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Spin invariant theory for the symmetric group

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ABSTRACT

We formulate a theory of invariants for the spin symmetric group in some suitable modules which involve the polynomial and exterior algebras. We solve the corresponding graded multiplicity problem in terms of specializations of the Schur Q-functions and a shifted q-hook formula. In addition, we provide a bijective proof for a formula of the principal specialization of the Schur Q-functions.

1. Introduction

1.1

The symmetric group S_n acts on $V = \mathbb{C}^n$ and then on the symmetric algebra S^*V naturally. It is well known that the algebra of S_n-invariants on S^*V is a polynomial algebra in n generators of degree $1, 2, \ldots, n$. More generally, consider the graded multiplicity of a given Specht module S^λ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n in the graded algebra S^*V, which has a generating function $P_\lambda(t) := \sum_{j\geq 0} m_\lambda(S^jV)t^j$. Kirillov [4] has obtained the following elegant formula for $P_\lambda(t)$ (also compare [16]):

$$P_\lambda(t) = \frac{t^{n(\lambda)}}{\prod_{(i,j)\in\lambda}(1-t^{h_{ij}})},$$

where h_{ij} is the hook length associated to a cell (i, j) in the Young diagram of λ, and

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i. \tag{1.1}$$

The generating function for the bi-graded S_n-invariants in $S^*V \otimes \wedge^*V$ was computed in [15]; see (4.2). More generally, Kirillov and Pak [5] obtained the bi-graded multiplicity of the Specht module S^λ for any λ in $S^*V \otimes \wedge^*V$; see (4.1).

1.2

According to [12], the symmetric group S_n affords a double cover $\widetilde{S}_n$:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \widetilde{S}_n \longrightarrow S_n \longrightarrow 1.$$

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Let us write $\mathbb{Z}_2 = \{1, z\}$. The spin (or projective) representation theory of S_n, or equivalently, the representation theory of the spin group algebra $\mathbb{C}S_n^- = \mathbb{C}\widetilde{S}_n/\langle z+1\rangle$, has been systematically developed by Schur (see [1] for an excellent modern exposition via a systematic use of superalgebras). Rich algebraic combinatorics of Schur Q-functions and shifted tableaux have been developed by Sagan [10] and Stembridge [17] (also see [8]) in relation to the irreducible spin representations and characters of S_n.

1.3

The goal of this paper is to formulate and prove the spin analogue of the graded multiplicity formulas in 1.1.

The results of this paper, though looking classical, have not appeared in the literature to our knowledge; however, it is expected that such simple results, once formulated, can be also derived by other approaches. It strongly suggests that the spin invariant theory of Weyl groups, or of finite groups in general, in the sense of this paper is a very interesting research direction to pursue. It is also natural to ask for the spin double counterpart, a spin analogue of Kostka polynomials, an interpretation of the graded multiplicity for the spin coinvariant algebra as generic degrees for (quantum) Hecke–Clifford algebras, etc. We hope to return to these topics at another occasion.

1.4

It is known [2,14,18,19] (see [6, Chap. 13]) that the representation theory of spin symmetric group (super)algebra $\mathbb{C}S_n^-$ is super-equivalent to its counterpart for Hecke–Clifford (super)algebra $\mathcal{H}_n := \mathcal{C}_n \rtimes \mathbb{C}S_n$; see Section 3.2 for notations and precise formulations. (All the algebras and modules in this paper are understood to admit a $\mathbb{Z}_2$-graded structure; however we will avoid using the terminology of supermodules.) Let D_-^λ denote the simple $\mathbb{C}S_n^-$-module and D^λ denote the simple $\mathcal{H}_n$-module, associated to a strict partition λ of n. The Clifford algebra $\mathcal{C}_n$ is naturally a simple module over the algebra $\mathcal{H}_n$ (which is identified with $D^{(n)}$), and it is the counterpart of the basic spin $\mathbb{C}S_n^-$-module $\mathcal{B}_n := D_-^{(n)}$.

In Proposition 3.2 we show that, for an arbitrary S_n-module M, the multiplicity problem for a simple $\mathbb{C}S_n^-$-module D_-^λ in $\mathcal{B}_n \otimes M$ is essentially equivalent to the multiplicity problem for a simple $\mathcal{H}_n$-module D^λ in $\mathcal{C}_n \otimes M$. Therefore, in this paper, we shall mainly work with the algebra $\mathcal{H}_n$, keeping in mind that the results can be transferred to the setting for $\mathbb{C}S_n^-$.

1.5

Our first main result provides the graded multiplicity of the simple $\mathcal{H}_n$-module D^λ in $\mathcal{C}_n \otimes S^*V$ for $V = \mathbb{C}^n$. For a partition λ of n with length $\ell(\lambda)$, we set

$$\delta(\lambda) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even,} \\ 1, & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

If λ is moreover a strict partition, we denote by λ^* the shifted diagram of λ, by c_{ij} the content, and by h_{ij}^* the shifted hook length of the cell $(i,j) \in \lambda^*$ (see Section 2 for precise definitions).

Theorem A. *Let λ be a strict partition of n. The graded multiplicity of D^λ in the $\mathcal{H}_n$-module $\mathcal{C}_n \otimes S^*V$ is*

$$2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} \frac{t^{n(\lambda)} \prod_{(i,j)\in\lambda^*} (1+t^{c_{ij}})}{\prod_{(i,j)\in\lambda^*} (1-t^{h_{ij}^*})}. \tag{1.2}$$

The lowest degree term in (1.2) is $2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} t^{n(\lambda)}$, thanks to the contribution $2^{\ell(\lambda)}$ from the product over the main diagonal of λ^* in the numerator. Theorem A can be reformulated in terms of the graded multiplicity of a coinvariant algebra which is isomorphic to a graded regular module of $\mathcal{H}_n$; see Theorem 3.5. In the spirit of a classical theorem of Borel which identifies the coinvariant algebra of a Weyl group with the cohomology ring of the corresponding flag variety, the coinvariant algebra of $\mathcal{H}_n$ should be regarded as the cohomology ring (which has yet to be developed) of the flag variety for the queer Lie supergroup.

To prove Theorem A, we first obtain an expression of the graded multiplicity in terms of the principal specialization of the Schur Q-function, $Q_\lambda(t^\bullet) := Q_\lambda(1, t, t^2, \ldots)$, and then apply the following formula.

Theorem B. *For a strict partition λ of n, we have*

$$Q_\lambda(t^\bullet) = \frac{t^{n(\lambda)} \prod_{(i,j)\in\lambda^*} (1+t^{c_{ij}})}{\prod_{(i,j)\in\lambda^*} (1-t^{h_{ij}^*})}. \tag{1.3}$$

It is well known (cf. [10,17]) that a Schur Q-function can be written as a sum over the so-called marked shifted tableaux of a given shape. We establish in Theorem 2.2 a bijection between marked shifted tableaux and certain new combinatorial objects which we call *colored shifted tableaux*. Theorem B is an easy consequence of such a bijection.

We show in Proposition 2.5 that the formula (1.3) is equivalent to another formula of [9, Proposition 3.1], who derived it from a Schur function identity of [3].

