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On the spectral radii and the signless Laplacian spectral radii of c-cyclic graphs with fixed maximum degree ☆

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ABSTRACT

If G is a connected undirected simple graph on n vertices and $n+c-1$ edges, then G is called a c-cyclic graph. Specially, G is called a tricyclic graph if $c=3$. Let $\Delta(G)$ be the maximum degree of G. In this paper, we determine the structural characterizations of the c-cyclic graphs, which have the maximum spectral radii (resp. signless Laplacian spectral radii) in the class of c-cyclic graphs on n vertices with fixed maximum degree $\Delta \geqslant \frac{n+c+1}{2}$. Moreover, we prove that the spectral radius of a tricyclic graph G strictly increases with its maximum degree when $\Delta(G) \geqslant \left(1+\sqrt{6+\frac{2n}{3}}\right)^2$, and identify the first six largest spectral radii and the corresponding graphs in the class of tricyclic graphs on n vertices.

1. Introduction

Throughout the paper, $G=(V,E)$ is a connected undirected simple graph with $|V|=n$ and $|E|=m$. If $m=n+c-1$, then G is called a c-cyclic graph. Specially, if $c=0,1,2$ or 3, then G is called a *tree*, a *unicyclic graph*, a *bicyclic graph*, or a *tricyclic graph*, respectively. Let $\mathcal{C}(n)$ be the class of tricyclic graphs with n vertices. Let $N_G(v)$ denote the neighbor set of vertex v in G, then $d_G(v)=|N_G(v)|$ is called the degree of v of G. If there is no confusion, we write $N_G(v)$ as $N(v)$, and $d_G(v)$ as $d(v)$. Let $\Delta(G)$, Δ for short, be the maximum degree of G. Let $\mathcal{S}(n,\Delta,c)$ be the class of connected c-cyclic graphs on n vertices with fixed maximum degree Δ.

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Let the adjacency matrix, degree matrix of G be $A(G)$ and $D(G)$, respectively. The *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. Denote the spectral radii of $A(G)$, $Q(G)$ by $\rho(G)$ and $\mu(G)$, respectively. The *characteristic polynomial* of $A(G)$ is denoted as $\Phi(G, x)$=det$(xI - A(G))$. Thus, $\rho(G)$ is equal to the maximum root of $\Phi(G, x) = 0$.

When G is connected, by the Perron–Frobenius Theorem of non-negative matrices, $\rho(G)$ and $\mu(G)$ have multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$, and there also exists a unique positive unit eigenvector corresponding to $\mu(G)$. In this paper, we use $f = (f(v_1), \ldots, f(v_n))^T$ to indicate the unique positive unit eigenvector corresponding to $\rho(G)$ or $\mu(G)$, and call f the *Perron vector* of G.

It is a interesting problem concerning graphs with maximal or minimal spectral radii over a given class of graphs. As early as in 1985, Brualdi and Hoffman [1] investigated the maximum spectral radius of the adjacency matrix of a (not necessarily connected) graph in the set of all graphs with a given number of vertices and edges. Their work was followed by other people, in the connected graph case as well as in the general case. Recently, the spectral radii of trees, unicyclic graphs and bicyclic graphs on n vertices with fixed maximum degree were discussed in [2–6], respectively. In this paper, we extend the results of [4,5] to the general c-cyclic graphs by determining the structural characterizations of the c-cyclic graphs, which have the maximum spectral radii (resp. signless Laplacian spectral radii) in the class of c-cyclic graphs on n vertices with fixed maximum degree $\Delta \geqslant \frac{n+c+1}{2}$. Moreover, we prove that the spectral radius of a tricyclic graph G strictly increases with its maximum degree when $\Delta(G) \geqslant \left(1 + \sqrt{6 + \frac{2n}{3}}\right)^2$, and identify the first six largest spectral radii and the corresponding graphs in the class of tricyclic graphs on n vertices.

2. The c-cyclic graphs with maximum spectral radii or signless Laplacian spectral radii in $\mathcal{S}(n, \Delta, c)$

Let $G - u$ or $G - uv$ denote the graph that obtained from G by deleting the vertex $u \in V(G)$ or the edge $uv \in E(G)$, respectively. Similarly, denote by $G + uv$ the graph obtained from G by adding an edge $uv \notin E(G)$.

Lemma 2.1 [7,8]. *Let u, v be two vertices of the connected graph G, and $w_1, w_2, \ldots, w_k$ $(1 \leqslant k \leqslant d(v))$ be some vertices of $N(v) \setminus N(u)$. Let $G' = G + uw_1 + \cdots + uw_k - vw_1 - \cdots - vw_k$. Suppose f is the Perron vector of G, if $f(u) \geqslant f(v)$, then $\rho(G') > \rho(G)$ and $\mu(G') > \mu(G)$.*

Lemma 2.2 [9,10]. *Let $G = (V, E)$ be a connected graph such that $u_1v_1 \in E$, $u_2v_2 \in E$, $v_1v_2 \notin E$, $u_1u_2 \notin E$. Let $G' = G + v_1v_2 + u_1u_2 - u_1v_1 - u_2v_2$. Suppose f is the Perron vector of G, if $f(v_1) \geqslant f(u_2)$ and $f(v_2) \geqslant f(u_1)$, then $\rho(G') \geqslant \rho(G)$ and $\mu(G') \geqslant \mu(G)$, where the equalities hold if and only if $f(v_1) = f(u_2)$ and $f(v_2) = f(u_1)$.*

Lemma 2.3. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in $\mathcal{S}(n, \Delta, c)$, and $u, v \in V(G)$. Suppose f is the Perron vector of G, if $d(u) > d(v)$, then $f(u) > f(v)$.*

Proof. On the contrary, suppose there exist $u, v \in V(G)$ such that $d(u) > d(v)$, but $f(u) \leqslant f(v)$. Suppose $d(u) - d(v) = k$. Let P_{uv} be the shortest path from u to v. Then, there must exist k vertices, say $w_1, \ldots, w_k$, such that $w_1, \ldots, w_k \in N(u) \backslash N(v)$ and $w_1, \ldots, w_k \notin V(P_{uv})$. Let $G_1 = G - uw_1 - \cdots - uw_k + vw_1 + \cdots + vw_k$. By Lemma 2.1, $\rho(G) < \rho(G_1)$ and $\mu(G) < \mu(G_1)$. But $G_1 \in \mathcal{S}(n, \Delta, c)$, it is a contradiction. Thus, $f(u) > f(v)$. □

Let G be a c-cyclic graph, where $c \geqslant 1$. The *base* of G, denoted by $\hat{G}$, is the unique minimal c-cyclic subgraph of G. It is easy to see that $\hat{G}$ is the unique c-cyclic subgraph of G such that $\hat{G}$ contains no pendant vertices, while G can be obtained from $\hat{G}$ by attaching trees to some vertices of $\hat{G}$.

