Delta_v}/\Pi_v)}{\#\,\mathrm{Mat}_{m_v}(O_{\Delta_v}/\Pi_v)} = \frac{\prod_{i=1}^{m_v}(N(v)^{id_v} - 1)}{N(v)^{d_v m_v(m_v+1)/2}}.$$

Therefore, we get

$$\operatorname{vol}\bigl(G'(O_v)\bigr) = \frac{\prod_{i=1}^{m_v}(N(v)^{id_v} - 1)}{N(v)^{m_v d_v(m_v d_v+1)/2}}. \tag{5.4}$$

From (5.2) and (5.4), we get

$$\lambda_v = \prod_{\substack{1\leqslant i\leqslant r-1\\ d_v\nmid i}} \bigl(N(v)^i - 1\bigr).$$

This proves Proposition 3.1.

6. Alternative approach via zeta functions

This section is an analytic proof for Theorem 1.1. Keep the notation as in Section 1 and Section 2.1. Particularly we have chosen the definite central division algebra B over K of dimension r^2 and a maximal A-order R in B. Fix complete representatives $c_1, \dots, c_h$ for the double coset space $G'(K)\backslash G'(\mathbb{A}_K^\infty)/G'(\widehat{A})$ where G' is the group scheme over A defined as before. For $1 \leqslant i \leqslant h$, let

$$I_i := B \cap c_i\widehat{R} \quad \text{and} \quad R_i := B \cap c_i\widehat{R}c_i^{-1},$$

where $\widehat{R} := R \otimes_A \widehat{A}$, the pro-finite completion of R. Then $I_1, \dots, I_h$ are complete representatives of right ideal classes of R, and R_i is the left order of I_i for each i. The *inverse* of I_i is

$$I_i^{-1} := B \cap \widehat{R}c_i^{-1}.$$

One has $I_i^{-1} \cdot I_i = R$ and $I_i \cdot I_i^{-1} = R_i$. The units group $R_i^\times$ of R_i is equal to $G'(K) \cap c_i G'(\widehat{A})c_i^{-1}$ and

$$\operatorname{Mass}\bigl(G'(\widehat{A})\bigr) = \sum_{i=1}^{h} \frac{1}{\#(R_i^\times)}.$$

6.1. Partial zeta functions

For $1 \leqslant i \leqslant h$, define the *partial zeta function*

$$\zeta_i(s) := \sum_{I \sim I_i, I \subset R} \frac{1}{|N_{B/K}(I)|^s}.$$

Here $N_{B/K}(I)$ is the fractional ideal of A generated by the reduced norm $N_{B/K}(\alpha)$ of elements α in I, and $|\mathfrak{m}|$ is the cardinality of $A/\mathfrak{m}$ for a non-zero ideal $\mathfrak{m} \subset A$. From the definition of $\zeta_i(s)$ one has

$$\zeta_i(s) = \sum_{\text{ideals } \mathfrak{m} \subset A} \frac{b_i(\mathfrak{m})}{|\mathfrak{m}|^s},$$

where $b_i(\mathfrak{m}) := \#\{I \subset R\colon\ I \sim I_i \text{ with } N_{B/K}(I) = \mathfrak{m}\}$.

Proposition 6.1. *The function $\zeta_i(s)$ converges absolutely for* $\mathrm{Re}(s) > r$ *for all $i = 1, \dots, h$ and has a meromorphic continuation to the whole complex plane with a simple pole at $s = r$. Moreover, one has*

$$\zeta_i(0) = -\frac{1}{\#(R_i^\times)}.$$

Proof. Given a right ideal $I \subset R$ with $I \sim I_i$ and $N_{B/K}(I) = \mathfrak{m}$. There exists a unique $\alpha \in I_i^{-1}$, up to multiplying elements in $R_i^\times$ from the right, such that $I = \alpha I_i$ and $N_{B/K}(\alpha) N_{B/K}(I_i) = \mathfrak{m}$. Hence

$$\#\big(R_i^\times\big) \cdot b_i(\mathfrak{m}) = \#\big\{\alpha \in I_i^{-1}\colon\ N_{B/K}(\alpha) N_{B/K}(I_i) = \mathfrak{m}\big\}.$$