1.6

Another result of this paper is a formula for the bi-graded multiplicity of the simple $\mathcal{H}_n$-module D^λ in $\mathcal{C}_n \otimes S^*V \otimes \wedge^*V$:

$$\sum_{p=0}^{\infty}\sum_{q=0}^{n} t^p s^q m_\lambda(\mathcal{C}_n \otimes S^pV \otimes \wedge^q V).$$

We shall adopt the following short-hand notation for a specialization of Schur Q-function in 2-variables s and t:

$$Q_\lambda(t^\bullet; st^\bullet) := Q_\lambda(1, t, t^2, \ldots; s, st, st^2, \ldots).$$

Theorem C. *Let λ be a strict partition of n. The bi-graded multiplicity of D^λ in the $\mathcal{H}_n$-module $\mathcal{C}_n \otimes S^*V \otimes \wedge^*V$ is*

$$2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} Q_\lambda(t^\bullet; st^\bullet). \tag{1.4}$$

Setting $s = 0$, we recover Theorem A from Theorem C. On the other hand, setting $t = 0$, we obtain a graded multiplicity formula of D^λ in $\mathcal{C}_n \otimes \wedge^*V$; see Corollary 4.5. We may also consider a Koszul $\mathbb{Z}$-grading which counts the standard generators of S^*V as degree 2 and the standard generators of $\wedge^*V$ as degree 1. It follows by Theorem C that, for the Koszul grading, the graded multiplicity in $\mathcal{C}_n \otimes S^*V \otimes \wedge^*V$ is given by the same formula (1.2) above. It will be nice to obtain a closed formula for $Q_\lambda(t^\bullet; st^\bullet)$.

1.7

The paper is organized as follows. In Section 2, we provide a bijective proof of Theorem B. The graded multiplicities in $\mathcal{C}_n \otimes S^*V$ are studied and Theorem A is proved in Section 3. In Section 4, we study the bi-graded multiplicities in $\mathcal{C}_n \otimes S^*V \otimes \wedge^*V$, and prove Theorem C.

2. Principal specialization of Schur Q-functions

In this section, we shall provide a bijective proof for Theorem B, after first recalling some basics on strict partitions and Schur Q-functions [10,17,7].

2.1. Strict partitions and shifted diagrams

Let $n \in \mathbb{Z}_+$. We denote a composition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n by $\lambda \models n$, and denote a partition λ of n by $\lambda \vdash n$. A partition λ will be identified with its Young diagram, that is, $\lambda = \{(i, j) \in \mathbb{Z}^2 \mid 1 \le i \le \ell(\lambda), 1 \le j \le \lambda_i\}$. To each cell $(i, j) \in \lambda$, we associate its content $c_{ij} = j - i$ and hook length $h_{ij} = \lambda_i + \lambda'_j - i - j + 1$, where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition of λ.

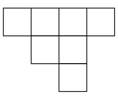
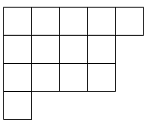
Suppose that the main diagonal of the Young diagram λ contains r cells. Let $\alpha_i = \lambda_i - i$ be the number of cells in the ith row of λ strictly to the right of (i, i), and let $\beta_i = \lambda'_i - i$ be the number of cells in the ith column of λ strictly below (i, i), for $1 \le i \le r$. We have $\alpha_1 > \alpha_2 > \cdots > \alpha_r \ge 0$ and $\beta_1 > \beta_2 > \cdots > \beta_r \ge 0$. Then the Frobenius notation for a partition is $\lambda = (\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r)$. For example, if $\lambda = (5, 4, 3, 1)$, then $\alpha = (4, 2, 0)$, $\beta = (3, 1, 0)$ and hence $\lambda = (4, 2, 0|3, 1, 0)$ in the Frobenius notation.

Suppose that λ is a strict partition of n, denoted by $\lambda \vdash_s n$. Let λ^* be the associated shifted Young diagram, that is,

$$\lambda^* = \{(i, j) \mid 1 \le i \le \ell(\lambda), i \le j \le \lambda_i + i - 1\}$$

which is obtained from the ordinary Young diagram by shifting the kth row to the right by $k - 1$ squares, for each k. Given $\lambda \vdash_s n$ with $\ell(\lambda) = \ell$, define its double partition $\widetilde{\lambda}$ to be $\widetilde{\lambda} = (\lambda_1, \ldots, \lambda_\ell|\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_\ell - 1)$ in Frobenius notation. Clearly, the shifted Young diagram λ^* coincides with the part of $\widetilde{\lambda}$ that lies above the main diagonal. For each cell $(i, j) \in \lambda^*$, denote by h^*_{ij} the associated hook length in the Young diagram $\widetilde{\lambda}$, and set the content $c_{ij} = j - i$.

For example, let $\lambda = (4, 2, 1)$. The corresponding shifted diagram and double diagram are

$\lambda^* =$  $\widetilde{\lambda} =$ 

The hook lengths and contents for each cell in λ are respectively as follows:

$$\begin{array}{|c|c|c|c|}\hline 6&5&4&1\\ \hline \multicolumn{1}{c|}{}&3&2\\ \cline{2-3} \multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&1\\ \cline{3-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 0&1&2&3\\ \hline \multicolumn{1}{c|}{}&0&1\\ \cline{2-3} \multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&0\\ \cline{3-3}\end{array}$$

2.2. Schur Q-functions

Let λ be a strict partition with $\ell(\lambda) = \ell$. Suppose $m \geq \ell$. The associated Schur Q-function $Q_\lambda(z_1, z_2, \ldots, z_m)$ is defined by

$$Q_\lambda(z_1, z_2, \ldots, z_m) = 2^\ell \sum_{w \in S_m/S_{m-\ell}} w\left(z_1^{\lambda_1} \cdots z_\ell^{\lambda_\ell} \prod_{1\leq i\leq \ell} \prod_{i<j\leq m} \frac{z_i + z_j}{z_i - z_j}\right), \tag{2.1}$$

where the symmetric group S_m acts by permuting the variables $z_1, \ldots, z_m$ and $S_{m-\ell}$ is the subgroup acting on $z_{\ell+1}, \ldots, z_m$. The definition of $Q_\lambda(z_1, z_2, \ldots, z_m)$ stabilizes as m goes to infinity, and we write $Q_\lambda(z) = Q_\lambda(z_1, z_2, \ldots)$, the symmetric functions in infinitely many variables $z = (z_1, z_2, \ldots)$. For $y = (y_1, y_2, \ldots)$, the following identity holds (see [7, III, Section 8]):

$$\prod_{i,j} \frac{1 + y_i z_j}{1 - y_i z_j} = \sum_{\lambda:\ \text{strict}} 2^{-\ell(\lambda)} Q_\lambda(y) Q_\lambda(z). \tag{2.2}$$

It will be convenient to introduce another family of symmetric functions $q_\nu(z)$ for any composition $\nu = (\nu_1, \nu_2, \ldots)$ as follows:

$$\begin{aligned} &q_0(z) = 1, \\ &q_r(z) = Q_{(r)}(z), \quad \text{for } r \geq 1, \\ &q_\nu(z) = q_{\nu_1}(z) q_{\nu_2}(z) \cdots . \end{aligned}$$

The generating function for $q_r(z)$ is

$$\sum_{r\geq 0} q_r(z) u^r = \prod_i \frac{1 + z_i u}{1 - z_i u}. \tag{2.3}$$

We will write $q_r = q_r(z)$, etc., whenever there is no need to specify the variables. Let $\Gamma_{\mathbb{C}}$ be the $\mathbb{C}$-algebra generated by q_r, $r \geq 1$, that is,

$$\Gamma_{\mathbb{C}} = \mathbb{C}[q_1, q_2, \ldots]. \tag{2.4}$$

Then Q_λ for strict partitions λ form a basis of $\Gamma_{\mathbb{C}}$.

