Multivalued Differential Equations in Banach Spaces

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Abstract—New existence principles and results are presented for differential inclusions in Banach spaces. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

This paper has two main goals. First, we establish new existence principles for the differential inclusion

\[ x' \in F(t, x), \quad \text{a.e. on } [0, T], \]
\[ x(0) = x_0 \in E. \]  

(1.1)

Here \( F : [0, T] \times E \to 2^E \), where \( E = (E, \| \cdot \|) \) is a real Banach space (note \( 2^E \) denotes the family of all nonempty subsets of \( E \)). The principles will then give us a new existence theory for (1.1). Our results extend well-known results in the literature (see [1, Section 9]). Second, we will use what we have derived for (1.1) to study the existence of solutions \( y : [0, T] \to K \subseteq E \) (so-called viable solutions) to the differential inclusion

\[ y'(t) \in \phi(t, y(t)), \quad \text{a.e. } t \in [0, T], \]
\[ y(0) = y_0 \in K. \]  

(1.2)

Here \( K \) is a proximate retract and \( \phi : [0, T] \times K \to 2^E \). The results presented give partial answers to some of the questions raised by Deimling in [1, Section 9].

2. DIFFERENTIAL INCLUSIONS

In this section, we discuss in detail the differential inclusion

\[ x' \in F(t, x), \quad \text{a.e. on } [0, T], \]
\[ x(0) = x_0 \in E, \]  

(2.1)

where \( F : [0, T] \times E \to C(E) \) (here \( C(E) \) denotes the family of all nonempty, compact subsets of \( E \)). We look for solutions to (2.1) in \( W^{1,1}([0, T], E) \). Recall \( W^{1,1}([0, T], E) \) is the set of
continuous functions \( u \) such that there exists \( v \in L^1([0,T], E) \) with \( u(t) - u(0) = \int_0^t v(s) \, ds \) for all \( t \in [0,T] \). (Notice if \( u \in W^{1,1}([0,T], E) \), then \( u \) is differentiable almost everywhere on \([0,T] \), \( u' \in L^1([0,T], E) \), and \( u(t) - u(0) = \int_0^t u'(s) \, ds \), for \( t \in [0,T] \).) Before we specify conditions on \( F \), we first recall some well-known concepts [2]. Let \( E_1 \) and \( E_2 \) be two Banach spaces, \( X \) a nonempty closed subset of \( E_1 \), and \( S \) a measurable space (respectively, \( S = I \times E \), where \( I \) is a real interval, and \( A \subseteq S \) is \( \mathcal{L} \otimes \mathcal{B} \) measurable if \( A \) belongs to the \( \sigma \)-algebra generated by all sets of the form \( N \times D \) where \( N \) is Lebesgue measurable and \( D \) is Borel measurable). Let \( H : X \to E_2 \) and \( G : S \to E_2 \) be two multifunctions with nonempty closed values. The function \( G \) is measurable (respectively, \( \mathcal{L} \otimes \mathcal{B} \) measurable) if the set \( \{ t \in S : G(t) \cap B \neq \emptyset \} \) is measurable for any closed \( B \) in \( E_2 \). The function \( H \) is lower semicontinuous (l.s.c) (respectively, upper semicontinuous (u.s.c.)) if the set \( \{ x \in X : H(x) \cap B \neq \emptyset \} \) is open (respectively, closed) for any open (respectively, closed) set \( B \) in \( E_2 \).

When we examine (2.1), we assume \( F : [0,T] \times E \to C(E) \) satisfies some of the following conditions (to be specified later).

\begin{align}
\text{(i)} & \quad t \mapsto F(t, x) \text{ is measurable for every } x \in E. \\
\text{(ii)} & \quad x \mapsto F(t, x) \text{ is u.s.c. for a.e. } t \in [0, T]. \\
\text{(iii)} & \quad t \mapsto F(t, x) \text{ is measurable for every } x \in E. \\
\text{(iv)} & \quad x \mapsto F(t, x) \text{ is continuous for a.e. } t \in [0, T].
\end{align}

For each \( r > 0 \), there exists \( h_r \in L^1[0,T] \) such that \( \|F(t, x)\| \leq h_r(t) \) for a.e. \( t \in [0,T] \) and every \( x \in E \) with \( \|x\| \leq r \).