Let deg be the usual degree map on the divisor group $\mathrm{Div}(K)$ of K, i.e. $\deg v = [\kappa(v) : \mathbb{F}_q]$ for any place v of K. Choose the valuation v_∞ on K_∞ normalized so that for $a \in K_\infty$

$$v_\infty(a) := \deg \infty \cdot \mathrm{ord}_\infty(a),$$

and the valuation V_∞ on $B_\infty := B \otimes_K K_\infty$ (remembering B_∞ is a division algebra) with

$$V_\infty(\alpha) := v_\infty\big(N_{B/K}(\alpha)\big)$$

for $\alpha \in B_\infty$. Identifying fractional ideals of A with divisors of K supported outside ∞, we get

$$\#\big(R_i^\times\big) \cdot \zeta_i(s) = \sum_{\ell=0}^{\infty} a_i(\ell) q^{-\ell s},$$

where

$$\begin{aligned} a_i(\ell) &:= \#\big(R_i^\times\big) \cdot \sum_{\substack{\text{ideals } \mathfrak{m} \subset A \\ \deg \mathfrak{m} = \ell}} b_i(\mathfrak{m}) \\ &= \#\big\{\alpha \in I_i^{-1}\colon\ V_\infty(\alpha) = -\ell + \deg N_{B/K}(I_i)\big\}. \end{aligned}$$

Since $V_\infty(\alpha) \equiv 0 \mod \deg\infty$ for all $\alpha \in B^\times$,

$$a_i(\ell) = 0 \quad \text{if } -\ell + \deg N_{B/K}(I_i) \not\equiv 0 \mod \deg\infty.$$

Set $\mathcal{O}_{B_\infty} := \{w \in B_\infty\colon\ V_\infty(w) \geqslant 0\}$, the maximal compact subring of B_∞. Fix an element Π_∞ in $\mathcal{O}_{B_\infty}$ with $V_\infty(\Pi_\infty) = \deg\infty$. From the Riemann–Roch theorem for the function field K [9, Chapter VI] one can deduce that for a sufficient large integer n_0

$$B_\infty = I_i^{-1} + \Pi_\infty^{-n_0}\mathcal{O}_{B_\infty}.$$

Note that for any integer μ in $\mathbb{Z}$ one has

$$\#\big(\Pi_\infty^{-\mu-1}\mathcal{O}_{B_\infty}/\Pi_\infty^{-\mu}\mathcal{O}_{B_\infty}\big) = N(\infty)^r.$$

Therefore when $\mu \geqslant n_0$,

$$\#\left(\frac{\{\alpha \in I_i^{-1};\ V_\infty(\alpha) \geqslant -(\mu+1)\deg\infty\}}{\{\alpha \in I_i^{-1};\ V_\infty(\alpha) \geqslant -\mu\deg\infty\}}\right) = N(\infty)^r.$$

Choose n_0 large enough so that $n_0\deg\infty \geqslant -\deg N_{B/K}(I_i)$ and set

$$\ell_i := n_0\deg\infty + \deg N_{B/K}(I_i)$$

and

$$C_i := \#\big\{\alpha \in I_i^{-1}\colon\ V_\infty(\alpha) \geqslant -n_0\deg\infty\big\}.$$

From the definition of a_i one has that for any positive integer μ

$$a_i(\ell_i + \mu\deg\infty) = \#\big\{\alpha \in I_i^{-1}\colon\ V_\infty(\alpha) = -(n_0+\mu)\deg\infty\big\}.$$

By induction on μ we obtain

$$a_i(\ell_i + \mu\deg\infty) = \big(N(\infty)^r - 1\big)N(\infty)^{r(\mu-1)}C_i.$$

Therefore

$$\begin{aligned}\#\big(R_i^\times\big)\cdot\zeta_i(s) &= \sum_{\ell=0}^{\ell_i} a_i(\ell)q^{-\ell s} + q^{-\ell_i s}\sum_{\mu=1}^{\infty} a_i(\ell_i+\mu\deg\infty)N(\infty)^{-\mu s} \\ &= \sum_{\ell=0}^{\ell_i} a_i(\ell)q^{-\ell s} + C_i\cdot q^{-\ell_i s}\cdot\frac{N(\infty)^r-1}{N(\infty)^r}\cdot\sum_{\mu=1}^{\infty}N(\infty)^{-\mu(s-r)}.\end{aligned}$$

This shows that $\zeta_i(s)$ converges absolutely for $\mathrm{Re}(s) > r$ with a simple pole at $s = r$, and the meromorphic continuation of ζ_i is:

$$\#\big(R_i^\times\big)\cdot\zeta_i(s) = \sum_{\ell=0}^{\ell_i} a_i(\ell)q^{-\ell s} + C_i\cdot q^{-\ell_i s}\cdot\frac{N(\infty)^r-1}{N(\infty)^r}\cdot\frac{N(\infty)^{(r-s)}}{1-N(\infty)^{(r-s)}}.$$

