Mock finitely generated Gorenstein injective modules and isolated singularities

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Abstract

In this paper, we study mock finitely generated modules and Gorenstein injective modules which are a generalization of finitely generated modules and injective modules respectively. We also discuss invariants for minimal injective resolvents which are analogous to Bass' invariants for minimal injective resolutions. We use these invariants to characterize Gorenstein isolated singularities in terms of mock finitely generated Gorenstein injective modules.

1. Introduction

R will denote a commutative noetherian ring.

A linear map $\psi : E \to M$ with $E$ an injective $R$-module and $M$ an $R$-module is said to be an injective precover of $M$ if for any linear map $E' \to M$ with $E'$ injective, the diagram

$$
\begin{array}{ccc}
E' & \to & E \\
\downarrow & & \downarrow \psi \\
E & \to & M
\end{array}
$$

can be completed to a commutative diagram.

If furthermore, the diagram

$$
\begin{array}{ccc}
E' & \to & E \\
\downarrow & & \downarrow \psi \\
E & \to & M
\end{array}
$$

can only be completed by automorphisms of $E$, then $\psi : E \to M$ is called an injective cover. So if an injective cover exists, it is unique up to isomorphism.

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In [2], it was shown that a ring $R$ is noetherian if and only if every $R$-module has an injective precover and if and only if every $R$-module has an injective cover. Hence for an $R$-module $M$, we can construct a complex

$$\cdots \to E^{-1} \to E^{-0} \to M \to 0$$

where $E^{-0} \to M$, $E^{-1} \to \text{Ker}(E^{-0} \to M)$, $E^{-i-2} \to \text{Ker}(E^{-i-1} \to E^{-i})$, $i \geq 1$, are injective precovers. This complex is called an injective resolvent of $M$. If furthermore, the maps are injective covers, then the complex is called a minimal injective resolvent.

We note that an injective resolvent is not exact in general but that the functor $\text{Hom}_R(E, -)$ with $E$ an injective $R$-module makes it exact. So for $R$-modules $M$ and $N$, $\text{Hom}_R(N, M)$ is left balanced by injective $R$-modules (see [3]). Therefore, left derived functors of $\text{Hom}_R(N, M)$, denoted by $\text{Ext}_R^k(N, M)$, can be computed using either an injective resolvent of $M$ or an injective resolution of $N$. We are using the topologists' notation $\text{Ext}_R^k(N, M)$ for $\text{Ext}_R^i(N, M)$ noting that $\text{Ext}_R^0(N, M)$ is not the same as $\text{Ext}_R^0(N, M)$. In fact, the natural map

$$\text{Ext}_R^0(N, M) \to \text{Hom}_R(N, M)$$

is not an isomorphism in general. The kernel and cokernel of this map will be denoted by $\text{Ext}_R^0(N, M)$ and $\overline{\text{Ext}}_R^0(N, M)$ respectively.

An $R$-module $M$ is said to be mock finitely generated if for any finitely generated $R$-module $N$, each of $\text{Ext}_R^k(N, M)$, $\overline{\text{Ext}}_R^k(N, M)$, $\overline{\text{Ext}}_R^0(N, M)$ and $\overline{\text{Ext}}_R^0(N, M)$ are finitely generated $R$-modules. All finitely generated $R$-modules are mock finitely generated (Proposition 3.2). Furthermore, if $R$ is Gorenstein, that is, $\text{id}_R < \infty$, then every Gorenstein injective cosyzygy of a mock finitely generated $R$-module is also mock finitely generated (Lemma 2.9).

An $R$-module $M$ is said to be Gorenstein injective if every injective resolvent of $M$ is exact and every injective resolution of $M$ is an injective resolvent (see [4] for equivalent definitions). We have an abundant supply of Gorenstein injective $R$-modules. For instance, if $R$ is a Gorenstein ring of dimension $d$, that is, $\text{id}_R = d$, then every $n$th cosyzygy, $n \geq d$, of an $R$-module is Gorenstein injective (see [4, Theorem 4.2]). Hence in this case, every $n$th cosyzygy of a finitely generated $R$-module is a mock finitely generated Gorenstein injective $R$-module for $n \geq d$. We note that these cosyzygies are rarely finitely generated.