Lemma 2.4. *Let $G = (V, E)$ be a graph of $\mathcal{S}(n, \Delta, c)$, where $\Delta \geqslant \frac{n+c+1}{2}$. If $d(u) = \Delta$, then u is the unique vertex with degree Δ.*

Proof. On the contrary, suppose there exists another vertex v such that $d(u) = d(v) = \Delta$. We consider the next two cases.
Case 1. $uv \notin E(G)$.

Since G is a c-cyclic graph, u and v have at most $c + 1$ common neighbor vertices. Thus, G has at least $d(u) + d(v) - (c + 1) + 2 \geqslant n + 2$ vertices, it is a contradiction.
Case 2. $uv \in E(G)$.

Since G is a c-cyclic graph, u and v have at most c common neighbor vertices. Thus, G has at least $d(u) + d(v) - c \geqslant n + 1$ vertices, it is a contradiction.

This completes the proof of this result. □

Let G be a connected graph and T_v be a tree such that T_v is attached to a vertex v of G. The vertex v is called the *root* of T_v, and T_v is called a root tree of G. Throughout this paper, we assume that T_v does not include the root v.

Lemma 2.5. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in $\mathcal{S}(n, \Delta, c)$. If $\Delta \geqslant \frac{n+c+1}{2}$ and $c \geqslant 1$, then there are at most two vertices in $\hat{G}$ having root trees.*

Proof. On the contrary, suppose there exist three vertices in $\hat{G}$, say u, v, w, having root trees T_u, T_v and T_w, respectively. Let f be the Perron vector of G. Without loss of generality, suppose $d(u) = \max\{d(u), d(w), d(v)\}$ and $f(v) \geqslant f(w)$. Choose $w_1 \in N(w) \cap V(T_w)$, then $w_1 \notin N(v)$. Let $G_1 = G - ww_1 + vw_1$. Lemma 2.4 implies that $d(w) < \Delta$ and $d(v) < \Delta$, thus $G_1 \in S(n, \Delta, c)$. By Lemma 2.1, we can conclude that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction. Thus, the result follows. □

Given $u, v \in V(G)$, the symbol $dist(u, v)$ is used to denote the *distance* between u and v, namely, the length of (number of edges in) the shortest path that connects u and v in G.

Lemma 2.6. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in $\mathcal{S}(n, \Delta, c)$, where $\Delta \geqslant \frac{n+c+1}{2}$ and $c \geqslant 1$. If $d(u) = \Delta$, $w \in N(u)$ and $w \notin V(\hat{G})$, then $d(w) = 1$.*

Proof. On the contrary, suppose there is a vertex $w \in N(u)$ such that $w \notin V(\hat{G})$, but $d(w) \geqslant 2$. Since $w \notin V(\hat{G})$, w must be in some root tree of G. Let $dist(w, v) = \min\{dist(w, v_1), v_1 \in V(\hat{G})\}$. Let f be the Perron vector of G. We consider the next two cases:
Case 1. $v = u$.

Then, $u \in V(\hat{G})$, and hence there exists vertex $u_1 \in V(\hat{G}) \cap N(u)$.
Subcase 1.1. $f(w) \geqslant f(u_1)$.

Then, there exists vertex $u_2 \in V(\hat{G}) \cap N(u_1)$ such that $u_2 \notin N(w)$. Let $G_1 = G - u_1u_2 + wu_2$. It is easy to see that $G_1 \in \mathcal{S}(n, \Delta, c)$. On the other hand, Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction.
Subcase 1.2. $f(w) < f(u_1)$.

Then, there exists vertex $w_1 \in N(w) \setminus \{u\}$. Let $G_1 = G - w_1w + w_1u_1$. It is easy to see that $G_1 \in \mathcal{S}(n, \Delta, c)$. While Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction.
Case 2. $v \neq u$.

Then, $u \notin V(\hat{G})$. Otherwise, if $u \in V(\hat{G})$, then $w \in V(\hat{G})$, a contradiction. Noting that there exists a vertex $v_1 \in V(\hat{G}) \cap N(v)$, if $f(w) \geqslant f(v)$, we let $G_1 = G - vv_1 + wv_1$ to obtain a contradiction. And noting that there exists a vertex $w_1 \in N(w) \cap N(T_w)$, if $f(w) < f(v)$, we let $G_1 = G - ww_1 + vw_1$ to obtain a contradiction. □

Lemma 2.7. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in* $\mathcal{S}(n, \Delta, c)$*, where* $\Delta \geqslant \frac{n+c+1}{2}$ *and* $c \geqslant 1$*. Suppose f is the Perron vector of G, if* $d(u) = \Delta$*,* $v \in N(u)$ *and* $w \notin N(u) \cup \{u\}$*, then* $f(v) \geqslant f(w)$*.*

Proof. On the contrary, suppose there are vertices $v \in N(u)$ and $w \notin N(u) \cup \{u\}$ such that $f(v) < f(w)$. We consider the next two cases:
Case 1. $v \notin V(\hat{G})$.