There exists \( \gamma \geq 0 \) with \( \gamma T < 1 \) and with \( \alpha(F([0,t] \times \Omega)) \leq \gamma \alpha(\Omega) \) for any bounded subset \( \Omega \) of \( E \); here \( \alpha \) denotes the Kuratowskii measure of noncompactness.

There exists \( k \in L^1([0,T]) \) with \( \lim_{h \to 0^+} \alpha(F(J_{t,h} \times \Omega)) \leq k(t)\alpha(\Omega) \) for \( t \in [0,T] \) and for any bounded subset \( \Omega \) of \( E \); here \( J_{t,h} = [t-h,t] \cap [0,T] \).

We first state a result from [3] for u.s.c. type maps.

**Theorem 2.1.** (See [3].) Let \( E = (E, \|\cdot\|) \) be a separable Banach space with \( F : [0,T] \times E \to \mathcal{C}(E) \) (here \( \mathcal{C}(E) \) denotes the family of nonempty, compact, convex subsets of \( E \)). Assume (2.2), (2.5), and (2.6) hold. Define the operator

\[ N : C([0,T], E) \to 2^{C([0,T], E)} \]

by \( N = S \circ \mathcal{F} \), where

\[ \mathcal{F} : C([0,T], E) \to 2^{L^1([0,T], E)} \]

is given by

\[ \mathcal{F}(y) = \{ v \in L^1([0,T], E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [0,T] \} \]

and

\[ S : L^1([0,T], E) \to C([0,T], E) \]

is given by

\[ Sv(t) = x_0 + \int_0^t v(s) \, ds. \]
Suppose there exists a nonempty, closed, convex, equicontinuous set $X$ of $C([0,T], E)$ such that $X$ is mapped into itself by the multi-$N$ and also $N(X)$ is a subset of a bounded set in $C([0,T], E)$ (2.10) holds. Then (2.1) has a solution $u \in W^{1,1}([0,T], E)$.

Our next result is new and in the spirit of results presented by Deimling [1].

**Theorem 2.2.** Let $E = (E, \| \cdot \|)$ be a separable Banach space with $F : [0,T] \times E \rightarrow CK(E)$. Assume (2.2), (2.5), and (2.7) hold. Define the operator $N : C([0,T], E) \rightarrow 2^{C([0,T], E)}$ by $N = S \circ F$ ($F$ and $S$ are given in (2.8) and (2.9), respectively) and suppose (2.10) holds. Then (2.1) has a solution $u \in W^{1,1}([0,T], E)$.

**Proof.** Let $K_0 = X$, $K_{n+1} = \cap_{n \geq 0} N(K_n)$ for $n \geq 0$ and $K_\infty = \cap_{n \geq 0} K_n$. A standard argument (see [1, p. 118]) using (2.7) shows $K_\infty \neq \emptyset$ is a convex, compact set. In addition, it is easy to see (since $N$ takes $X$ into $X$) that $N : K_\infty \rightarrow 2^{K_\infty}$. In addition (follow the argument in [3]), $N : K_\infty \rightarrow Cc(K_\infty)$ has closed graph (here $Cc(K_\infty)$ denotes the family of nonempty, closed, convex subsets of $K_\infty$). Now [4, p. 465] implies $N|_{K_\infty}$ is u.s.c. Consequently, $N|_{K_\infty} : K_\infty \rightarrow CK(K_\infty)$ is u.s.c. and $K_\infty$ is convex and compact. Ky Fan’s Fixed Point Theorem [5] implies $N$ has a fixed point in $K_\infty$. □

Next we state a result from [3] for l.s.c. type maps.