The aim of this paper is to study mock finitely generated Gorenstein injective modules. In Section 3, we introduce and study invariants for minimal injective resolvents that are analogous to Bass' invariants for minimal injective resolutions. In Section 4, we use the preliminary results in Section 2 and the invariants to characterize a Gorenstein isolated singularity $R$, that is, a Gorenstein local ring $R$ such that $R_P$ is regular for each prime ideal $P$ that is not maximal, in terms of mock finitely generated Gorenstein injective modules. For instance, we will show that a Gorenstein local ring is an isolated singularity if and only if every reduced mock finitely generated Gorenstein injective $R$-module is artinian (Theorem 4.1). Recall that a module is said to be reduced if it has no nonzero injective submodule.
2. Preliminary results

We start with the following:

**Lemma 2.1.** Let $M$ be an $R$-module and $C^j$ be a $j$th cosyzygy of an $R$-module $N$. Then

$$\text{Ext}^i_R(C^j, M) \cong \text{Ext}_{i+j}^R(N, M)$$

for all $i \geq 1$.

**Proof.** We consider the short injective resolution $0 \rightarrow C^{j-1} \rightarrow E^{j-1} \rightarrow C^j \rightarrow 0$ for $j > 1$. Then we have a long exact sequence

$$\cdots \rightarrow \text{Ext}_{i+1}^j(E^{j-1}, M) \rightarrow \text{Ext}_i(C^{j-1}, M) \rightarrow \cdots$$

(see [4]). So $\text{Ext}_i(C^j, M) \cong \text{Ext}_{i+1}(C^{j-1}, M)$ for $i \geq 1$. and thus inductively

$$\text{Ext}_i(C^j, M) \cong \text{Ext}_{i+j}(N, M). \quad \square$$

Now let $K^{-i}(M)$ denote the $i$th syzygy of the minimal injective resolvent of an $R$-module $M$, and let $\mathcal{C}_R$ be the full subcategory of $R$-modules whose injective resolvents are exact. Then we have

**Lemma 2.2.** If $M \in \mathcal{C}_R$, then

$$\text{Ext}_i^R(N, M) \cong \text{Ext}_i^k(N, K^{-i+2}(M)) \quad \text{for all } i \geq 1$$

and for all $R$-modules $N$.

**Proof.** Let $0 \rightarrow K^{-i-2} \rightarrow E^{-i-1} \xrightarrow{\beta} E^{-i} \xrightarrow{\alpha} E^{-i+1} \rightarrow \cdots$ be a partial minimal injective resolvent of $M$. We note that this sequence is exact by assumption. Then it follows from the complex

$$\cdots \rightarrow \text{Hom}(N, E^{-i-1}) \xrightarrow{\tilde{\beta}} \text{Hom}(N, E^{-i}) \xrightarrow{\tilde{\alpha}} \text{Hom}(N, E^{-i+1}) \rightarrow \cdots$$

that

$$\text{Ext}_i^R(N, M) = \frac{\text{Ker } \tilde{\alpha}}{\text{Im } \tilde{\beta}} = \text{Ext}_i^k(N, K^{-i-2}(M)). \quad \square$$
Corollary 2.3. The following are equivalent for each \( M \in \mathfrak{C}_R \).

1. \( \text{Ext}^i(N, M) = 0 \) for all \( i \geq 1 \) and for all \( N \in \mathfrak{C}_R \).
2. \( \text{Ext}^i(N, M) = 0 \) for all \( N \in \mathfrak{C}_R \).
3. \( M \) is an injective \( R \)-module.

Proof. (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (1) are trivial.

(2) \( \Rightarrow \) (3) By Lemma 2.2, \( \text{Ext}^1(N, K^{-3}(M)) = 0 \) for all \( N \in \mathfrak{C}_R \). But \( K^{-2}(M) \in \mathfrak{C}_R \). So \( \text{Ext}^1(K^{-2}(M), K^{-3}(M)) = 0 \) and thus the exact sequence \( 0 \to K^{-3}(M) \to E^{-2} \to K^{-2}(M) \to 0 \) splits. Thus \( K^{-2}(M) \) is injective and so \( K^{-1}(M) \) is also injective. But then \( M \) is injective. \( \square \)

Lemma 2.4. Let \( M \) be a reduced \( R \)-module and \( \psi : E \to M \) be the injective cover of \( M \). Then \( E \) is the injective envelope of the kernel of \( \psi \).