By Lemma 2.6, we have $d(v) = 1$. Let P_{uw} be the shortest path from u to w such that $w_1 \in V(P_{uw})$ and $ww_1 \in E(P_{uw})$. Note that $w \notin N(u)$. Hence, $w_1 \neq u$. By Lemmas 2.3 and 2.4, we can conclude that $f(u) > f(w_1)$. Let $G_1 = G - uv - ww_1 + vw_1 + uw$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. By Lemma 2.2, we can conclude that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction.
Case 2. $v \in V(\hat{G})$.

Then, $d(v) \geqslant 2$. Let P_{vw} be the shortest path from v to w. We claim that there must exist vertex $v_1 \notin V(P_{vw})$ such that $v_1 \in N(v) \setminus N(w)$. Otherwise, if $v_1 \in N(v)$, then $v_1 \in N(w)$ holds for every $v_1 \notin V(P_{vw})$. Note that $d(v) \geqslant 2$. Hence, $|P_{vw}| \leqslant 2$. It is a contradiction to $v \in N(u)$ and $w \notin N(u)$ (We only need to consider the cases of $vw \in E(G)$ or $vw \notin E(G)$). Thus, there exists vertex $v_1 \notin V(P_{vw})$ such that $v_1 \in N(v) \setminus N(w)$. Let $G_1 = G - vv_1 + wv_1$. By Lemma 2.4, $G_1 \in S(n, \Delta, c)$. By Lemma 2.1, it follows that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction. □

Lemma 2.8. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in* $\mathcal{S}(n, \Delta, c)$*, where* $\Delta \geqslant \frac{n+c+1}{2}$ *and* $c \geqslant 1$*. If* $x \in V(\hat{G})$ *and there is a root tree* T_x*, then* T_x *is a star, and x is the center vertex of* T_x*.*

Proof. If $x = u$, the result follows from Lemma 2.6. Next we assume that $x \neq u$ and T_x is not a star. Let $y \in V(T_x)$ such that $dist(x, y) = \max\{dist(x, v), v \in V(T_x) \text{ and } d(v) \geqslant 2\}$. Let f be the Perron vector of G. We consider the next two cases:
Case 1. $f(y) \geqslant f(x)$.

Then, there must exist vertex $x_1 \in N(x) \cap V(\hat{G})$ such that $d(x_1) \geqslant 2$ and $x_1 \notin N(y)$. We claim that $y \neq u$. Otherwise, assume that $y = u$. By the choice of y, there exists some pendant vertex y_1 such that $y_1 \in N(y)$. Then, $f(y_1) \geqslant f(x_1)$ follows from Lemma 2.7, while Lemma 2.3 implies that $f(y_1) < f(x_1)$, a contradiction. Thus, $y \neq u$.

Let $G_1 = G - xx_1 + yx_1$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction.
Case 2. $f(y) < f(x)$.

By Lemmas 2.3 and 2.4, $y \neq u$. By the choice of y, there must exist some pendant vertex $y_1 \in N(y)$. Let $G_1 = G - yy_1 + xy_1$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction. □

Denote by $\omega(G)$ the number of vertices of G, namely, $\omega(G) = |V(G)|$.

Lemma 2.9. *Let G be a graph in* $\mathcal{S}(n, \Delta, c)$*, where* $\Delta \geqslant \frac{n+c+1}{2}$ *and* $c \geqslant 1$*. If* $n \geqslant 3c$ *and* $d(u) = \Delta$*, then there must exist some vertex* $w \notin V(\hat{G})$ *such that* $w \in N(u)$*.*

Proof. On the contrary, suppose that $w \in N(u)$ implies $w \in V(\hat{G})$. Two cases should be considered as follows.
Case 1. $G = \hat{G}$.

Then, $2n + 2c - 2 = \sum_{i=1}^{n} d_G(v_i) \geqslant 2(n-1) + \Delta \geqslant 2(n-1) + \frac{n+c+1}{2} \geqslant 2(n-1) + \frac{4c+1}{2} = 2n + 2c - 2 + \frac{1}{2}$, a contradiction.
Case 2. $G \neq \hat{G}$.

Then, G has one pendant vertex v_1. Let $G_1 = G - v_1$. It is easy to see that G_1 is a graph in $\mathcal{S}(n-1, \Delta, c)$ because $v_1 \notin N(u)$. If G_1 does not have any pendant vertices, then $G_1 = \hat{G}$. If G_1 has one pendant vertex v_2. Let $G_2 = G_1 - v_2$. It is easy to see that G_2 is a graph in $\mathcal{S}(n-2, \Delta, c)$ because $v_2 \notin N_{G_1}(u)$.

Repeat the above process, and suppose $\omega(\hat{G}) = b$. Then, we can conclude that $\hat{G} \in \mathcal{S}(b, \Delta, c)$. Thus, we have $2b+2c-2 = \sum_{i=1}^{n} d_{\hat{G}}(v_i) \geqslant 2(b-1)+\Delta \geqslant 2(b-1)+\frac{n+c+1}{2} \geqslant 2(b-1)+\frac{4c+1}{2} = 2b+2c-2+\frac{1}{2}$, a contradiction. Thus, the result follows. □

If $v \in V(G)$ and $d(v) \neq 1$, then v is called a *non-pendant vertex* of G. As usually, let P_n, C_n and K_n be the path, cycle and complete graph on n vertices, respectively. Let $2K_2$ be the graph on 4 vertices, which is the union of two edges. Here is the main result of this section.