**Theorem 2.3.** (See [3].) Let $E = (E, \| \cdot \|)$ be a separable Banach space and let $F : [0,T] \times E \rightarrow C(E)$ satisfy (2.5), (2.6), and either (2.3) or (2.4). Define the operator $N : C([0,T], E) \rightarrow 2^{C([0,T], E)}$ by $N = S \circ F$ ($F$ and $S$ are given in (2.8) and (2.9), respectively), and suppose (2.10) holds. Then (2.1) has a solution $u \in W^{1,1}([0,T], E)$.

**Proof.** Now [3] implies $F$ has a continuous selection $f : C([0,T], E) \rightarrow L^1([0,T], E)$. Now consider the problem

$$y'(t) = f(t, y(t)), \quad t \in [0,T],$$

$$y(0) = x_0. \quad (2.11)$$

Define $N_1 : C([0,T], E) \rightarrow C([0,T], E)$ by

$$N_1y(t) = x_0 + \int_0^t f(s, y(s)) \, ds$$

and let $K_0$, $K_1$, and $K_\infty$ be as in Theorem 2.2. Again we have $N_1 : K_\infty \rightarrow K_\infty$ with $K_\infty \neq \emptyset$ convex and compact. Also $N_1 : K_\infty \rightarrow K_\infty$ is continuous. Schauder’s Fixed Point Theorem [5] implies (2.11) has a solution. Consequently, (2.1) has a solution. □

It is easy to see how the theorems in this section can be used to establish new existence results. We illustrate the generality of our existence principles with the following existence result.

**Theorem 2.5.** Let $E = (E, \| \cdot \|)$ be a separable Banach space.

1. Let $F : [0,T] \times E \rightarrow CK(E)$ satisfy (2.2), (2.5), and (2.7). In addition, suppose

$$\alpha \in L^1[0,T] \text{ and } \psi : [0, \infty) \rightarrow (0, \infty) \text{ a nondecreasing, continuous function such that } \| F(s, u) \| \leq \alpha(s) \psi(\| u \|) \quad \text{for a.e. } s \in [0,T] \text{ and all } u \in E$$

and

$$\int_0^T \alpha(s) \, ds < \int_{\| u_0 \|}^{\infty} \frac{dx}{\psi(x)} \quad (2.13)$$

hold. Then (2.1) has a solution $u \in W^{1,1}([0,T], E)$. 

Let $F : [0, T] \times E \to C(E)$ satisfy (2.5), (2.7), (2.12), (2.13), and either (2.3) or (2.4). Then (2.1) has a solution $u \in W^{1,1}([0, T], E)$.

**Proof.** Let

$$X = \{ y \in C([0, T], E) : \|y(t)\| \leq b(t), \text{ for } t \in [0, T] \}
\text{ and } \|y(t) - y(s)\| \leq |b(t) - b(s)|, \text{ for } t, s \in [0, T] \},$$

where

$$b(t) = I^{-1} \left( \int_{0}^{t} \alpha(s) \, ds \right) \quad \text{and} \quad I(z) = \int_{\|x_0\|}^{z} \frac{dx}{\psi(x)}.$$

Notice $X$ is a closed, convex, bounded, equicontinuous subset of $C([0, T], E)$. Let $N = S \circ F$ ($F$ and $S$ are given in (2.8) and (2.9), respectively). The result follows immediately from Theorem 2.2 (in Case (I)) and Theorem 2.4 (in Case (II)) once we show $N$ maps $X$ into $X$. To see this, take $y \in X$. We must show $Ny \in X$. Notice first that

$$\|Ny(t)\| \leq \|x_0\| + \int_{0}^{t} \alpha(s) \psi(\|y(s)\|) \, ds \leq \|x_0\| + \int_{0}^{t} \alpha(s) \psi(b(s)) \, ds$$

$$= \|x_0\| + \int_{0}^{b(t)} \frac{dx}{\psi(x)} = \int_{0}^{b(t)} \alpha(s) \, ds.$$

Also for $u \in Ny$ there exists $v \in L^1([0, T], E)$ with $u(t) = x_0 + \int_{0}^{t} v(s) \, ds$ and $v(t) \in F(t, y(t))$ a.e. on $[0, T]$. Now for $t, s \in [0, T]$ with $t > s$, we have

$$\|u(t) - u(s)\| \leq \int_{s}^{t} \alpha(x) \psi(\|y(x)\|) \, dx \leq \int_{s}^{t} \alpha(x) \psi(b(x)) \, dx = \int_{s}^{t} b'(x) \, dx = b(t) - b(s).$$

Consequently, $Ny \in X$.