Proof. We simply note that \( E = E(\text{Ker} \psi) \oplus E' \) where \( E' \) is also an injective \( R \)-module. But \( E' \) gets mapped isomorphically to a submodule of \( M \). So \( E' = 0 \) since \( M \) is reduced. \( \square \)

Proposition 2.5. Let \( R \) be a Gorenstein ring and \( P \in \text{Spec} \ R \). If \( M \) is a Gorenstein injective \( R \)-module, then \( M_P \) is also a Gorenstein injective \( R_P \)-module. Furthermore if \( M \) is reduced, then \( M_P \) is reduced.

Proof. Let \( \cdots \to E^{-2} \to E^{-1} \to E^0 \to M \to 0 \) be the minimal injective resolution of \( M \). Then it is exact since \( M \) is Gorenstein injective. But kernels of injective covers are reduced. So if \( K = \text{Ker}(E^{-0} \to M) \), then \( \cdots \to E^{-2} \to E^{-1} \to K \to 0 \) is a minimal injective resolution by Lemma 2.4. Hence the exact sequence \( \cdots \to E^{-2}_P \to E^{-1}_P \to K_P \to 0 \) is also a minimal injective resolution. If \( E^{-i}_P = 0 \) for some \( i \geq 1 \), then \( K_P \) is injective and so the exact sequence \( 0 \to K_P \to E^{-0}_P \to M_P \to 0 \) splits and thus \( M_P \) is injective. If \( E^{-1}_P \) are not zero for \( i > 1 \), then setting \( d = \text{id}_R \), we see that \( M_P \) is a \( d \)th cosyzygy and thus is Gorenstein injective. \( \square \)

Corollary 2.6. If \( R \) is a Gorenstein isolated singularity, then every Gorenstein injective \( R \)-module is locally injective on the punctured spectrum.

Proof. Let \( M \) be a Gorenstein injective \( R \)-module, Then \( M_P \) is a Gorenstein injective \( R_P \)-module for each \( P \in \text{Spec} \ R \) by the proposition above. But if \( P \) is not maximal, then \( \text{gl dim} \ R_P < \infty \) and so \( M_P \) is injective. \( \square \)

The following is well known.

Lemma 2.7. If \( M \) is an \( R \)-module and \( N \) is a finitely generated \( R \)-module, then

\[ \text{Ext}^i_R(N, M)_P \cong \text{Ext}^i_{R_P}(N_P, M_P) \quad \text{for all } P \in \text{Spec} \ R. \]

We are not in a position to prove the following:

**Theorem 2.8.** Let $R$ be a Gorenstein ring and $M$ be a Gorenstein injective $R$-module. If $C$ is a cosyzygy of a finitely generated $R$-module, then

1. $\text{Ext}^i_R(C, M)_P \cong \text{Ext}^i_R(C, M_p)$ for all $i > 1$ and for all $P \in \text{Spec } R$.
2. If $M$ is mock finitely generated, then $\text{Ext}^i_R(C, M)$ is a finitely generated $R$-module for all $i \geq 1$.

**Proof.** Let $C$ be a $j$th cosyzygy of a finitely generated $R$-module $N$. Then

$$\text{Ext}^i_R(C, M) \cong \text{Ext}^i_R(N, M)$$

by Lemma 2.1 and so $\text{Ext}^i_R(C, M)$ is finitely generated if $M$ is mock finitely generated proving part (2) of the theorem. But $\text{Ext}^i_R(N, M) \cong \text{Ext}^i_R(N, K^{-i-j-2}(M))$ by Lemma 2.2. So

$$\text{Ext}^i_R(C, M)_P \cong \text{Ext}^i_R(N, K^{-i-j-2}(M))_P$$

by Lemma 2.7

$$\cong \text{Ext}^i_R(N_P, M_P)$$

(by Lemma 2.2 since $M_P \in \mathcal{C}_R$).

But then the result follows from Lemma 2.1. □

**Remark.** It is now easy to see that if $N$ is a finitely generated $R$-module and $M$ is a Gorenstein injective $R$-module, then $\text{Ext}^i_R(N, M)_P \cong \text{Ext}^i_R(N_P, M_P)$. Hence in this case, $\text{Ext}^i_R(N, M)_P \cong \text{Ext}^i_R(N_P, M_P)$ for all $i \geq 0$.

**Lemma 2.9.** Let $R$ be a Gorenstein ring. Then every Gorenstein injective cosyzygy of a mock finitely generated $R$-module is also mock finitely generated.