Theorem 2.1. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in $\mathcal{S}(n, \Delta, c)$, where $c \geqslant 1$, $n \geqslant 3c$ and $\Delta \geqslant \frac{n+c+1}{2}$. Suppose u and v are the vertices of G which share the maximum degree and the second maximum degree of $\hat{G}$, respectively. Then, G must satisfy the following conditions:*

(1) *Every non-pendant vertex is adjacent to u.*
(2) *$\hat{G}$ has no induced subgraphs that are isomorphic to $2K_2$, P_4 or C_4.*
(3) *G is obtained from $\hat{G}$ by attaching $\Delta + 1 - \omega(\hat{G})$ pendant vertices to u, and $n - \Delta - 1$ pendant vertices to v, respectively.*

Proof. Let f be the Perron vector of G. Suppose $u_0 \in V(G)$ such that $d(u_0) = \Delta$. Since $\Delta \geqslant \frac{n+c+1}{2}$, u_0 is the unique vertex of G with degree Δ by Lemma 2.4. By Lemmas 2.6 and 2.9, there must exist some pendant vertex $x \in N(u_0)$. Now assume that there exists a non-pendant vertex y such that $y \notin N(u_0)$. Lemma 2.7 implies that $f(x) \geqslant f(y)$, while Lemma 2.3 implies that $f(x) < f(y)$, a contradiction. Thus, every non-pendant vertex is adjacent to u_0. Thus, (1) follows, and hence $u_0 = u$.

Assume $\hat{G}$ has $2K_2$ as an induced subgraph. Let $V(2K_2)=\{v_1, v_2, v_3, v_4\}$ and $E(2K_2)=\{v_1v_2, v_3v_4\}$. By (1) we can conclude that $u \notin \{v_1, v_2, v_3, v_4\}$ and $uv_i \in E(G)$, where $1 \leqslant i \leqslant 4$. Without loss of generality, suppose $f(v_1) \geqslant f(v_3)$. Let $G_1 = G - v_3v_4 + v_1v_4$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. But Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction. Now suppose $\hat{G}$ has P_4 as an induced subgraph. Let $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and $E(P_4) = \{v_1v_2, v_2v_3, v_3v_4\}$. By (1) we can conclude that $u \notin \{v_1, v_2, v_3, v_4\}$ and $uv_i \in E(G)$, where $1 \leqslant i \leqslant 4$. Without loss of generality, suppose $f(v_2) \geqslant f(v_3)$. Let $G_1 = G - v_3v_4 + v_2v_4$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. But Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction. It can be proved similarly that G has no induced subgraphs that are isomorphic to C_4. Thus, (2) follows.

Now we prove (3). By Lemmas 2.5 and 2.8, G is obtained from $\hat{G}$ by attaching some pendant vertices to at most two vertices, say w_1, w_2, of $\hat{G}$, respectively. Since $n \geqslant 3c$, by Lemmas 2.6 and 2.9 we can conclude that $w_1 = u$. Next we shall show that $w_2 = v$ if there is a root tree T_{w_2}. On the contrary, suppose $w_2 \neq v$, then $d_{\hat{G}}(w_2) < d_{\hat{G}}(v)$.

If $f(w_2) \geqslant f(v)$, then there must exist $x \in N(v) \cap V(\hat{G})$ such that $x \notin N(w_2)$. By (1), $u \notin \{x, v, w_2\}$, $uw_2 \in E(G)$ and $uv \in E(G)$. Let $G_1 = G - xv + xw_2$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. But Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction.

If $f(w_2) < f(v)$, then there must exist $y \in N(w_2) \cap V(T_{w_2})$. By (1), $u \notin \{y, v, w_2\}$, $uw_2 \in E(G)$ and $uv \in E(G)$. Let $G_1 = G - yw_2 + yv$. Then, $G_1 \in \mathcal{S}(n, \Delta, c)$. But Lemma 2.1 implies that $\rho(G_1) > \rho(G)$ and $\mu(G_1) > \mu(G)$, a contradiction.

Thus, $w_2 = v$ and hence (3) follows from (1). □

Remark 2.1. In some literature (for instance, see [11]), if G has no $2K_2$, P_4 or C_4 as an induced subgraph, then G is called a *split graph*. By Theorem 2.1, if G is the graph with maximum spectral radius or signless Laplacian spectral radius in $\mathcal{S}(n, \Delta, c)$, where $c \geqslant 1$, $n \geqslant 3c$ and $\Delta \geqslant \frac{n+c+1}{2}$, then $\hat{G}$ is a split graph.

In the following, as shown in Fig. 1, let $M(n, \Delta, c)$ be the c-cyclic graph on n vertices and maximum degree Δ, where $c \geqslant 1$ and $\Delta \geqslant \frac{n+c+1}{2}$, and let H_1,H_2, H_3, H_4 and H_5 be the tricyclic graphs as shown in Fig. 2. For convenience, we write $M(n, \Delta, 3)$ as $F_1(\Delta)$. Let $F_2(\Delta)$ be the tricyclic graph obtained from K_4 by attaching $\Delta - 3$ and $n - 1 - \Delta$ pendant vertices to two vertices of K_4, respectively.

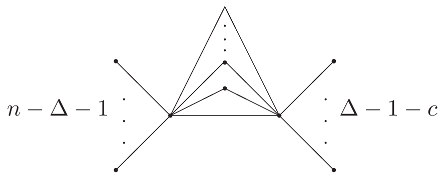

Fig. 1. The c-cyclic graph $M(n, \Delta, c)$.

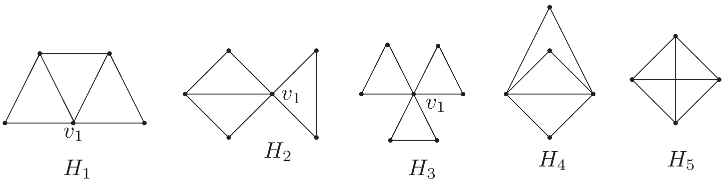

Fig. 2. The tricyclic graphs H_i, $i = 1, 2, \ldots, 5$.