2.3. Marked shifted tableaux and Schur Q-functions

Denote by $\mathbf{P}'$ the ordered alphabet $\{1' < 1 < 2' < 2 < 3' < 3 \cdots\}$. The symbols $1', 2', 3', \ldots$ are said to be marked, and we shall denote by $|a|$ the unmarked version of any $a \in \mathbf{P}'$; that is, $|k'| = |k| = k$ for each $k \in \mathbb{N}$. For a strict partition λ, a *marked shifted tableau* T of shape λ, or a *marked shifted λ-tableau* T, is an assignment $T : \lambda^* \to \mathbf{P}'$ satisfying:

(M1) The letters are weakly increasing along each row and column.
(M2) The letters $\{1, 2, 3, \ldots\}$ are strictly increasing along each column.
(M3) The letters $\{1', 2', 3', \ldots\}$ are strictly increasing along each row.

For a marked shifted tableau T of shape λ, let α_k be the number of cells $(i, j) \in \lambda^*$ such that $|T(i, j)| = k$ for $k \geq 1$. The sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ is called the *weight* of T. The Schur Q-function associated to λ can be interpreted as (see [10,17,7])

$$Q_\lambda(x) = \sum_T x^T, \tag{2.5}$$

where the summation is over all marked shifted λ-tableaux, and $x^T = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ if T has weight $(\alpha_1, \alpha_2, \alpha_3, \ldots)$. Denote by $|T| = \sum_{k\geq 1} k\alpha_k$ if the weight of T is $(\alpha_1, \alpha_2, \ldots)$.

Example 2.1. Suppose $\lambda = (5, 4, 2)$. The following is an example of a marked shifted tableau of shape λ and its weight is $(2, 5, 4)$:

$$T = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 2' & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & 3' & 3 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3' & 3 & \multicolumn{1}{|c}{} \\ \cline{3-4} \end{array}$$

A *shifted reverse plane tableau* S of shape λ is a labeling of cells in the shifted diagram λ^* with nonnegative integers so that the rows and columns are weakly increasing. Denote by $|S|$ the summation of the entries in S. It is known (cf. [11, Theorem 6.2.1]) that

$$\sum_S t^{|S|} = \prod_{(i,j)\in\lambda} \frac{1}{1-t^{h^*_{ij}}}, \tag{2.6}$$

summed over all shifted reverse plane tableaux of shape λ.

2.4. A bijection theorem

Let λ be a strict partition. A *colored shifted tableau* C is an assignment $C : \lambda^* \to \mathbf{P}'$ such that the associated assignment $\overline{C} : \lambda^* \to \mathbb{Z}_+$ defined by

$$\overline{C}(i,j) = \begin{cases} |C(i,j)| - j, & \text{if } C(i,j) \text{ is marked,} \\ |C(i,j)| - i, & \text{if } C(i,j) \text{ is unmarked} \end{cases}$$

is a shifted reverse plane tableau of shape λ. The weight of a colored shifted tableau is defined in the same way as for marked shifted tableaux. Denote by $|C| = \sum_{k\ge 1} k\alpha_k$ if the weight of C is $(\alpha_1, \alpha_2, \ldots)$.

Theorem 2.2. *Suppose that λ is a strict partition of n and $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a composition of n. Then there exists a bijection between the set of marked shifted λ-tableaux of weight α and the set of colored shifted λ-tableaux of weight α.*

Proof. Suppose that T is a marked shifted tableau of shape λ and weight $\alpha = (\alpha_1, \alpha_2, \ldots)$. Set $m = \max\{|T(i,j)| \mid (i,j) \in \lambda^*\}$. For each $1 \le k \le m$, $\lambda^{k,*} = \{(i,j) \in \lambda^* \mid |T(i,j)| \le k\}$ is a shifted diagram of a strict partition λ^k, and $\lambda^1 \subseteq \lambda^2 \subseteq \cdots \subseteq \lambda^m$.

We shall construct by induction on k a chain of colored shifted tableaux T^k of shape λ^k and weight $\alpha^k = (\alpha_1, \ldots, \alpha_k)$, for $1 \le k \le m$. Set $T^1 : \lambda^{1,*} \to \mathbf{P}'$ to be the restriction of T to $\lambda^{1,*}$. Since T is a marked shifted tableau, λ^1 is a one-row partition and hence T^1 is already a colored shifted tableau of weight $\alpha^1 = (\alpha_1)$.

Suppose that T^{k-1} is a colored shifted tableau of shape λ^{k-1} and weight $\alpha^{k-1} = (\alpha_1, \ldots, \alpha_{k-1})$. In order to construct T^k from T^{k-1}, we start with an intermediate tableau T_k defined by

$$T_k : \lambda^{k,*} \longrightarrow \mathbf{P}'$$
$$(i,j) \mapsto \begin{cases} T^{k-1}(i,j), & \text{if } (i,j) \in \lambda^{(k-1),*} \\ T(i,j), & \text{if } (i,j) \in \lambda^{k,*}/\lambda^{(k-1),*}. \end{cases}$$

There is at most one cell labeled by k' in each row of T_k since T satisfies (M3). Suppose that the cells labeled by k' in T_k are $(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)$ with $i_1 < i_2 < \cdots < i_p$. Start with the topmost cell (i_1, j_1) and label its left and upper neighboring cells $(i_1, j_1 - 1)$ and $(i_1 - 1, j_1)$, if they exist, as

$$\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{k'} \\ \hline \end{array}$$

(In case when either the left or the upper neighboring cell is missing, the exchange procedure below is simplified in an obvious manner). Set

$$\bar{b} = \begin{cases} |b| - (j_1 - 1), & \text{if } b \text{ is marked,} \\ |b| - i_1, & \text{otherwise;} \end{cases}$$

$$\bar{c} = \begin{cases} |c| - j_1, & \text{if } c \text{ is marked,} \\ |c| - (i_1 - 1), & \text{otherwise.} \end{cases}$$

If $k - j_1 < \bar{b}$ or $k - j_1 < \bar{c}$, exchange k' and b or c in T_k as follows:

Case I $(\bar{b} \ge \bar{c})$: $\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{k'} \\ \hline \end{array} \longrightarrow \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \hline \multicolumn{1}{|c|}{k'} & \multicolumn{1}{c|}{b} \\ \hline \end{array}$

Case II $(\bar{c} > \bar{b})$: $\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{k'} \\ \hline \end{array} \longrightarrow \begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{k'} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{c} \\ \hline \end{array}$

Note that b is unmarked in Case I and c is unmarked in Case II. Hence the resulting diagram

I: $\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{c} \\ \hline \multicolumn{1}{|c|}{k'} & \multicolumn{1}{c|}{b} \\ \hline \end{array}$ or II: $\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{k'} \\ \hline \multicolumn{1}{|c|}{b} & \multicolumn{1}{c|}{c} \\ \hline \end{array}$

satisfies the requirement for colored shifted tableaux. Keep repeating the above procedure for the new cells occupied with this k', until it stops. Then move on to apply the same procedure above to the cells $(i_2, j_2), \ldots, (i_p, j_p)$ one by one, and denote by T^k the resulting tableau in the end.