### 3. Differential Inclusions on Proximate Retracts

In this section, we study the existence of viable solutions $x : [0, T] \to K \subseteq E$ to the differential inclusion

$$x'(t) \in \phi(t, x(t)), \quad \text{a.e. } t \in [0, T],$$

$$x(0) = y_0 \in K. \quad (3.1)$$

By a solution (viable) to (3.1), we mean an $x \in W^{1,1}([0, T], E)$ with $x' \in \phi(t, x)$ a.e. on $[0, T]$, $x(0) = y_0$ and $x(t) \in K$ for $t \in [0, T]$. Throughout this section we assume

$$K \text{ is a proximate retract.} \quad (3.2)$$

**Definition 3.1.** (See [6].) A nonempty closed subset $K$ of $E$ is said to be a proximate retract if there exists an open neighborhood $U$ of $K$ in $E$ and a continuous (single-valued) mapping $r : U \to K$ (called a metric retraction) such that the following two conditions are satisfied:

(i) $r(x) = x$, for all $x \in K$,

(ii) $\|r(x) - x\| = \text{dist}(x, K)$, for all $x \in U$. 

REMARK 3.1. Any closed, convex subset of a uniformly convex Banach space is a proximate retract.

REMARK 3.2. Now since we can take a sufficiently small \( U \) (for example, by restricting \( U \) to \( U \cap \{ y \in E : \text{dist}(y,K) < \delta \} \) for some given \( \delta > 0 \)), we may assume (and we do so) that \( \|r(u) - u\| \leq \delta \), for all \( u \in U \).

Throughout this section, we will assume \( \phi \) satisfies either
\[
\phi : [0,T] \times K \rightarrow CK(E) \text{ satisfies (2.2) and (2.5) (here } F \text{ is replaced by } K \) (3.3)
\]
or
\[
\phi : [0,T] \times K \rightarrow C(E) \text{ satisfies (2.5) and either (2.3) or (2.4) (here } F \text{ is replaced by } \phi \text{ and } E \text{ is replaced by } K \) (3.4)
\]

Now let \( U \) be a fixed neighborhood of \( K \) (chosen as in Remark 3.2) and let \( \lambda \) be an Urysohn function for \( (K,E\setminus U) \) with \( \lambda(x) = 1 \) if \( x \in K \) and \( \lambda(x) = 0 \) if \( x \notin U \). Let \( r : U \rightarrow K \) be a metric retraction. Define \( \tilde{\phi} : [0,T] \times E \rightarrow C(E) \) by
\[
\tilde{\phi}(t,x) = \begin{cases} \lambda(x)\phi(t,r(x)), & \text{if } x \in U, \\ \{0\}, & \text{if } x \notin U. \end{cases}
\]

REMARK 3.3. If \( \phi \) satisfies (3.3), then \( \tilde{\phi} \) satisfies (2.2) and (2.5) (with \( F \) replaced by \( \tilde{\phi} \)). A similar remark applies if \( \phi \) satisfies (3.4).

Assume also that
\[
\phi(t,x) \subseteq T_k(x), \quad \text{for all } x \in K \text{ and a.e. } t \in [0,T],
\]

where
\[
T_k(x) = \left\{ v \in E : \liminf_{t \rightarrow 0^+} \frac{\text{dist}(x + tv, K)}{t} = 0 \right\}
\]
is the Bouligand tangent cone to \( K \) at \( x \).

Essentially the same reasoning as in [7, p. 177] establishes the following result.

THEOREM 3.1. Let \( a > 0 \). Assume (3.5) holds. If \( x \in W^{1,1}([0,a], E) \) is such that \( x'(t) \in \tilde{\phi}(t,x(t)) \) for a.e. \( t \in [0,a] \) and \( x(0) = K \), then \( x(t) \in K \) for all \( t \in [0,a] \).