Corollary 2.1. *Let G be the graph with the maximum spectral radius or signless Laplacian spectral radius in $\mathcal{S}(n, \Delta, c)$, where $c \geqslant 1$ and $\Delta(G) \geqslant \frac{n+c+1}{2}$.*

(1) *If G is a unicyclic graph, then $G \cong M(n, \Delta, 1)$.*
(2) *If $n \geqslant 6$ and G is a bicyclic graph, then $G \cong M(n, \Delta, 2)$.*
(3) *If $n \geqslant 9$ and G is a tricyclic graph, then $G \cong F_1(\Delta)$ or $G \cong F_2(\Delta)$.*

Proof. We only prove (3), because the other cases can be proved similarly. By Theorem 2.1, every non-pendant vertex is adjacent to the vertex with maximum degree of $\hat{G}$. Thus, there are only five possible candidates of $\hat{G}$ (see Fig. 2) because $\hat{G}$ is also a tricyclic graph. Moreover, $\hat{G}$ has no induced subgraphs that are $2K_2$ and P_4 by (2) of Theorem 2.1, then we have $G \cong F_1(\Delta)$ or $G \cong F_2(\Delta)$. $\square$

Remark 2.2. For the cases of spectral radii of unicyclic and bicyclic graphs, the corresponding results of Corollary 2.1 had been obtained in [4] and [5], respectively.

Theorem 2.2. *Let G be the graph with the maximum spectral radius (resp. signless Laplacian spectral radius) in $\mathcal{S}(n, \Delta, c)$. If $\Delta \leqslant n-2$, then there must exist some graph $G_1 \in \mathcal{S}(n, \Delta+1, c)$ such that $\rho(G) < \rho(G_1)$ (resp. $\mu(G) < \mu(G_1)$).*

Proof. Let f be the Perron vector of G. Suppose $x \in V(G)$ such that $d(x) = \Delta$ and $f(x) = \max\{f(u), d(u) = \Delta\}$. Since $\Delta(G) \leqslant n-2$, there must exist $y \in N(x)$ and $z \notin N(x)$ such that $yz \in E(G)$.

If $d(x) > d(y)$, Lemma 2.3 implies that $f(x) > f(y)$. Let $G_1 = G - yz + xz$. Then, $G_1 \in \mathcal{S}(n, \Delta+1, c)$. By Lemma 2.1, we have $\rho(G) < \rho(G_1)$ and $\mu(G) < \mu(G_1)$.

If $d(x) = d(y)$, by the choice of x we have $f(x) \geqslant f(y)$. Let $G_1 = G - yz + xz$. Then, $G_1 \in \mathcal{S}(n, \Delta+1, c)$. By Lemma 2.1, we have $\rho(G) < \rho(G_1$ and $\mu(G) < \mu(G_1)$.

Thus, the result follows. $\square$

3. A relation between $\rho(G)$ and $\Delta(G)$ of a graph G in $\mathcal{C}(n)$

The following result is often used to calculate the characteristic polynomials of graphs.

Lemma 3.1 [12] (Schwenk's formulas)**.** *Let G be a (simple) graph. Denote by C_v the set of all cycles in G containing a vertex v. Then,*

$$\Phi(G, x) = x\Phi(G-v, x) - \sum_{w \sim v} \Phi(G-v-w, x) - 2\sum_{C \in C_v} \Phi(G - V(C), x).$$

Theorem 3.1. *Let G be the tricyclic graph with the maximum spectral radius in* $\mathcal{S}(n, \Delta, 3)$*, where* $n \geqslant 9$ *and* $\Delta \geqslant \frac{n}{2} + 2$*. (1) If* $\Delta \leqslant n - 5$*, then* $G \cong F_1(\Delta)$*. (2) If* $n - 4 \leqslant \Delta \leqslant n - 1$*, then* $G \cong F_2(\Delta)$*.*

Proof. Let $f_1(x) = x^4 - (n+2)x^2 - 6x - \Delta^2 + 3\Delta - n + n\Delta - 11$, $f_2(x) = x^5 - x^4 - (n+1)x^3 + (n-7)x^2 + (n\Delta + 2\Delta - n - \Delta^2 - 5)x + 3n + \Delta^2 - n\Delta - 3 - 2\Delta$. By Lemma 3.1, we have

$$\Phi(F_1(\Delta), x) = x^{n-4} f_1(x); \quad \Phi(F_2(\Delta), x) = x^{n-6}(x+1) f_2(x). \tag{1}$$

By Eq. (1), $\rho(F_1(\Delta))$ is equal to the maximum root of $f_1(x) = 0$, and $\rho(F_2(\Delta))$ is equal to the maximum root of $f_2(x) = 0$. Set $\gamma_1(x) = x^3 - 3x^2 - \Delta x + 2n + \Delta - 14$, and denote by α_1 the maximum root of $\gamma_1(x) = 0$. Let $\gamma_2(x) = (\Delta + 7 - n)x^2 - 2(n - 4 - \Delta)x + (n\Delta - \Delta^2 - 7n + 31)$. It is easy to see that

$$f_2(x) = f_1(x)(x-1) + \gamma_1(x). \tag{2}$$

$$f_1(x) = (x+3)\gamma_1(x) + \gamma_2(x); \quad f_2(x) = (x^2 + 2x - 2)\gamma_1(x) + (x-1)\gamma_2(x). \tag{3}$$

By Corollary 2.1, we have $G \cong F_1(\Delta)$ or $G \cong F_2(\Delta)$. We consider the next three cases:
Case 1: $\Delta \leqslant n - 7$.

Since $\frac{n}{2} + 2 \leqslant \Delta \leqslant n - 7$, we have $n \geqslant 18$ and $\Delta \geqslant 11$. When $x > \sqrt{\Delta} > 1 + \sqrt{1 + \frac{\Delta}{3}}$, Since $\gamma_1'(x) = 3x^2 - 6x - \Delta > 0$, it follows that $\gamma_1(x) > \gamma_1(\sqrt{\Delta}) = 2(n - 7 - \Delta) \geqslant 0$. Combining with Eq. (2), we have $f_2(x) > 0$ when $x \geqslant \rho(F_1(\Delta)) > \sqrt{\Delta}$. Moreover, note that $\lim_{x \longrightarrow +\infty} f_2(x) = +\infty$, hence $\rho(F_2(\Delta)) < \rho(F_1(\Delta))$. Then, $G \cong F_1(\Delta)$.
Case 2: $n - 4 \leqslant \Delta \leqslant n - 1$.