We claim that T^k is a colored shifted tableau. By induction hypothesis, T^{k-1} is a colored shifted tableau. Clearly, the exchange procedure above by definition ensures that the requirement being a colored shifted tableau is already fulfilled for the cells in T^k other than those occupied by k. So it remains to check the conditions on each cell $(i, j) \in \lambda^{k,*}$ with $T^k(i, j) = k$. Assume that the cell $(i, j-1)$ in T^k, if it exists, is labeled by $d \in \mathbf{P}'$. Note that $j - 1 \geq i$ and $|d| \leq k$. If d is unmarked, then $|d| - i \leq k - i$. If d is marked, then $|d| - (j - 1) \leq |d| - i \leq k - i$. Similarly, assume that the cell $(i - 1, j)$ in T^k, if it exists, is labeled by $e \in \mathbf{P}'$. For unmarked e, we have $|e| < k$ or equivalently $|e| - (i - 1) \leq k - i$, since there is at most one unmarked k in each column of T^k. For marked e, we have $|e| \leq k$ and hence $|e| - j \leq k - i$, since $j \geq i$. This proves the claim.

Hence, we have constructed a colored shifted tableau T^m of the same shape and weight as for T which we started with.

We claim the exchange procedure above from T to T^m is reversible. It suffices to show that the above procedure from T^{k-1} to T^k is invertible for each k. Denote by $T^{k,0}$ the resulting tableau after removing cells labeled by unmarked k from T^k. There exists at most one cell labeled by marked k' in each row of $T^{k,0}$, and suppose that these cells are $(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)$ in $T^{k,0}$ with $i_1 > i_2 > \cdots > i_p$. Start with the lowest cell (i_1, j_1) and suppose that its right and lower neighboring cells $(i_1, j_1 + 1)$ and $(i_1 + 1, j_1)$, if they exist, in $T^{k,0}$ are labeled by $b, c \in \mathbf{P}'$ as follows:

$$\begin{array}{|c|c|}\hline k' & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Set

$$\tilde{b} = \begin{cases} |b| - (j_1 + 1), & \text{if } b \text{ is marked,} \\ |b| - i_1, & \text{otherwise,} \end{cases}$$

$$\tilde{c} = \begin{cases} |c| - j_1, & \text{if } c \text{ is marked,} \\ |c| - (i_1 + 1), & \text{otherwise.} \end{cases}$$

If $k' > b$ or $k' > c$, exchange k' and b or c in $T^{k,0}$ as follows:

$$\text{Case I } (\tilde{b} \leq \tilde{c}): \quad \begin{array}{|c|c|}\hline k' & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline b & k' \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\text{Case II } (\tilde{b} > \tilde{c}): \quad \begin{array}{|c|c|}\hline k' & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline c & b \\ \hline k' & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Keep repeating the above procedure to the new cell occupied by this k', until it stops. Then move on to the cells $(i_2, j_2), \ldots, (i_p, j_p)$ one by one and apply the same procedure. Denote by $T^{k,1}$ the resulting tableau in the end. Removing the cells labeled by k' from $T^{k,1}$, we recover the tableau T^{k-1}. □

Example 2.3. Suppose $\lambda = (5, 4, 2)$ and T is the marked shifted tableau given by Example 2.1. Then the colored shifted tableau corresponding to T is T^3, where

$$T^1 = \begin{array}{|c|c|}\hline 1' & 1 \\ \hline\end{array} \qquad T^2 = \begin{array}{|c|c|c|c|c|}\hline 1' & 2' & 1 & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & \multicolumn{2}{c}{} \\ \cline{2-3}\end{array}$$

$$T^3 = \begin{array}{|c|c|c|c|c|}\hline 1' & 2' & 3' & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & 2 & 3 \\ \cline{2-5} \multicolumn{2}{c|}{} & 3' & 3 & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array}$$

2.5. Proof of Theorem B

Proof. It follows by Theorem 2.2 that

$$\sum_T t^{|T|} = \sum_C t^{|C|}, \tag{2.7}$$

where the first summation is over all marked shifted λ-tableaux T and the second summation is over all colored shifted λ-tableaux C. The left-hand side of (2.7) is equal to $Q_\lambda(t, t^2, t^3, \ldots)$ by (2.5).

It follows from the definition of colored shifted tableaux and then (2.6) that

$$\begin{aligned}\sum_C t^{|C|} &= \Big(\prod_{(i,j)\in\lambda^*} (t^i + t^j)\Big) \sum_S t^{|S|} \\ &= \frac{\prod_{(i,j)\in\lambda^*} (t^i + t^j)}{\prod_{(i,j)\in\lambda^*} (1 - t^{h^*_{ij}})},\end{aligned}$$

where the summation on S is taken over all shifted reverse plane tableaux of shape λ.

Putting everything together, we obtain that

$$Q_\lambda(t^\bullet) = \frac{1}{t^n} Q_\lambda(t, t^2, t^3, \ldots) = \frac{1}{t^n} \sum_T t^{|T|}$$
$$= \frac{1}{t^n} \frac{\prod_{(i,j)\in\lambda^*} (t^i + t^j)}{\prod_{(i,j)\in\lambda^*} (1 - t^{h^*_{ij}})} = \frac{t^{n(\lambda)} \prod_{(i,j)\in\lambda^*} (1 + t^{c_{ij}})}{\prod_{(i,j)\in\lambda^*} (1 - t^{h^*_{ij}})}. \tag{2.8}$$

This completes the proof of Theorem B. □

Remark 2.4. It follows from the proof above that Theorem B can be restated as

$$Q_\lambda(t^\bullet) = \frac{\prod_{(i,j)\in\lambda^*} (t^{i-1} + t^{j-1})}{\prod_{(i,j)\in\lambda^*} (1 - t^{h^*_{ij}})}.$$

2.6. Another formula for $Q_\lambda(t^\bullet)$

For $k \in \mathbb{N}$, we set

$$(a; t)_k = (1 - a)(1 - at) \cdots (1 - at^{k-1}).$$

Rosengren [9, Proposition 3.1] has obtained the following formula for $Q_\lambda(t^\bullet)$, starting from a Schur function identity of Kawanaka:

$$Q_\lambda(t^\bullet) = \prod_{1\le i\le \ell(\lambda)} \frac{(-1; t)_{\lambda_i}}{(t; t)_{\lambda_i}} \prod_{1\le i<j\le \ell(\lambda)} \frac{t^{\lambda_j} - t^{\lambda_i}}{1 - t^{\lambda_i+\lambda_j}}. \tag{2.9}$$

Proposition 2.5. *The formula (1.3) is equivalent to Rosengren's formula (2.9).*

Proof. Set $\ell = \ell(\lambda)$. It is known (cf. [7, III, Section 8, Ex. 12]) that in the ith row of λ^*, the hook lengths h^*_{ij} for $i \le j \le \lambda_i + i - 1$ are $1, 2, \ldots, \lambda_i, \lambda_i + \lambda_{i+1}, \lambda_i + \lambda_{i+2}, \ldots, \lambda_i + \lambda_\ell$ with exception $\lambda_i - \lambda_{i+1}, \lambda_i - \lambda_{i+2}, \ldots, \lambda_i - \lambda_\ell$. Hence we have

$$\prod_{(i,j)\in\lambda^*} \frac{1}{1 - t^{h^*_{ij}}} = \frac{1}{\prod_{1\le i\le \ell} (t; t)_{\lambda_i}} \prod_{1\le i<j\le \ell} \frac{1 - t^{\lambda_i - \lambda_j}}{1 - t^{\lambda_i+\lambda_j}}. \tag{2.10}$$

The equivalence between (1.3) and (2.9) can now be deduced by applying (2.10) and noting that the contents c_{ij} for $i \le j \le \lambda_i + i - 1$ are $0, 1, \ldots, \lambda_i - 1$. □

Remark 2.6. By (2.5), $Q_\lambda(1^m)$ is equal to the number of marked shifted Young tableaux of shape λ with filling by letters $\le m$. On the other hand, it follows from [13, Theorem 4] that $2^{\frac{\delta(\lambda)-\ell(\lambda)}{2}} Q_\lambda(1^m)$ gives the dimension of the irreducible representation of the queer Lie superalgebra $\mathfrak{q}(m)$ of highest weight λ.