**Proof.** Let $C$ be an $i$th cosyzygy of a mock finitely generated $R$-module $M$ and let $N$ be an $R$-module. Then $\text{Ext}^i(N, C) \cong \text{Ext}^{i+1}(N, M)$. So $\text{Ext}^i(N, C)$ is finitely generated for all finitely generated $R$-modules $N$. Hence $C$ is mock finitely generated by Corollary 6.4 of [4]. □

Now let $C^i(M)$ denote the $i$th cosyzygy of the minimal injective resolution of an $R$-module $M$ and let $\mathcal{D}_R$ denote the full subcategory of $R$-modules $M$ such that every injective resolution of $M$ is an injective resolvent. Then we have the following:
Lemma 2.10. If \( M \in \mathcal{D}_R \), then
\[
\text{Ext}_R^i(N, M) \cong \text{Ext}_R^i(N, C^{i+2}(M))
\]
for all \( i \geq 1 \) and for all \( R \)-modules \( N \).

**Proof.** We consider the partial minimal injective resolution \( 0 \to M \to E^0 \to E^1 \to \cdots \to E^i \to E^{i+1} \to C^{i+2}(M) \to 0 \). This is also a partial injective resolvent of \( C^{i+2}(M) \) since \( M \in \mathcal{D}_R \). So the result follows. \( \square \)

Theorem 2.11. Let \( R \) be a Gorenstein ring, \( M \) be a Gorenstein injective \( R \)-module, and \( N \) be a cosyzygy of a finitely generated \( R \)-module. Then
1. \( \text{Ext}_R^i(N, M)_P \cong \text{Ext}_R^i(N_P, M_P) \) for all \( i \geq 1 \) for all \( P \in \text{Spec} \, R \).
2. If \( M \) is mock finitely generated, then \( \text{Ext}_R^i(N, M) \) is a finitely generated \( R \)-module for all \( i \geq 1 \).

**Proof.** \( M \) Gorenstein injective means \( M \in \mathcal{D}_R \cap \mathcal{I}_R \). So
\[
\text{Ext}_R^i(N, M)_P \cong \text{Ext}_R^i(N, C^{i+2}(M))_P \quad \text{(by Lemma 2.10)}
\]
\[
\cong \text{Ext}_R^i(N_P, C^{i+2}(M)_P) \quad \text{(by Theorem 2.8 since } C^{i+2}(M) \text{ is also Gorenstein injective)}.
\]

But \( M_P \) is Gorenstein injective by Proposition 2.5. So \( M_P \in \mathcal{D}_{R_P} \). Hence
\[
\text{Ext}_R^i(N_P, C^{i+2}(M)_P) \cong \text{Ext}_R^i(N_P, M_P) \quad \text{(by Lemma 2.10)}.
\]
If \( M \) is mock finitely generated, then \( C^{i+2}(M) \) is also mock finitely generated by Lemma 2.9. But \( \text{Ext}_R^i(N, M) \cong \text{Ext}_R^i(N, C^{i+2}(M)) \) and so \( \text{Ext}_R^i(N, M) \) is finitely generated. \( \square \)

3. Invariants

The following lemma is well known. We include a proof here for completeness.

**Lemma 3.1.** If \( M \) is a finitely generated \( R \)-module and \( \mathcal{P} \) is a prime ideal of \( R \) with \( \text{Hom}(E(R/\mathcal{P}), M) \neq 0 \), then \( \mathcal{P} \) is a maximal ideal.

**Proof.** Let \( \varphi \in \text{Hom}(E(R/P), M) \), \( \varphi \neq 0 \). By replacing \( M \) with \( \text{im}(\varphi) \), we may assume \( \varphi \) is surjective. And by going modulo a maximal submodule, we may assume \( M \) is simple. Hence we may assume \( M = R/\mathcal{M} \) for a maximal ideal \( \mathcal{M} \).