Because of Theorem 3.1, we now concentrate our study on the differential inclusion
\[
x'(t) \in \tilde{\phi}(t,x(t)), \quad \text{a.e. } t \in [0,T],
\]
\[
x(0) = y_0 \in K.
\]

Notice any solution of (3.6) is a viable solution of (3.1); to see this, notice if \( x \) is a solution of (3.6), then \( x(t) \in K \) for all \( t \in [0,T] \) by Theorem 3.1 so \( \phi(t,x(t)) = \lambda(x(t))\phi(t,r(x(t))) = \phi(t,x(t)) \).

Conversely, if \( y \) is a viable solution of (3.1), then \( y \) is a solution of (3.6).

Now suppose there is a constant \( M \) with \( \|y\|_0 = \sup_{[0,T]} \|y(t)\| < M \) for any possible viable solution to (3.1). Let \( \epsilon > 0 \) be given and let \( \tau_\epsilon : E \rightarrow [0,1] \) be the Urysohn function for
\[
(B(0,M), E\setminus B(0,M + \epsilon))
\]
such that \( \tau_\epsilon(x) = 1 \) if \( \|x\| \leq M \), and \( \tau_\epsilon(x) = 0 \) if \( \|x\| \geq M + \epsilon \). Let \( \tilde{\phi}_\epsilon(t,x) = \tau_\epsilon(x)\tilde{\phi}(t,x) \) and we look at the differential inclusion
\[
x'(t) \in \tilde{\phi}_\epsilon(t,x(t)), \quad \text{a.e. } t \in [0,T],
\]
\[
x(0) = y_0.
\]
THEOREM 3.2. Let \( E = (E, \|\cdot\|) \) be a separable Banach space and assume (3.2) and (3.5) hold. In addition, suppose \( \phi : [0, T] \times K \to C(E) \) satisfies either (3.3) or (3.4), and also assume there is a constant \( M \) with \( \|y\|_0 < M \) for any possible viable solution \( y \in W^{1,1}([0, T], E) \) to (3.1). Let \( \epsilon > 0 \) be given and let \( \tau_\epsilon, \phi_\epsilon \) be as above. Suppose

\[
\text{there exists } k \in L^1[0, T] \text{ with } \lim_{h \to 0^+} \alpha(\phi_\epsilon(J_{t,h} \times \Omega)) \leq k(t)\alpha(\Omega) \text{ for } t \in (0, T] \text{ and for any bounded subset } \Omega \text{ of } E; \text{ here } J_{t,h} = [t - h, t] \cap [0, T]
\]  

(3.8)

holds. Define the operator \( N_\epsilon : C([0, T], E) \to 2^{C([0, T], E)} \) by \( N_\epsilon = S \circ \mathcal{F}_\epsilon \); here \( S \) is given as in (2.9) and

\[
\mathcal{F}_\epsilon : C([0, T], E) \to 2^{L^1([0, T], E)}
\]

is given by

\[
\mathcal{F}_\epsilon(y) = \left\{ v \in L^1([0, T], E) : v(t) \in \phi_\epsilon(t, y(t)) \text{ a.e. } t \in [0, T] \right\}.
\]  

(3.9)

Assume

there exists a nonempty, closed, convex, bounded, equicontinuous set \( X \) of \( C([0, T], E) \) such that \( X \) is mapped into itself by the multi-N_\epsilon

(3.10)

and

\[
\|w\|_0 < M \text{ for any possible solution } w \in W^{1,1}([0, T], E) \text{ to (3.7)}
\]  

(3.11)

hold. Then (3.1) has a viable solution \( u \) with \( \|u\|_0 < M \).

PROOF. From Theorem 2.2 or Theorem 2.4 (note (2.5) is satisfied with \( F \) replaced by \( \phi_\epsilon \)), we have immediately that (3.7) has a solution \( y \). By assumption (3.11), \( \|y\|_0 < M \) and so by definition \( \phi_\epsilon(t, y(t)) = \phi(t, y(t)) \). Thus \( y \) is a solution of (3.6). Now Theorem 3.1 implies \( y(t) \in K \) for every \( t \in [0, T] \), and so \( y \) is a solution of (3.1).