3. The graded multiplicity in $\mathcal{C}_n \otimes S^*V$

The goal of this section is to establish Theorem A. In addition, a tensor identity in Lemma 3.1 allows us to translate a multiplicity problem for $\mathcal{H}_n$ to $\mathbb{C}S^-_n$, and vice versa (see Proposition 3.2).

3.1. Some basics about superalgebras

We shall recall some basic notions of superalgebras, referring the reader to [6, Chapter 12]. Let us denote by $\bar{v} \in \mathbb{Z}_2$ the parity of a homogeneous vector v of a vector superspace. A superalgebra $\mathcal{A}$ is a $\mathbb{Z}_2$-graded associative algebra. An $\mathcal{A}$-module always means a $\mathbb{Z}_2$-graded left $\mathcal{A}$-module in this paper. A homomorphism $f : V \to W$ of $\mathcal{A}$-modules V and W means a linear map such that $f(av) = (-1)^{\bar{f}\bar{a}} af(v)$. Note that this and other such expressions only make sense for homogeneous a, f and the meaning for arbitrary elements is attained by extending linearly from the homogeneous case. Let V be a finite dimensional $\mathcal{A}$-module. Let ΠV be the same underlying vector space but with the opposite $\mathbb{Z}_2$-grading. The new action of $a \in \mathcal{A}$ on $v \in \Pi V$ is defined in terms of the old action by $a \cdot v := (-1)^{\bar{a}} av$. Denote by $\mathcal{A}$-smod the category of finite dimensional $\mathcal{A}$-modules.

Given two superalgebras $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is naturally a superalgebra. Suppose that V is an $\mathcal{A}$-module and W is a $\mathcal{B}$-module. Then the tensor space $V \otimes W$ affords an $\mathcal{A} \otimes \mathcal{B}$-module, denoted by $V \boxtimes W$, via

$$(a \otimes b)(v \otimes w) = (-1)^{\bar{b}\bar{v}} av \otimes bw, \quad a \in \mathcal{A}, b \in \mathcal{B}, v \in V, w \in W.$$

3.2. Spin symmetric group algebras $\mathbb{C}S_n^-$ and Hecke–Clifford algebras $\mathcal{H}_n$

Recall that the spin symmetric group algebra $\mathbb{C}S_n^-$ is the algebra generated by $t_1, t_2, \ldots, t_{n-1}$ subject to the relations:

$$\begin{aligned}
&t_i^2 = 1, \quad 1 \le i \le n-1\\
&t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad 1 \le i \le n-2\\
&t_i t_j = -t_j t_i, \quad 1 \le i, j \le n-1, |i-j| \ge 1.
\end{aligned}$$

$\mathbb{C}S_n^-$ is a superalgebra with each t_i being odd, for $1 \le i \le n-1$.

Denote by $\mathcal{C}_n$ the Clifford superalgebra generated by the odd elements $c_1, \ldots, c_n$, subject to the relations $c_i^2 = 1$, $c_i c_j = -c_j c_i$ for $1 \le i \ne j \le n$. Observe that $\mathcal{C}_n$ is a simple superalgebra and there is a unique (up to isomorphism) irreducible $\mathcal{C}_n$-module U_n.

Define the Hecke–Clifford algebra $\mathcal{H}_n = \mathcal{C}_n \rtimes \mathbb{C}S_n$ to be the superalgebra generated by odd elements $c_1, \ldots, c_n$ and even elements $s_1, \ldots, s_{n-1}$, subject to the relations:

$$\begin{aligned}
&s_i^2 = 1, \qquad s_i s_j = s_j s_i, \quad 1 \le i, j \le n-1, |i-j| > 1,\\
&s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \le i \le n-2,\\
&c_i^2 = 1, \qquad c_i c_j = -c_j c_i, \quad 1 \le i \ne j \le n,\\
&s_i c_i = c_{i+1} s_i, \qquad s_i c_j = c_j s_i, \quad 1 \le i, j \le n-1, j \ne i, i+1.
\end{aligned}$$

There is a superalgebra isomorphism (cf. [13,19]):

$$\begin{aligned}
&\mathbb{C}S_n^- \otimes \mathcal{C}_n \longrightarrow \mathcal{H}_n\\
&c_i \mapsto c_i, \quad 1 \le i \le n,\\
&t_j \mapsto \tfrac{1}{\sqrt{-2}} s_j (c_j - c_{j+1}), \quad 1 \le j \le n-1.
\end{aligned} \tag{3.1}$$

The two exact functors

$$\mathfrak{F}_n := - \boxtimes U_n : \ \mathbb{C}S_n^-\text{-smod} \to \mathcal{H}_n\text{-smod},$$

$$\mathfrak{G}_n := \mathrm{Hom}_{\mathcal{C}_n}(U_n, -) : \ \mathcal{H}_n\text{-smod} \to \mathbb{C}S_n^-\text{-smod}$$

define Morita super-equivalence between the superalgebras $\mathcal{H}_n$ and $\mathbb{C}S_n^-$ (cf. Kleshchev [6, Proposition 13.2.2] for precise details).

It is known [2,13,18] (cf. [6]) that for each strict partition λ of n, there exists an irreducible $\mathcal{H}_n$-module D^λ and $\{D^\lambda \mid \lambda \vdash_s n\}$ forms a complete set of non-isomorphic irreducible $\mathcal{H}_n$-modules. We have a complete set of non-isomorphic irreducible $\mathbb{C}S_n^-$-modules $\{D_-^\lambda \mid \lambda \vdash_s n\}$, and by [6, Proposition 13.2.2],

$$\mathfrak{G}_n(D^\lambda) = \begin{cases} D_-^\lambda, & \text{if } n \text{ or } \ell(\lambda) \text{ is even},\\ D_-^\lambda \bigoplus \Pi D_-^\lambda, & \text{otherwise.} \end{cases} \tag{3.2}$$

Denote the trivial representation by $\mathbf{1}$ and the sign representation of S_n by sgn. Note that $\mathcal{C}_n \cong \mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} \mathbf{1}$ is the irreducible $\mathcal{H}_n$-module $D^{(n)}$ [6, Lemma 22.2.4]. It follows from (3.2) that the irreducible $\mathbb{C}S_n^-$-module $\mathcal{B}_n := D_-^{(n)}$ satisfies that

$$\mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n) \cong \begin{cases} \mathcal{B}_n, & \text{if } n \text{ is even},\\ \mathcal{B}_n \bigoplus \Pi \mathcal{B}_n, & \text{if } n \text{ is odd.} \end{cases} \tag{3.3}$$

It can be shown that the $\mathbb{C}S_n^-$-module $\mathcal{B}_n$ coincides with the basic spin representation L_n defined in [1, 2C].