If \( \mathcal{P} \not\subset \mathcal{M} \), let \( r \in \mathcal{P} \) and \( r \notin \mathcal{M} \). Then for each \( z \in R/\mathcal{M} \), \( z \neq 0 \), \( rz \neq 0 \). But for each \( x \in E(R/\mathcal{P}) \), \( r^n x = 0 \) for some \( n \geq 1 \). So for \( \varphi(x) = z \) we would have \( r^n z = 0 \), a
contradiction. Hence $\mathcal{P} \subseteq \mathcal{M}$. If $P \neq \mathcal{M}$, let $r \in \mathcal{M}$ and $r \notin \mathcal{P}$. Then $rE(R/\mathcal{P}) = E(R/\mathcal{P})$ but $\Omega(R/\mathcal{M}) = 0$. So there is no surjective map $\varphi : E(R/\mathcal{P}) \to R/\mathcal{M}$. Hence $\mathcal{P} = \mathcal{M}$. □

**Proposition 3.2.** Every finitely generated $R$-module is mock finitely generated and has an injective cover that is a direct sum of finitely many copies of $E(R/\mathcal{M})$ over finitely many maximal ideals $\mathcal{M}$.

**Proof.** Let $M$ be a finitely generated $R$-module and $\mathcal{M}$ be a maximal ideal of $R$. Let $M' = \{ x \in M \mid \mathcal{M}^nx = 0 \text{ for some } n \geq 1 \}$. Then $M'$ is anartinian $R$-module and

$$\text{Hom}(E(R/\mathcal{M}), M) = \text{Hom}(E(R/\mathcal{M}), M') \cong \text{Hom}\left(\frac{E(R/\mathcal{M})}{\mathcal{M}'E(R/\mathcal{M})}, M'\right)$$

where $\mathcal{M}'M' = 0$. But $E(R/\mathcal{M})/\mathcal{M}'E(R/\mathcal{M})$ is artinian and annihilated by $\mathcal{M}'$ and so has finite length. Hence $\text{Hom}(E(R/\mathcal{M}), M)$ is finitely generated. We note that if $\text{Hom}(E(R/\mathcal{M}), M) \neq 0$, then $\mathcal{M} \in \text{Ass}(M)$.

Now let $N$ be a finitely generated $R$-module and $0 \to N \to E^0 \to E^1 \to \cdots$ be the minimal injective resolution of $N$. Then $E^i = \bigoplus_{P \in \text{Spec}R} \mu^i(P, N) E(R/P)$ by Matlis [5] were $\mu^i$ are Bass invariants. But $\mu^i(P, N) < \infty$ for each $i$ by Bass [1]. Furthermore, $\text{Ass}(M)$ has only finitely many primes since $M$ is finitely generated. Hence it follows from Lemma 3.1 that $\text{Hom}(E^i, M) \cong \bigoplus_{\mathcal{M} \in \text{Ass}(M)} \mu^i(\mathcal{M}, N) \text{Hom}(E(R/\mathcal{M}), M)$. But $\text{Hom}(E(R/\mathcal{M}), M)$ is finitely generated from the above. So $\text{Hom}(E^i, M)$ is finitely generated for each $i \geq 0$. Thus $\text{Ext}_i(N, M)$ is finitely generated for each $i \geq 0$. We now use the exact sequence

$$0 \to \text{Ext}_0(N, M) \to \text{Ext}_0(N, M) \to \text{Ext}_0(N, M) \to \text{Ext}_0(N, M)$$

to get that $\text{Ext}_0(N, M) = \text{Ext}_0(N, M)$ are also finitely generated. Hence $M$ is mock finitely generated.

It follows from Lemma 3.1 that the injective cover of $M$ is direct sum of copies of $E(R/\mathcal{M})$ over maximal ideals $\mathcal{M}$. But each such $\mathcal{M}$ is an associated prime ideal as noted above. So the injective cover is a direct sum of copies of $E(R/\mathcal{M})$ over finitely many maximal ideals. We now show that there are only finitely many copies of $E(R/\mathcal{M})$ for each such maximal ideal.

We first recall that $\text{Hom}(E(R/\mathcal{M}), E(M) = (\hat{R}_\mathcal{M})^n$ for some $n$ since $\text{Hom}(E(R/\mathcal{M}), E(R/\mathcal{M})) = \hat{R}_\mathcal{M}$ and $\mu^0(M) < \infty$. Hence $\text{Hom}(E(R/\mathcal{M}), M)$ is a finitely generated $\hat{R}_\mathcal{M}$-module. Let $\varphi_1, \varphi_2, \ldots, \varphi_s \in \text{Hom}(E(R/\mathcal{M}), M)$ be generators as an $\hat{R}_\mathcal{M}$-module. Then if $\varphi \in \text{Hom}(E(R/\mathcal{M}), M)$, then $\varphi = \sum_{i=1}^s \varphi_i \sigma_i$ for some $\sigma_1, \sigma_2, \ldots, \sigma_s \in \text{Hom}(E(R/\mathcal{M}), E(R/\mathcal{M}))$ since $\hat{R}_\mathcal{M} \cong \text{Hom}(E(R/\mathcal{M}), E(R/\mathcal{M}))$. This means that we can complete