Since $\lim_{x \longrightarrow -\infty} \gamma_1(x) = -\infty$, $\gamma_1(0) = 2n + \Delta - 14 > 0$, $\gamma_1(\sqrt{\Delta}) = 2(n - 7 - \Delta) < 0$, $\lim_{x \longrightarrow +\infty} \gamma_1(x) = +\infty$, we have $\alpha_1 > \sqrt{\Delta}$. When $x > \sqrt{\Delta} > 2$, note that $\gamma_2'(x) = 2(\Delta + 7 - n)x - 2(n - \Delta - 4) > 6(\Delta + 6 - n) > 0$, hence $\gamma_2(x) > \gamma_2(\sqrt{\Delta}) = (\Delta - n + 4)(7 + 2\sqrt{\Delta}) + 3 > 0$. By Eq. (3), we have $f_1(x) > 0$ and $f_2(x) > 0$ when $x \geqslant \alpha_1 > \sqrt{\Delta}$. Thus, $\rho(F_1(\Delta)), \rho(F_2(\Delta)) \in (\sqrt{\Delta}, \alpha_1)$. Once again, Eq. (2) implies that $f_2(\rho(F_1(\Delta))) = \gamma_1(\rho(F_1(\Delta))) < 0$. Combining with $\lim_{x \longrightarrow +\infty} f_2(x) = +\infty$, we have $\rho(F_1(\Delta)) < \rho(F_2(\Delta))$, the result follows.
Case 3: $n - 6 \leqslant \Delta \leqslant n - 5$.

Here we only consider the case of $\Delta = n - 5$, because the case of $\Delta = n - 6$ can be proved similarly. By $\frac{n}{2} + 2 \leqslant \Delta = n - 5$, we have $n \geqslant 14$. When $n = 14$, by Eq. (1) the result follows. Next we may suppose that $n \geqslant 15$. Since $\gamma_1(\sqrt{n}) = 5\sqrt{n} - 19 > 0$, by the discussion of Case 2 we can conclude that $\sqrt{n-5} < \alpha_1 < \sqrt{n}$. When $\sqrt{n-5} < x < \sqrt{n}$, since $\gamma_2'(x) > 0$, we have $\gamma_2(x) = 2(x^2 - x - n + 3) < \gamma_2(\sqrt{n}) = 2(3 - \sqrt{n}) < 0$. By Eq. (3), it follows that $f_1(\alpha_1) = \gamma_2(\alpha_1) < 0$, $f_2(\alpha_1) = (\alpha_1 - 1)\gamma_2(\alpha_1) < 0$, hence $\rho(F_1(\Delta)), \rho(F_2(\Delta)) > \alpha_1$. Therefore, when $x \geqslant \rho(F_1(\Delta)) > \alpha_1$, Eq. (2) implies that $f_2(x) \geqslant \gamma_1(x) > 0$. Thus, $\rho(F_2(\Delta)) < \rho(F_1(\Delta))$, the result follows. □

In the following, let $S_1 = F_2(n-1)$, $S_2 = F_1(n-1)$, S_3 be the graph obtained from H_1 by attaching $n - 5$ pendant vertices to v_1, S_4 be the graph obtained from H_2 by attaching $n - 6$ pendant vertices to v_1, S_5 be the graph obtained from H_3 by attaching $n - 7$ pendant vertices to v_1, and $S_6 = F_2(n-2)$.

In 1981, Cvetković [13] indicated 12 directions in further investigations of graph spectra, one of which is classifying and ordering graphs. After then, ordering graphs with various properties by their spectra, becomes an attractive topic (see [16–20]). There are many corresponding results of order of trees, unicyclic and bicyclic graphs via their spectral radii [16–20], while few results on the tricyclic graphs. Up to now, to our best knowledge, only the tricyclic graph which has the maximum spectral radius in $\mathcal{C}(n)$ had been determined.

Theorem 3.2 [14]. *Let G be the graph with the maximum spectral radius in $\mathcal{C}(n)$, then $G \cong S_1$.*

By Theorem 3.1, next we shall extend the order of Theorem 3.2 to the first six largest tricyclic graphs.

Theorem 3.3. *Suppose $n \geqslant 18$. If $G \in \mathcal{C}(n) \setminus \{S_1, S_2, ..., S_6\}$, then $\rho(G) < \rho(S_6) < \rho(S_5) < \cdots < \rho(S_1)$.*

Let $K_{1,n-1}$ be the star on n vertices. The proof of Theorem 3.3 needs the next Lemma.

Lemma 3.2 [15]. *If G is a connected graph on n vertices and m edges, then $\rho(G) \leqslant \sqrt{2m-n+1}$, where equality holds if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.*

The proof of Theorem 3.3. Since $n \geqslant 18$, Theorems 2.2 and 3.1 imply that $\rho(G) < \rho(S_6)$ because $S_1, S_2, ..., S_5$ are all the tricyclic graphs on n vertices with maximum degree $n-1$. By Theorem 3.2, we only need to show that $\rho(S_6) < \rho(S_5) < \cdots < \rho(S_2)$.

By Lemma 3.1 and Eq. (1), we have

$$\Phi(S_2, x) = x^{n-4}(x^4 - (n+2)x^2 - 6x + 3(n-5)).$$
$$\Phi(S_3, x) = x^{n-6}(x^2 + x - 1)(x^4 - x^3 - nx^2 + (n-7)x + (n-5)).$$
$$\Phi(S_4, x) = x^{n-6}(x+1)(x^5 - x^4 - (n+1)x^3 + (n-5)x^2 + (2n-4)x - 2n + 12).$$
$$\Phi(S_5, x) = x^{n-8}(x-1)^2(x+1)^3(x^3 - x^2 - (n-1)x + n - 7).$$
$$\Phi(S_6, x) = x^{n-6}(x+1)(x^5 - x^4 - (n+1)x^3 + (n-7)x^2 + (3n-13)x - n + 5).$$

When $x \geqslant \sqrt{n-1}$, since $\Phi(S_3, x) - \Phi(S_2, x) = x^{n-6}(3x^2 + 2x - n + 5) \geqslant x^{n-6}(2n + 2 + 2\sqrt{n-1}) > 0$, it follows that $\rho(S_3) < \rho(S_2)$.