3.3. The multiplicity problem of $\mathcal{H}_n$ vs $\mathbb{C}S_n^-$

Given a $\mathbb{C}S_n$-module M and a $\mathbb{C}S_n^-$-module E, the tensor product $E \otimes M$ affords a $\mathbb{C}S_n^-$-module as follows:

$$t_j(u \otimes x) = (t_j u) \otimes (s_j x), \quad 1 \le j \le n-1, u \in E, x \in M. \tag{3.4}$$

Meanwhile, the tensor product $F \otimes M$ of a $\mathcal{H}_n$-module F and a $\mathbb{C}S_n$-module M naturally affords a $\mathcal{H}_n$-module with

$$c_i(u \otimes x) = (c_i u) \otimes x, \qquad s_j(u \otimes x) = (s_j u) \otimes (s_j x), \tag{3.5}$$

for $1 \le i \le n$, $1 \le j \le n-1$, $u \in F$, $x \in M$.

Lemma 3.1 (*A Tensor Identity*). *Suppose that M is a $\mathbb{C}S_n$-module. Then we have an isomorphism of $\mathbb{C}S_n^-$-modules: $\mathfrak{G}_n(\mathcal{C}_n)\otimes M \cong \mathfrak{G}_n(\mathcal{C}_n \otimes M)$; that is,*

$$\mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n) \otimes M \cong \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n \otimes M).$$

Proof. Observe that by (3.1) the action of $\mathbb{C}S_n^-$ on $\mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n \otimes M)$ is given by

$$(t_j * f)(u) = \left(\frac{1}{\sqrt{-2}} s_j(c_j - c_{j+1})\right)(f(u)), \quad f \in \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n \otimes M), u \in U_n \tag{3.6}$$

while by (3.4) the $\mathbb{C}S_n^-$-module structure of $\mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n) \otimes M$ is given by

$$t_j * (f \otimes x) = (t_j * f) \otimes (s_j x), \quad f \in \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n), x \in M. \tag{3.7}$$

Define a map

$$\phi : \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n) \otimes M \longrightarrow \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n \otimes M),$$
$$f \otimes x \mapsto (u \mapsto f(u) \otimes x).$$

Clearly ϕ is injective and thus an isomorphism of vector spaces by a dimension counting argument. It remains to show that ϕ is a $\mathbb{C}S_n^-$-module homomorphism. Indeed, for $u \in U_n, f \in \mathrm{Hom}_{\mathcal{C}_n}(U_n, \mathcal{C}_n)$ and $x \in M$, we have

$$\begin{aligned}
\phi(t_j * (f \otimes x))(u) &= \phi(t_j * f \otimes s_j x)(u) \quad \text{by (3.7)} \\
&= (t_j * f)(u) \otimes s_j x \\
&= \left(\left(\frac{1}{\sqrt{-2}} s_j(c_j - c_{j+1})\right) f(u)\right) \otimes s_j x \\
&= \left(\frac{1}{\sqrt{-2}} s_j(c_j - c_{j+1})\right)(f(u) \otimes x) \quad \text{by (3.5)} \\
&= (t_j * \phi(f \otimes x))(u) \quad \text{by (3.6)}. \quad \square
\end{aligned}$$

Proposition 3.2. *Suppose that M is a $\mathbb{C}S_n$-module. Let m_λ and m_λ^- be the multiplicities of D^λ and D_-^λ in the $\mathcal{H}_n$-module $\mathcal{C}_n \otimes M$ and $\mathbb{C}S_n^-$-module $\mathcal{B}_n \otimes M$, respectively. Then,*

$$m_\lambda^- = \begin{cases} m_\lambda, & \text{if } n \text{ is even}, \\ m_\lambda, & \text{if } n \text{ is odd and } \ell(\lambda) \text{ is odd}, \\ \frac{1}{2} m_\lambda, & \text{if } n \text{ is odd and } \ell(\lambda) \text{ is even}. \end{cases}$$

Proof. It follows by definition that

$$\mathcal{C}_n \otimes M \cong \bigoplus_{\lambda \vdash_s n} m_\lambda D^\lambda, \qquad \mathcal{B}_n \otimes M \cong \bigoplus_{\lambda \vdash_s n} m_\lambda^- D_-^\lambda. \tag{3.8}$$

By (3.3) and Lemma 3.1, we have

$$\mathfrak{G}_n(\mathcal{C}_n \otimes M) \cong \begin{cases} \mathcal{B}_n \otimes M, & \text{if } n \text{ is even}, \\ 2\mathcal{B}_n \otimes M, & \text{if } n \text{ is odd}. \end{cases}$$

This together with (3.8) implies that

$$\bigoplus_{\lambda \vdash_s n} m_\lambda \mathfrak{G}_n(D^\lambda) \cong \begin{cases} \bigoplus_{\lambda \vdash_s n} m_\lambda^- D_-^\lambda, & \text{if } n \text{ is even}, \\ \bigoplus_{\lambda \vdash_s n} 2m_\lambda^- D_-^\lambda, & \text{if } n \text{ is odd}. \end{cases}$$

The proposition now follows by comparing the multiplicities of D_-^λ on both sides and using (3.2). $\square$

3.4. Proof of Theorem A

The symmetric group S_n acts naturally on the ($\mathbb{Z}_+$-graded) symmetric algebra on $V = \mathbb{C}^n$:

$$S^*V = \bigoplus_{j \geq 0} S^j V.$$

As S_n-modules, we will identify S^*V with the algebra of polynomials in n variables over $\mathbb{C}$. Note that $\mathcal{C}_n \otimes S^*V = \mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} S^*V$ is naturally a $\mathbb{Z}_+$-graded $\mathcal{H}_n$-module, with the grading inherited from the one on S^*V.

Lemma 3.3. *We have the following isomorphism of $\mathcal{H}_n$-modules for $j \geq 0$:*

$$\mathcal{C}_n \otimes S^j V \cong \bigoplus_{\nu \models n, n(\nu)=j} \mathrm{ind}_{\mathbb{C}S_\nu}^{\mathcal{H}_n} \mathbf{1}.$$

Proof. Identify $S^*V \equiv \mathbb{C}[x_1, \dots, x_n]$. The definition (1.1) of $n(\nu)$ makes sense for any composition ν. The representatives of the S_n-orbits on the set of all monomials of degree j in $\mathbb{C}[x_1, \dots, x_n]$ can be chosen to be

$$(x_1 \dots x_{\nu_1})^0 (x_{\nu_1+1} \dots x_{\nu_1+\nu_2})^1 (x_{\nu_1+\nu_2+1} \dots x_{\nu_1+\nu_2+\nu_3})^2 \dots$$

where $\nu = (\nu_1, \nu_2, \dots)$ runs over all compositions of n such that $n(\nu) = j$. Then, as $\mathbb{C}S_n$-modules,

$$S^j V \cong \bigoplus_{\nu \models n, n(\nu)=j} \mathrm{ind}_{\mathbb{C}S_\nu}^{\mathbb{C}S_n} \mathbf{1}.$$

Hence, as $\mathcal{H}_n$-modules, we have

$$\mathcal{C}_n \otimes S^j V \cong \bigoplus_{\nu \models n, n(\nu)=j} \mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} \mathrm{ind}_{\mathbb{C}S_\nu}^{\mathbb{C}S_n} \mathbf{1} \cong \bigoplus_{\nu \models n, n(\nu)=j} \mathrm{ind}_{\mathbb{C}S_\nu}^{\mathcal{H}_n} \mathbf{1}. \quad \square$$

Below, we shall denote by $[u^n]f(u)$ the coefficient of u^n of a formal power series $f(u)$ in a variable u. We are ready to prove Theorem A in the Introduction.