$$\begin{array}{ccc}
E(R/\mathcal{M}) & \xrightarrow{\varphi} & E(R/\mathcal{M})^s \\
\downarrow & & \downarrow \\
M & \xrightarrow{\varphi_1, \ldots, \varphi_s} & M
\end{array}$$

to a commutative diagram.
Now let \( E \to M \) be the injective cover of \( M \) and let \( E = E_1 \oplus E_2 \) where \( E_1 \) is the direct sum of all copies of \( E(R/\mathfrak{M}) \) is some decomposition of \( E \) into indecomposable injective \( R \)-modules. Then by the above, we can complete

\[
\begin{array}{ccc}
E_1 & \to & E(R/\mathfrak{M})^\ell
\end{array}
\]

\( \xrightarrow{(\varphi_1, \ldots, \varphi_\ell)} \)

\[
M
\]

to a commutative diagram. But then

\[
E = E_1 \oplus E_2
\]

\[
E(R/\mathfrak{M})^\ell \oplus E_2 \to M
\]

can be completed to a commutative diagram. So \( E(R/\mathfrak{M})^\ell \oplus E_2 \to M \) is an injective precover and hence \( E \) is a direct summand of \( E(R/\mathfrak{M})^\ell \oplus E_2 \). Thus the injective cover of \( M \) has finitely many copies of \( E(R/\mathfrak{M}) \).

Now, let \( \cdots \to E^{-1} \to E^{-0} \to M \to 0 \) be the minimal injective resolvent of an \( R \)-module \( M \). Then we will let \( v_i(P, M) \) denote the number of components of \( E^{-i} \) that are isomorphic to \( E(R/P) \). Thus

\[
E^{-i} \cong \bigoplus_{P \in \text{Spec} \ R} v_i(P, M)E(R/P).
\]

Then the following is analogous to Bass' result in [1] and its proof is the proof of Bass invariants applied to minimal injective resolvents.

**Proposition 3.3.** Let \( R \) be a local ring with maximal ideal \( \mathfrak{M} \) and residue field \( k \), and let \( M \) be a reduced \( R \)-module. Then

\[
v_i(\mathfrak{M}, M) = \dim_k \text{Ext}_R^i(k, M).
\]

If furthermore \( M \) is mock finitely generated, then \( v_i(\mathfrak{M}, M) < \infty \) for all \( i \).

**Proof.** Let \( \cdots \to E^{-i-1} \to E^{-i} \xrightarrow{d_i} \cdots \to E^{-1} \xrightarrow{d_1} E^{-0} \xrightarrow{d_0} M \to 0 \) be a minimal injective resolvent of \( M \). Then

\[
\text{Hom}_R(k, E^{-i}) \cong \text{Hom}
\]

\[
\left(k, \bigoplus_{P \in \text{Spec} \ R} v_i(P, M)E(R/P)\right)
\]

\[
\cong \text{Hom}(k, v_i(\mathfrak{M}, M)E(R/\mathfrak{M})) \oplus \text{Hom}
\]

\[
\left(k, \bigoplus_{P \not\in \mathfrak{M}} v_i(P, M)E(R/P)\right)
\]

\[
\cong v_i(\mathfrak{M}, M)k
\]
since
\[ \text{Hom}(k, E(R/P)) = \begin{cases} k & \text{if } P = \mathcal{M}, \\ 0 & \text{if } P \neq \mathcal{M}. \end{cases} \]

So \( v_i(\mathcal{M}, M) = \dim_k \text{Hom}(k, E^-i) \).