When $x \geqslant \sqrt{n-1}$, Since $\Phi(S_4, x) - \Phi(S_3, x) = x^{n-6}(3x^2 + 6x - n + 7) \geqslant x^{n-6}(2n + 4 + 6\sqrt{n-1}) > 0$, we have $\rho(S_4) < \rho(S_3)$.

When $x \geqslant \sqrt{n-1}$, since $\Phi(S_5, x) - \Phi(S_4, x) = x^{n-8}(3x^4 + 4x^3 - (n-2)x^2 - 6x + n - 7) > x^{n-8}(x^2(3(n-1)-(n-2)) + x(4(n-1)-6) + n - 7) = x^{n-8}((2n-1)x^2 + (4n-10)x + n - 7) > 0$, we have $\rho(S_5) < \rho(S_4)$.

Next we shall show that $\rho(S_6) < \rho(S_5)$.

When $n = 18$, by $\Phi(S_5, x)$ and $\Phi(S_6, x)$, it is easily checked that $\rho(S_6) < \rho(S_5)$. Next we may suppose that $n \geqslant 19$. Let $f_3(x) = x^3 - x^2 - (n-1)x + n - 7$, $f_4(x) = x^5 - x^4 - (n+1)x^3 + (n-7)x^2 + (3n-13)x - n + 5$. Then, $\rho(S_5)$ and $\rho(S_6)$ are equal to the maximum roots of $f_3(x) = 0$ and $f_4(x) = 0$, respectively. Let $\gamma_3(x) = -2x^2 + (n-11)x + n - 9$. Then,

$$f_4(x) = (x^2 - 2)f_3(x) + \gamma_3(x). \tag{4}$$

By Lemma 3.2, it follows that $\sqrt{n-2} < \rho(S_6) < \sqrt{n+5}$, and $\sqrt{n-1} < \rho(S_5) < \sqrt{n+5}$. When $\sqrt{n-1} < x < \sqrt{n+5}$, we have $\gamma_3(x) > \min\{\gamma_3(\sqrt{n-1}), \gamma_3(\sqrt{n+5})\} = \min\{(n-11)\sqrt{n-1} - (n+7), (n-11)\sqrt{n+5} - (n+19)\} > 0$. Thus, when $\rho(S_5) \leqslant x < \sqrt{n+5}$, we have $f_4(x) \geqslant \gamma_3(x) > 0$ by Eq. (4). Hence, $\rho(S_6) < \rho(S_5)$. $\square$

Note that $S_1, S_2, ..., S_5$ are all the tricyclic graphs on n vertices with maximum degree $n-1$. Thus, we may consider the next problem: Whether the spectral radius of a tricyclic graph strictly increases with its maximum degree. The answer is positive when Δ is enough large because we have

Theorem 3.4. *Suppose G, G' are two tricyclic graphs on n vertices. If $\Delta(G) \geqslant \left(1 + \sqrt{6 + \frac{2n}{3}}\right)^2$ and $\Delta(G) > \Delta(G')$, then $\rho(G) > \rho(G')$.*

The proof of Theorem 3.4 needs the next Lemma.

Lemma 3.3. *If* $\Delta \geqslant \left(1+\sqrt{6+\frac{2n}{3}}\right)^2 - 1$, *then* $\rho(F_1(\Delta)) \leqslant \sqrt{\Delta+1}$.

Proof. By Eq. (1), $\rho(F_1(\Delta))$ is equal to the maximum root of $f_1(x)=0$. When $x \geqslant \sqrt{\Delta+1} > \sqrt{\frac{2n}{3}}$, since $f_1''(x) = 12x^2 - 2n - 4 > 0$, we have

$$f_1'(x) = 4x^3 - 2(n+2)x - 6 > 4\left(\sqrt{\frac{2n}{3}}\right)^3 - 2(n+2)\sqrt{\frac{2n}{3}} - 6 = \left(\frac{2n}{3} - 4\right)\sqrt{\frac{2n}{3}} - 6 > 0.$$

Hence, when $x \geqslant \sqrt{\Delta+1} \geqslant 1+\sqrt{6+\frac{2n}{3}}$, it follows that

$$\begin{aligned} f_1(x) &\geqslant (\Delta+1)^2 - (n+2)(\Delta+1) - 6\sqrt{\Delta+1} + (n-\Delta+3)\Delta - n - 11 \\ &= 3\Delta - 6\sqrt{\Delta+1} - 2n - 12 \\ &= 3\left(\sqrt{\Delta+1}-1\right)^2 - 2n - 18 \\ &\geqslant 0. \end{aligned}$$

Therefore, $\rho(F_1(\Delta)) \leqslant \sqrt{\Delta+1}$. □

The proof of Theorem 3.4. In the proof of this result, we write $\Delta(G')$ as Δ', and $\Delta(G)$ as Δ. Let $a = \left(1+\sqrt{6+\frac{2n}{3}}\right)^2 - 1$. We consider the next two cases.

Case 1. $\Delta \leqslant n-5$.

Subcase 1.1. $\Delta' \geqslant a$.

Since $\Delta' \geqslant a > \frac{n}{2}+2$, by Theorem 3.1 and Lemma 3.3 we can conclude that $\rho(G') \leqslant \rho(F_1(\Delta')) \leqslant \sqrt{\Delta'+1}$. On the other hand, since $\Delta > \Delta'$, we have $\rho(G) > \sqrt{\Delta} \geqslant \sqrt{\Delta'+1}$, hence the result follows.

Subcase 1.2. $\Delta' < a$.

Note that $a > \frac{n}{2}+2$. By Theorem 2.2 and Lemma 3.3 we can conclude that $\rho(G') < \rho(F_1(a)) \leqslant \sqrt{a+1}$. Moreover, since $\Delta \geqslant \left(1+\sqrt{6+\frac{2n}{3}}\right)^2 = a+1$, it follows that $\rho(G) > \sqrt{\Delta} \geqslant \sqrt{a+1}$, the result also follows.