Proof of Theorem A. Denote by $K(\mathcal{H}_n\text{-smod})$ the complexified Grothendieck group of the category $\mathcal{H}_n$-smod. Recall the definition of the algebra $\Gamma_\mathbb{C}$ from (2.4). There exists an isomorphism called the characteristic map [13,18,2]

$$\mathrm{ch} : \bigoplus_{n \geq 0} K(\mathcal{H}_n\text{-smod}) \longrightarrow \Gamma_\mathbb{C},$$

which sends D^λ to $2^{-\frac{\ell(\lambda)-\delta(\lambda)}{2}} Q_\lambda$ for all strict partitions λ. It is known that

$$\mathrm{ch}\big(\mathrm{ind}_{\mathbb{C}S_\nu}^{\mathcal{H}_n} \mathbf{1}\big) = q_\nu, \quad \forall \nu \models n. \tag{3.9}$$

By Lemma 3.3, (3.9) and (2.3), we have

$$\begin{aligned}\sum_j t^j \mathrm{ch}(\mathcal{C}_n \otimes S^j V) &= \sum_j t^j \sum_{\nu \models n, n(\nu)=j} q_\nu(z) \\ &= \sum_{\nu \models n} \prod_{r \geq 0} q_{\nu_{r+1}}(z) (t^r)^{\nu_{r+1}} \\ &= [u^n] \prod_{r \geq 0} \sum_{s \geq 0} q_s(z) (t^r u)^s \\ &= [u^n] \prod_{i \geq 1, r \geq 0} \frac{1 + z_i t^r u}{1 - z_i t^r u}.\end{aligned}$$

Recall the notation $Q_\lambda(t^\bullet)$ from the Introduction. It follows from (2.2) that

$$\prod_{i \geq 1, j \geq 0} \frac{1 + z_i t^j u}{1 - z_i t^j u} = \sum_{\lambda:\text{ strict}} 2^{-\ell(\lambda)} u^{|\lambda|} Q_\lambda(t^\bullet) Q_\lambda(z).$$

Hence

$$\sum_j t^j \mathrm{ch}(\mathcal{C}_n \otimes S^j V) = \sum_{\lambda \vdash_s n} 2^{-\ell(\lambda)} Q_\lambda(t^\bullet) Q_\lambda(z).$$

Since the characteristic map ch is an isomorphism, we have an isomorphism of $\mathcal{H}_n$-modules:

$$\mathcal{C}_n \otimes S^* V \cong \bigoplus_{\lambda \vdash_s n} 2^{-\frac{\delta(\lambda)+\ell(\lambda)}{2}} Q_\lambda(t^\bullet) D^\lambda.$$

This together with Theorem B implies Theorem A. $\square$

The following corollary follows directly from Theorem A and Proposition 3.2.

Corollary 3.4. *The graded multiplicity of D_-^λ in the $\mathbb{C}S_n^-$-module $\mathcal{C}_n \otimes S^*V$ is given by (1.2) unless n is odd and $\ell(\lambda)$ is even; in this case, the graded multiplicity is*

$$2^{-\frac{\ell(\lambda)}{2}-1} \frac{t^{n(\lambda)} \prod_{(i,j) \in \lambda^*} (1 + t^{c_{ij}})}{\prod_{(i,j) \in \lambda^*} (1 - t^{h_{ij}^*})}.$$

3.5. A graded regular $\mathcal{H}_n$-module

It is well known that the algebra $(S^*V)^{S_n}$ of S_n-invariants in S^*V is a free polynomial algebra whose Hilbert series $P(t)$ is given by

$$P(t) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}. \tag{3.10}$$

Define the ring of coinvariants $(S^*V)_{S_n}$ to be the quotient of S^*V by the ideal generated by the homogeneous invariant polynomials of positive degrees. It is well known that S^*V is a free module over the algebra $(S^*V)^{S_n}$, and so we have an isomorphism of graded $\mathbb{C}S_n$-modules

$$S^*V \cong (S^*V)_{S_n} \otimes_{\mathbb{C}} (S^*V)^{S_n}. \tag{3.11}$$

Theorem 3.5. *The graded multiplicity of D^λ in $\mathcal{C}_n \otimes (S^*V)_{S_n}$ is*

$$2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} \frac{t^{n(\lambda)}(1-t)(1-t^2)\cdots(1-t^n) \prod_{(i,j)\in\lambda^*}(1+t^{c_{ij}})}{\prod_{(i,j)\in\lambda^*}(1-t^{h^*_{ij}})}.$$

Proof. Follows directly from Theorem A, (3.10), and (3.11). □

Remark 3.6. Recall the isomorphism of $\mathcal{H}_n$-modules $D^{(n)} \cong \mathcal{C}_n$ (cf. [6]). It follows that the graded multiplicity of $\mathcal{C}_n$ in $\mathcal{C}_n \otimes (S^*V)_{S_n}$ is $(1+t)(1+t^2)\cdots(1+t^{n-1})$, and the graded multiplicity of $\mathcal{C}_n$ in $\mathcal{C}_n \otimes S^*V$ is

$$\frac{(1+t)(1+t^2)\cdots(1+t^{n-1})}{(1-t)(1-t^2)\cdots(1-t^n)}. \tag{3.12}$$

Remark 3.7. The number g^λ of standard shifted Young tableaux of shape λ is known to be (cf. [11,7])

$$g^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda^*} h^*_{ij}}.$$

By the isomorphism of $\mathbb{C}S_n$-modules $(S^*V)_{S_n} \cong \mathbb{C}S_n$, the $\mathcal{H}_n$-module $\mathcal{C}_n \otimes (S^*V)_{S_n}$ is isomorphic to the regular representation of $\mathcal{H}_n$. It is known (cf. [6]) that the multiplicity of D^λ in the regular representation of $\mathcal{H}_n$ is given by $\frac{1}{2^{\delta(\lambda)}}\dim D^\lambda$. By specializing $t = 1$ in Theorem 3.5, we recover the dimension formula $\dim D^\lambda = 2^{n-\frac{\ell(\lambda)-\delta(\lambda)}{2}} g^\lambda$.