From the definition of C_i one has $C_i = 1 + \sum_{\ell=0}^{\ell_i} a_i(\ell)$. Hence

$$\zeta_i(0) = -\frac{1}{\#(R_i^\times)}. \qquad \square$$

6.2. *Mass formula*

Define the *zeta function for the maximal order* R:

$$\zeta_R(s) := \sum_{\text{right ideals } I \subset R} \frac{1}{|N_{B/K}(I)|^s} = \sum_{i=1}^{h} \zeta_i(s).$$

Then $\zeta_R(s)$ also has meromorphic continuation and by Proposition 6.1 we have

$$\zeta_R(0) = -\mathrm{Mass}\big(G'(\widehat{A})\big).$$

Recall that for each place v of K, $B_v := B \otimes_K K_v$ is isomorphic to $\mathrm{Mat}_{m_v}(\Delta_v)$, where Δ_v is a central division algebra over K_v with $\dim_{K_v} \Delta_v = d_v^2$ and $m_v d_v = r$. Then $\zeta_R(s)$ can be expressed by the Dedekind zeta function ζ_K of K in the following:

Theorem 6.2. *We have*

$$\zeta_R(s) = \big(1 - N(\infty)^{-s}\big)\zeta_K(s) \cdot \prod_{i=1}^{r-1} \zeta_K(s-i) \cdot \prod_{v \in S}\Bigg(\prod_{\substack{1 \leqslant i \leqslant r-1 \\ d_v \nmid i}} \big(1 - N(v)^{i-s}\big)\Bigg),$$

where S is the finite set of ramified places for B.

Let $\zeta_A(s) := (1 - N(\infty)^{-s})\zeta_K(s)$, the zeta function for A. Then

$$\zeta_A(0) = -\frac{\#\,\mathrm{Pic}(A)}{q-1}$$

where $\mathrm{Pic}(A)$ is the ideal class group of A. Therefore the above theorem tells us

Corollary 6.3 *(Mass formula). We have*

$$\begin{aligned}\mathrm{Mass}\big(G'(\widehat{A})\big) &= \frac{\#\,\mathrm{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K(-i) \cdot \prod_{v \in S}\Bigg(\prod_{\substack{1 \leqslant i \leqslant r-1 \\ d_v \nmid i}} \big(1 - N(v)^{i}\big)\Bigg) \\ &= \frac{\#\,\mathrm{Pic}(A)}{q-1} \cdot \prod_{i=1}^{r-1} \zeta_K(-i) \cdot \prod_{v \in S}\Bigg(\prod_{\substack{1 \leqslant i \leqslant r-1 \\ d_v \nmid i}} \big(N(v)^{i} - 1\big)\Bigg).\end{aligned}$$

Proof. The first equality just follows from Theorem 6.2. Now, for each place $v \in S$, suppose $\mathrm{inv}(\Delta_v) = b_v/d_v$. Since Δ_v is a division ring, the integers b_v and d_v are relatively prime. It is well known that

$$\sum_{v \in S} b_v/d_v \equiv 0 \pmod{\mathbb{Z}}. \tag{6.1}$$

It follows that

$$\sum_{v\in S} r - m_v \equiv 0 \pmod 2. \tag{6.2}$$

Indeed, if r is odd, then each term $r - m_v$ is even. Suppose r is even. Let $S_1 \subset S$ be the subset consisting of places v such that m_v is odd. For each $v \in S_1$, the integer d_v is even and hence b_v is odd. Since r is even it follows from (6.1) that

$$\sum_{v\in S_1} m_v b_v \equiv 0 \pmod 2,$$

and hence $|S_1|$ is even.