COROLLARY 3.3. Let \( K \) be a closed, convex subset of a separable Hilbert space \( E = (E, \|\cdot\|) \) and assume (3.5) holds. Suppose \( \phi : [0, T] \times K \to C(E) \) satisfies either (3.3) or (3.4), and also assume there is a constant \( M \) with \( \|y\|_0 < M \) for any possible viable solution \( y \in W^{1,1}([0, T], E) \) to (3.1). In addition, suppose

\[
\text{there exists } k \in L^1[0, T] \text{ with } \lim_{h \to 0^+} \alpha(\phi(J_{t,h} \times \Omega)) \leq k(t)\alpha(\Omega) \text{ for } t \in (0, T] \text{ and for any bounded subset } \Omega \text{ of } K; \text{ here } J_{t,h} = [t - h, t] \cap [0, T]
\]  

(3.12)

holds. Let \( \epsilon > 0 \) be given and let \( \tau_\epsilon, \phi_\epsilon \) be as above. Define the operator \( N_\epsilon : C([0, T], E) \to 2^{C([0, T], E)} \) by \( N_\epsilon = S \circ \mathcal{F}_\epsilon \) (\( S \) and \( \mathcal{F}_\epsilon \) are given in (2.9) and (3.9), respectively). Assume (3.10) and (3.11) hold. Then (3.1) has a viable solution \( u \) with \( \|u\|_0 < M \).

PROOF. The result follows from Theorem 3.2 once we show (3.8) is true. To see this, notice \( r \) in this case is nonexpansive. Now if \( \Omega \) is a bounded subset of \( E \), then since

\[
\phi_\epsilon(J_{t,h} \times \Omega) \subseteq \overline{\phi(J_{t,h} \times \Omega) \cup \{0\}} \subseteq \overline{\overline{\phi(J_{t,h} \times r(\Omega)) \cup \{0\}}} \cup \{0\},
\]

we have

\[
\alpha(\phi_\epsilon(J_{t,h} \times \Omega)) \leq \alpha(\phi(J_{t,h} \times r(\Omega))).
\]

This, together with (3.12) and the fact that \( r \) is nonexpansive, yields

\[
\lim_{h \to 0^+} \alpha(\phi_\epsilon(J_{t,h} \times \Omega)) \leq \lim_{h \to 0^+} \alpha(\phi(J_{t,h} \times r(\Omega))) \leq k(t)\alpha(\Omega) \leq k(t)\alpha(\Omega).
\]

\]
We now use Corollary 3.3 to obtain a new and very general existence result for (3.1). Our result gives a partial answer to some questions raised in [1, Section 9].

**Theorem 3.4.** Let $K$ be a closed, convex subset of a separable Hilbert space $E = (E, \| \cdot \|)$, $0 \in K$, and assume (3.5) holds. Suppose $\phi : [0, T] \times K \to C(E)$ satisfies (3.12) and either (3.3) or (3.4). In addition, assume

\[
\alpha \in L^1[0,T] \text{ and } \psi : [0, \infty) \to (0, \infty) \text{ a nondecreasing, continuous function such that } \|\phi(s, u)\| \leq \alpha(s)\psi(\|u\|), \text{ for a.e. } s \in [0,T] \text{ and all } u \in K
\]

and

\[
\int_0^T \alpha(s) \, ds < \int_0^\infty \frac{dx}{\psi(x)}
\]

hold. Then (3.1) has a viable solution.