Now let \( \sigma \in \text{Hom}(k, E^-i) \) be a nonzero map. Then we first note that \( \sigma(k) \cong k \). But \( \text{Ker} \, d_i = K^{-i-1} \hookrightarrow E^{-i} \) is an essential extension by Lemma 2.4 noting that \( M \) is reduced and the kernel of an injective cover is reduced. Therefore \( \sigma(k) \cap \text{Ker} \, d_i \neq 0 \) and so \( \sigma(k) \subseteq \text{Ker} \, d_i \) since \( \sigma(k) \cong k \). Hence the map \( \text{Hom}(k, E^-i) \to \text{Hom}(k, E^{-i+1}) \) in the deleted complex

\[ \cdots \to \text{Hom}(k, E^-i) \to \text{Hom}(k, E^{-i+1}) \to \cdots \to \text{Hom}(k, E^{-0}) \to 0 \]

is the zero map. So \( \text{Ext}_i(k, M) \cong \text{Hom}(k, E^-i) \) and thus the result follows. \( \square \)

**Theorem 3.4.** Let \( R \) be a Gorenstein ring and \( M \) be a reduced Gorenstein injective \( R \)-module. Then for each \( P \in \text{Spec} \, R \),

\[ v_i(P, M) = \dim_{k(P)} \text{Ext}^R_i(k(P), M_P) = \dim_{k(P)} \text{Ext}^R_i(R/P, M)_P. \]

If \( M \) is furthermore mock finitely generated, then \( v_i(P, M) < \infty \) for all \( i \).

**Proof.** By Proposition 2.5, minimal injective resolvents of \( M \) are preserved by localizations. Hence \( v_i(P, M) = v_i(PR_P, M_P) \). Therefore, the result follows from Proposition 3.3 and the remark after Theorem 2.8. \( \square \)

**Remarks.** We note that it follows from Bass' work in [1] that if \( M \) is mock finitely generated, then all the Bass invariants, \( \mu^i(P, M) \), are finite, and that if \( P, Q \in \text{Spec} \, R \) are such that \( P \subseteq Q \) with no primes in between, then \( \mu^i(P, M) \neq 0 \) implies \( \mu^{i+1}(Q, M) \neq 0 \). Using \( \text{Ext}^R_i(N, M) \) instead of \( \text{Ext}^k_i(N, M) \) and Theorem 3.4 above, we similarly get that if \( M \) is a reduced mock finitely generated Gorenstein injective \( R \)-module and \( P, Q \) are as above, then \( v_i(P, M) \neq 0 \) implies \( v_{i-1}(Q, M) \neq 0 \).

4. Gorenstein Isolated Singularities

Our aim in this section is to prove the following result, some parts of which are analogous to the well known results concerning finitely generated maximal Cohen–Macaulay modules over Cohen–Macaulay rings (see Yoshino [7]).

**Theorem 4.1.** If \( R \) is a Gorenstein local ring of dimension \( d \), then the following are equivalent.

1. \( (R, \mathcal{M}, k) \) is an isolated singularity.
2. Every Gorenstein injective \( R \)-module is locally injective on the punctured spectrum of \( R \).
(3) Every reduced mock finitely generated Gorenstein injective $R$-module is artinian.

(4) Every $n$th cosyzygy of a minimal injective resolution of a mock finitely generated $R$-module is artinian for $n \geq d + 1$.

(5) Every $n$th syzygy of a minimal injective resolvent of a reduced mock finitely generated $R$-module is artinian for $n \geq d + 1$.

(6) $\text{Ext}_R^k(N, M)$ (respectively $\text{Ext}_f^k(N, M)$) is of finite length for all cosyzygies $N$ of finitely generated $R$-modules and for all mock finitely generated Gorenstein injective $R$-modules $M$.

(7) $\text{Ext}_R^k(N, M)$ (respectively $\text{Ext}_f^k(N, M)$) is of finite length for all Gorenstein injective $R$-modules $N$ and $M$ that are cosyzygies of finitely generated $R$-modules.

Proof. (1) $\Rightarrow$ (2) is Corollary 2.6.

(2) $\Rightarrow$ (3) Let $M$ be a reduced mock finitely generated Gorenstein injective $R$-module. Then $M_P$ is injective for all $P \in \text{Spec } R - \{\mathcal{M}\}$. But $M_P$ is reduced since $M$ is by Proposition 2.5. So $M_P = 0$. Therefore $\text{Supp } M = \{\mathcal{M}\}$. Hence the injective envelope of $M$ is a direct sum of copies of $E(k)$. But $\mu_0(\mathcal{M}, M) < \infty$ since $M$ is mock finitely generated. So $M$ is artinian.