The second equality follows from (6.2). □

Proof of Theorem 6.2. Write $\zeta_R(s)$ as

$$\sum_{\text{ideals } \mathfrak{m}\subset A} \frac{b(\mathfrak{m})}{|\mathfrak{m}|^s}$$

where $b(\mathfrak{m}) := \sum_{i=1}^h b_i(\mathfrak{m}) = \#\{\text{right ideals } I \subset R\colon\ N_{B/K}(I) = \mathfrak{m}\}$. Recall the following bijection

$$\begin{aligned} G'\big(\mathbb{A}_K^\infty\big)/G'(\widehat{A}) &\cong \{\text{right fractional ideals of } R\} \\ cG'(\widehat{A}) &\mapsto B \cap c\widehat{R}. \end{aligned}$$

The counting number $b(\mathfrak{m})$ is equal to the number of cosets $cG'(\widehat{A})$ such that $c \in \widehat{R}$ and the coset $K \cap N_{B/K}(c)\widehat{A} = \mathfrak{m}$. Write c as the form $(c_v)_{v\neq\infty}$ where $c_v \in G'(K_v)$. Then for each finite place v of K,

$$N_{B/K}(c_v) \cdot O_v = \mathfrak{m}_v \cdot O_v$$

where $\mathfrak{m}_v \subset A$ is the v-component of $\mathfrak{m}$. Therefore

$$b(\mathfrak{m}) = \prod_{v\neq\infty} b(\mathfrak{m}_v).$$

Let $\mathfrak{p}_v$ denote the ideal of A corresponding to the finite place v. Then

$$\zeta_B(s) = \prod_{v\neq\infty}\left(\sum_{\ell=0}^{\infty} \frac{b(\mathfrak{p}_v^\ell)}{N(v)^{\ell s}}\right).$$

Let O_{Δ_v} be the maximal compact subring in Δ_v and we fix an isomorphism $\varphi_v : B_v \to \mathrm{Mat}_{m_v}(\Delta_v)$ such that

$$R_v := R \otimes_A O_v = \varphi_v^{-1}\big(\mathrm{Mat}_{m_v}(O_{\Delta_v})\big).$$

Choose a generator Π_v of the maximal ideal in O_{Δ_v}. As in the case when Δ_v is a field, we have the "Iwasawa" decomposition for the units group $\mathrm{GL}_{m_v}(\Delta_v)$ of $\mathrm{Mat}_{m_v}(\Delta_v)$, i.e. every element in $\mathrm{GL}_{m_v}(\Delta_v)$ can be written as the form

$$\begin{pmatrix} \Pi_v^{\ell_1} & u_{12} & \cdots & u_{1m_v} \\ 0 & \Pi_v^{\ell_2} & \cdots & u_{2m_v} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Pi_v^{\ell_{m_v}} \end{pmatrix} \cdot U,$$

where $\ell_1, \ldots, \ell_{m_v} \in \mathbb{Z}$, $u_{ij} \in \Delta_v$ for $1 \leqslant i < j \leqslant m_v$, and the element U is in $\mathrm{GL}_{m_v}(O_{\Delta_v})$. So for $\ell \geqslant 0$, $b(\mathfrak{p}_v^\ell)$ is equal to the number of representatives of the form

$$\begin{pmatrix} \Pi_v^{\ell_1} & u_{12} & \cdots & u_{1m_v} \\ 0 & \Pi_v^{\ell_2} & \cdots & u_{2m_v} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Pi_v^{\ell_{m_v}} \end{pmatrix},$$

where $\sum_{i=1}^{m_v} \ell_i = \ell$, $\ell_i \geqslant 0$, and $u_{ij} \in O_{\Delta_v}/\Pi_v^{\ell_i} O_{\Delta_v}$ for $1 \leqslant i < j \leqslant m_v$. This gives

$$b\big(\mathfrak{p}_v^\ell\big) = \sum_{\substack{\ell_1+\cdots+\ell_{m_v}=\ell \\ \ell_i \geqslant 0}} \left(\prod_{i=1}^{m_v} N(v)^{d_v \ell_i (m_v - i)} \right) = \sum_{\substack{\ell_1+\cdots+\ell_{m_v}=\ell \\ \ell_i \geqslant 0}} \left(\prod_{i=1}^{m_v} N(v)^{d_v \ell_i (i-1)} \right).$$

Hence

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{b(\mathfrak{p}_v^\ell)}{N(v)^{\ell s}} &= \sum_{\ell=0}^{\infty} N(v)^{-\ell s} \left(\sum_{\substack{\ell_1+\cdots+\ell_{m_v}=\ell \\ \ell_i \geqslant 0}} \left(\prod_{i=1}^{m_v} N(v)^{d_v \ell_i (i-1)} \right) \right) \\ &= \prod_{i=1}^{m_v} \left(\sum_{\ell_i=0}^{\infty} N(v)^{(i-1) d_v \ell_i - \ell_i s} \right) \\ &= \prod_{i=1}^{m_v} \frac{1}{(1 - N(v)^{(i-1)d_v - s})}. \qquad \square \end{aligned}$$

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