**Proof.** Let

\[ X = \{ y \in C([0,T], E) : \|y(t)\| \leq b(t), \text{ for } t \in [0,T] \text{ and } \|y(t) - y(s)\| \leq |b(t) - b(s)|, \text{ for } t, s \in [0,T] \}, \]

where

\[ b(t) = I^{-1} \left( \int_0^t \alpha(s) \, ds \right) \quad \text{and} \quad I(x) = \int_{\|y_0\|}^{x} \frac{dx}{\psi(x)}. \]

Also let

\[ M_0 = I^{-1} \left( \int_0^T \alpha(s) \, ds \right) \quad \text{and} \quad M = M_0 + 1. \]

Notice $X$ is a closed, convex, bounded, equicontinuous subset of $C([0,T], E)$. Fix $\epsilon > 0$, and let $\tau_\epsilon, \tilde{\phi}_\epsilon$ be as above. Let $N_\epsilon = S \circ F_\epsilon$ ($S$ and $F_\epsilon$ are given in (2.9) and (3.9), respectively). We wish to apply Corollary 3.3.

First we show $N_\epsilon$ maps $X$ into $X$. Before we prove this, recall $r$ (the metric retraction) is nonexpansive, i.e., $\|r(x) - r(z)\| \leq \|x - z\|$, for all $x, z \in E$. In particular, since $0 \in K$ (so $r(0) = 0$), we have $\|r(x)\| \leq \|x\|$, for all $x \in E$. Let $y \in X$. Notice first (since $\tau_\epsilon : E \to [0,1]$) that

\[
\|N_\epsilon y(t)\| \leq \|y_0\| + \int_0^t \alpha(s)\psi(\|r(y(s))\|) \, ds
\]

\[
\leq \|y_0\| + \int_0^t \alpha(s)\psi(\|y(s)\|) \, ds
\]

\[
\leq \|y_0\| + \int_0^t \alpha(s)\psi(b(s)) \, ds = b(t).
\]

Also for $u \in N_\epsilon y$ and $t, s \in [0,T]$ with $t > s$, we have

\[
\|u(t) - u(s)\| \leq \int_s^t \alpha(x)\psi(\|r(y(x))\|) \, dx \leq \int_s^t \alpha(x)\psi(\|y(x)\|) \, dx
\]

\[
\leq \int_s^t \alpha(x)\psi(b(x)) \, dx = b(t) - b(s).
\]

Consequently, $N_\epsilon y \in X$, so (3.10) is satisfied.

Next, suppose $y$ is any possible viable solution of (3.1). Then we have

\[
\|y(t)\|' \leq \alpha(t)\psi(\|y(t)\|), \quad \text{a.e. on } \{ t \in [0,T] : \|y(t)\| > 0 \}.
\]
A standard argument (see [8]) implies \(|y(t)| \leq M_0\) for \(t \in [0, T]\) (here \(M_0\) is as in (3.15)). Thus, any possible viable solution \(y\) of (3.1) satisfies \(|y|_0 \leq M_0 < M\).

Finally, we show (3.11) is true. Let \(u\) be a solution of (3.7). Suppose there exists \(t \in (0, T]\) with \(|u(t)| \geq M\). Now since \(u(0) = y_0\), there exists \([0, t_0] \subseteq [0, T]\) with \(0 \leq |u(t)| < M\) for \(t \in (0, t_0)\) and \(|u(t_0)| = M\). Thus, \(u\) satisfies the differential inclusion

\[
x'(t) \in \phi(t, x(t)), \quad \text{a.e. } t \in [0, t_0],
\]

\[
x(0) = y_0.
\]

(3.16)

Now Theorem 3.1 (with \(a = t_0\)) implies any solution \(w\) of (3.16) satisfies \(w(t) \in K\) for \(t \in [0, t_0]\). Thus, in particular, \(u(t) \in K\) for \(t \in [0, t_0]\). Hence \(u\) satisfies

\[
x'(t) \in \phi(t, x(t)), \quad \text{a.e. } t \in [0, t_0],
\]

\[
x(0) = y_0.
\]

A standard argument (see [8]) implies \(|u(t)| \leq M_0\) for \(t \in [0, t_0]\). This is a contradiction. Thus, \(|u(t)| < M\) for \(t \in [0, T]\) and so (3.11) is true.

Consequently, all the conditions in Corollary 3.3 hold, and so (3.1) has a viable solution \(u\) with \(|u|_0 < M\).

\begin{flushright}
\textbf{REFERENCES}
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