4. The graded multiplicity in $\mathcal{C}_n \otimes S^*V \otimes \wedge^*V$

4.1. The S_n-module $S^*V \otimes \wedge^*V$

The S_n-action on $V = \mathbb{C}^n$ induces a natural S_n-action on the exterior algebra

$$\wedge^*V = \bigoplus_{q=0}^{n} \wedge^q V.$$

This gives rise to a $\mathbb{Z}_+ \times \mathbb{Z}_+$ bi-graded $\mathbb{C}S_n$-module structure on

$$S^*V \otimes \wedge^*V = \bigoplus_{p\geq 0, 0\leq q\leq n} S^pV \otimes \wedge^q V.$$

According to Kirillov and Pak [5], the bi-graded multiplicity of the Specht module S^λ for $\lambda \vdash n$ in $S^*V \otimes \wedge^*V$ is given by

$$\sum_{p=0}^{\infty}\sum_{q=0}^{n} t^p s^q m_\lambda(S^pV \otimes \wedge^q V) = \frac{\prod_{(i,j)\in\lambda}(t^{i-1}+st^{j-1})}{\prod_{(i,j)\in\lambda}(1-t^{h_{ij}})}, \tag{4.1}$$

which can be rewritten as

$$\frac{t^{n(\lambda)}\prod_{(i,j)\in\lambda}(1+st^{c_{ij}})}{\prod_{(i,j)\in\lambda}(1-t^{h_{ij}})}.$$

In particular, this recovers Solomon's formula [15] for the generating function for the bi-graded S_n-invariants in $S^*V \otimes \wedge^* V$:

$$\frac{(1+s)(1+st)\cdots(1+st^{n-1})}{(1-t)(1-t^2)\cdots(1-t^n)}. \tag{4.2}$$

The formal similarity between the graded multiplicities (1.2) and (4.1) in very different settings is rather striking. Also compare the similarity between (3.12) and (4.2).

4.2. Proof of Theorem C

Lemma 4.1. *The following holds as $\mathcal{H}_n$-modules:*

$$\mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} \mathrm{sgn} \cong \mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} \mathbf{1}.$$

Proof. Define a $\mathbb{C}$-linear map

$$\begin{aligned} &f : \mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} \mathrm{sgn} = \mathcal{C}_n \otimes \mathrm{sgn} \to \mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n} \mathbf{1} \\ &c \otimes 1 \mapsto c \cdot (c_1 c_2 \cdots c_n \otimes 1) \end{aligned}$$

It is straightforward to show that f is actually a $\mathcal{H}_n$-module isomorphism. □

Lemma 4.2. *For $p \geq 0$, $0 \leq q \leq n$, as $\mathcal{H}_n$-modules, we have*

$$\mathcal{C}_n \otimes S^p V \otimes \wedge^q V \cong \bigoplus_{\alpha,\beta} \mathrm{ind}_{\mathbb{C}(S_\alpha \times S_\beta)}^{\mathcal{H}_n} \mathbf{1},$$

summed over all $\alpha \models n-q$, $\beta \models q$ with $n(\alpha)+n(\beta)=p$.

Proof. Arguing similarly as in the proof of Theorem A, we have an isomorphism of $\mathbb{C}S_n$-modules:

$$S^p V \otimes \wedge^q V \cong \bigoplus_{\alpha,\beta} \mathrm{ind}_{\mathbb{C}(S_\alpha \times S_\beta)}^{\mathbb{C}S_n} (\mathbf{1} \otimes \mathrm{sgn}),$$

summed over all $\alpha \models n-q$, $\beta \models q$ with $n(\alpha)+n(\beta)=p$. Now the lemma follows by applying $\mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n}$ to the above isomorphism and using Lemma 4.1. □

We are ready to prove Theorem C from the Introduction.

Proof of Theorem C. It follows by (3.9) and Lemma 4.2 that

$$\mathrm{ch}(\mathcal{C}_n \otimes S^p V \otimes \wedge^q V) = \sum_{\alpha,\beta} q_\alpha(z) q_\beta(z),$$

summed over all $\alpha = (\alpha_1, \alpha_2, \ldots) \models n-q$, $\beta = (\beta_1, \beta_2, \ldots) \models q$ with $n(\alpha)+n(\beta)=p$. Hence,

$$\begin{aligned} \sum_{p\geq 0, 0\leq q\leq n} t^p s^q \mathrm{ch}(\mathcal{C}_n \otimes S^p V \otimes \wedge^q V) &= \sum_{p\geq 0, 0\leq q\leq n} t^p s^q \sum_{\alpha\models n-q, \beta\models q, n(\alpha)+n(\beta)=p} q_\alpha(z) q_\beta(z) \\ &= [u^n] \sum_{\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots} \prod_{r\geq 0} q_{\alpha_{r+1}}(z)(t^r u)^{\alpha_{r+1}} \prod_{k\geq 0} q_{\beta_{k+1}}(z)(t^k s u)^{\beta_{k+1}} \\ &= [u^n] \prod_{i\geq 1, r\geq 0} \frac{1+z_i t^r u}{1-z_i t^r u} \frac{1+z_i t^r s u}{1-z_i t^r s u} \\ &= \sum_{\lambda \vdash_s n} 2^{-\ell(\lambda)} Q_\lambda(t^\bullet; st^\bullet) Q_\lambda(z) \end{aligned}$$

where we have used the short-hand notation $Q_\lambda(t^\bullet; st^\bullet)$ from the Introduction and (2.2) in the last equation. Theorem C follows. □

Remark 4.3. It is an interesting open problem to find an explicit formula for $Q_\lambda(t^\bullet; st^\bullet)$. Consider a Koszul $\mathbb{Z}_+$-grading which counts the standard generators of S^*V as degree 2 and the standard generators of $\wedge^* V$ as degree 1. This corresponds precise to setting $t = s^2$, and hence, $Q_\lambda(t^\bullet; st^\bullet) = Q_\lambda(s^\bullet)$. Therefore, for the Koszul grading, the graded multiplicity of D^λ in $\mathcal{C}_n \otimes S^*V \otimes \wedge^* V$ is given by the same formula (1.4).

4.3. Some consequences of Theorem C

Recall the notation of $Q_\lambda(t^\bullet; st^\bullet)$ from the Introduction. By Proposition 3.2, we have the following corollary as a counterpart of Theorem C for $\mathbb{C}S_n^-$.

Corollary 4.4. *The bi-graded multiplicity of D_-^λ in the $\mathbb{C}S_n^-$-module $\mathcal{B}_n \otimes S^*V \otimes \wedge^*V$ is given by (1.4) unless n is odd and $\ell(\lambda)$ is even; in this case, the bi-graded multiplicity is*

$$2^{-\frac{\ell(\lambda)}{2}-1}Q_\lambda(t^\bullet; st^\bullet).$$

Corollary 4.5. *The graded multiplicity of D^λ in the $\mathcal{H}_n$-module $\mathrm{ind}_{\mathbb{C}S_n}^{\mathcal{H}_n}(\wedge^*V)$ is given by $2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}}Q_\lambda(1, s)$. Moreover,*

$$Q_\lambda(1, s) = \begin{cases} \frac{2^{\ell(\lambda)}(1+s)(s^l-s^k)}{1-s}, & \text{if } \lambda = (k, l) \text{ with } k > l \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first statement is obtained by setting $t = 0$ in Theorem C. By (2.1), we see that

$$Q_\lambda(z_1, z_2) = \begin{cases} \frac{2^{\ell(\lambda)}(z_1+z_2)(z_1^k z_2^l - z_1^l z_2^k)}{z_1-z_2}, & \text{if } \lambda = (k, l) \text{ with } k > l \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The corollary follows by setting $z_1 = 1$ and $z_2 = s$. □

Setting $s = 0$ in Theorem C, we recover Theorem A.

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