Case 2. $n-4 \leqslant \Delta \leqslant n-1$.

By Theorem 3.3, we only need to consider the cases of $n-4 \leqslant \Delta \leqslant n-2$.

Subcase 2.1. $\Delta = n-4$.

Then, $\Delta' \leqslant n-5$. By $n-4 \geqslant \left(1+\sqrt{6+\frac{2n}{3}}\right)^2$, it follows that $n \geqslant 79$. Since $n-5 > \frac{n}{2}+2$, by Theorems 2.2 and 3.1 we have $\rho(G') \leqslant \rho(F_1(n-5))$. Let $f_5(x) = x^4 - (n+2)x^2 - 6x + 7n - 51$. By Eq. (1), $\rho(F_1(n-5))$ equals to the maximum root of $f_5(x)=0$. When $x \geqslant \sqrt{n-4}$, since $f_5''(x) = 12x^2 - 2n - 4 \geqslant 10n - 52 > 0$, we have $f_5'(x) = 4x^3 - (2n+4)x - 6 \geqslant (2n-20)\sqrt{n-4} - 6 > 0$, and hence $f_5(x) \geqslant f_5(\sqrt{n-4}) = n - 27 - 6\sqrt{n-4} > 0$. Then, $\rho(G') \leqslant \rho(F_1(n-5)) < \sqrt{n-4} < \rho(G)$, the result follows.

Subcase 2.2. $\Delta = n-3$.

Then, $\Delta' \leqslant n-4$. By $n-3 \geqslant \left(1+\sqrt{6+\frac{2n}{3}}\right)^2$, it follows that $n \geqslant 75$. Thus, $n-4 > \frac{n}{2}+2$. By Theorems 2.2 and 3.1, we have $\rho(G') \leqslant \rho(F_2(n-4))$. Let $f_6(x) = x^5 - x^4 - (n+1)x^3 + (n-7)x^2 + (5n-29)x - 3(n-7)$. By Eq. (1), $\rho(F_2(n-4))$ equals to the maximum root of $f_6(x)=0$.

Since $\lim_{x \longrightarrow -\infty} f_6(x) = -\infty$, $f_6(-3) = 18n - 252 > 0$, $f_6(0) = 21 - 3n < 0$, $f_6(1) = 2n - 16 > 0$, $f_6(3) = 6 - 6n < 0$, and $f_6(\sqrt{n-3}) = (n-17)(\sqrt{n-3}-7) - 86 > 0$, we have $\rho(G') \leqslant \rho(F_2(n-4)) < \sqrt{n-3} < \rho(G)$.

Subcase 2.3. $\Delta = n-2$.

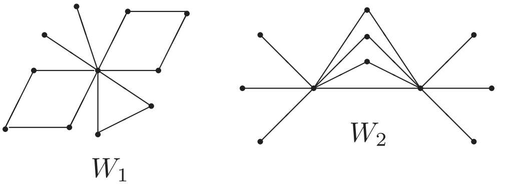

Fig. 3. The tricyclic graphs W_1 and W_2.

Then, $\Delta' \leqslant n-3$. By $n-2 \geqslant \left(1+\sqrt{6+\frac{2n}{3}}\right)^2$, it follows that $n \geqslant 71$. Thus, $n-3 > \frac{n}{2}+2$. By Theorems 2.2 and 3.1, we have $\rho(G') \leqslant \rho(F_2(n-3))$. Let $f_7(x) = x^5 - x^4 - (n+1)x^3 + (n-7)x^2 + (4n-20)x - 2(n-6)$. By Eq. (1), $\rho(F_2(n-3))$ equals to the maximum root of $f_7(x) = 0$.

If $n = 71$, since $\Delta = n-2$, G has T^* as its proper subgraph, where T^* is a tree of order 71 obtained by attaching one pendant vertex to one pendant vertex of the star $K_{1,69}$. Thus, $\rho(G) > \rho(T^*) > 8.3075 > 8.3069 > \rho(F_2(68)) \geqslant \rho(G')$, the result follows.

If $n \geqslant 72$, since $\lim_{x \longrightarrow -\infty} f_7(x) = -\infty$, $f_7(-2) = 2n-16 > 0$, $f_7(0) = 12-2n < 0$, $f_7(1) = 2n-16 > 0$, $f_7(3) = 24-8n < 0$, and $f_7(\sqrt{n-2}) = (n-14)(\sqrt{n-2}-7) - 76 > 0$, we have $\rho(G') \leqslant \rho(F_2(n-3)) < \sqrt{n-2} < \rho(G)$.

By combining the above arguments, this completes the proof of this result. $\square$

4. Remarks

Bearing Theorems 2.1 and 2.2 in mind, we find there are many similar properties between the spectral radius and signless Laplacian spectral radius of a graph. Thus, it is natural to consider the following question: "Whether the signless Laplacian spectral radius of a tricyclic graph strictly increases with its maximum degree when Δ is enough large". The answer is given by the next result.

Theorem 4.1 [21]. *Let G and G′ be two tricyclic graphs on n vertices. If* $\Delta(G) \geqslant \lceil \frac{n-1}{2} \rceil + 4$ *and* $\Delta(G) > \Delta(G')$, *then* $\mu(G) > \mu(G')$.

Actually, we obtained the similar result for general graphs as follows.

Theorem 4.2 [21]. *Let G and G′ be two connected graphs with n vertices and m edges. If* $\Delta(G) \geqslant m - \lfloor \frac{n-1}{2} \rfloor + 1$ *and* $\Delta(G) > \Delta(G')$, *then* $\mu(G) > \mu(G')$.

As the next Example shown, the bound $\lceil \frac{n-1}{2} \rceil + 4$ in Theorem 4.1 is best possible.

Example 4.1. Let W_1 and W_2 be the tricyclic graphs on 11 vertices as shown in Fig. 3. Though $\Delta(W_1) > \Delta(W_2)$, we have $\mu(W_1) < 9.153 < 9.199 < \mu(W_2)$.

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