(3) $\Rightarrow$ (4) If $M$ is an $n$th cosyzygy of a minimal injective resolution of a mock finitely generated $R$-module where $n \geq d + 1$, then $M$ is a reduced Gorenstein injective $R$-module by Enochs-Jenda [4, Theorem 4.2] and is mock finitely generated by Lemma 2.9. Hence $M$ is artinian.

(4) $\Rightarrow$ (1) We consider the minimal injective resolution $0 \to R/P \to E^0 \to E^1 \to \cdots \to E^d \to C^{d+1} \to 0$ where $P \in \text{Spec } R - \{\mathcal{M}\}$. Then $R/P$ is mock finitely generated by Proposition 3.2 and therefore $C^{d+1}$ is artinian by assumption. So $E^{d+1}_p = E(k)^r$ for some $r$. Hence $E^{d+1}_p = 0$. Thus $\text{id}(R/P)_P \leq d$ and so $R_P$ is regular.

(2) $\Rightarrow$ (5) Let $0 \to M \to E^{-n+1} \to E^{-n+2} \to \cdots \to E^{-1} \to E^0 \to N \to 0$ be a partial minimal injective resolvent of a reduced mock finitely generated $R$-module $N$. If $n \geq d + 1$, then $M$ is Gorenstein injective by [4, Theorem 4.3] and so $M_P$ is injective for all $P \in \text{Spec } R - \{\mathcal{M}\}$ by definition. But $M$ is reduced since it is a kernel of an injective cover and so $M_P = 0$ for such primes $P$. Thus the injective envelope, $E(M)$, of $M$ is a direct sum of copies of $E(k)$ as above. But $E(M) \cong E^{-n+1}$ by Lemma 2.4 and $\nu_{n-1}(\mathcal{M}, N) < \infty$ by Proposition 3.3. Hence $M$ is artinian.

(5) $\Rightarrow$ (1) We again consider the minimal injective resolution $0 \to R/P \to E^0 \to E^1 \to \cdots \to E^d \to C^{d+1} \to 0$. If $C^{d+1} = 0$, then we are done. If $C^{d+1} \neq 0$, then $\text{id}(R/P)_P = \infty$ since $\text{id}(R/P) \leq d$ or is infinite. But every cosyzygy of $R/P$ is mock finitely generated by Lemma 2.9 and Proposition 3.2. Furthermore, every $n$th cosyzygy, $n \geq d + 1$, is reduced by Proposition 3.1 of [4]. So since $C^{d+1}$ is Gorenstein injective, we see that $C^{d+1}$ is a $(d + 1)$th syzygy of a minimal injective resolvent of a reduced mock finitely generated $R$-module and hence is artinian. Thus $R_P$ is regular as before.

(2) $\Rightarrow$ (6) $\text{Ext}_R^k(N, M)$ is finitely generated and $\text{Ext}_R^k(N, M)_P \cong \text{Ext}_R^k(N_P, M_P)$ for all $P \in \text{Spec } R$ by Theorem 2.11. So if $P \neq \mathcal{M}$, then $\text{Ext}_R^k(N, M)_P = 0$. Thus $P \notin \text{ Supp } \text{Ext}_R^k(N, M)$ for all $P \neq \mathcal{M}$. Hence $\text{Ext}_R^k(N, M)$ is of finite length.
Similarly, $\text{Ext}^R_p(N, M)$ is finitely generated and $\text{Ext}^R_p(N, M)_P = 0$ for all $P \in \text{Spec } R - \{ \mathcal{M} \}$ by Theorem 2.8. Hence $\text{Ext}^R_p(N, M)$ is of finite length.

(6) $\Rightarrow$ (7) We simply note that $M$ is mock finitely generated by Lemma 2.9 and Proposition 3.2.

(7) $\Rightarrow$ (1) We once more consider the minimal injective resolution $0 \rightarrow R/P \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^d \rightarrow C^{d+1} \rightarrow 0$. In the short injective resolution $0 \rightarrow C^d \rightarrow E^d \rightarrow C^{d+1} \rightarrow 0$, both $C^d$ and $C^{d+1}$ are Gorenstein injective modules and so $\text{Ext}^R_p(C^d, C^{d+1}) \cong \text{Ext}^R_p(C^{d+1}, C^d)_P$ by Theorem 2.11. But $\text{Ext}^R_p(C^{d+1}, C^d)_P = 0$ for all $P \in \text{Spec } R - \{ \mathcal{M} \}$ by assumption. Thus the sequence $0 \rightarrow C^d_P \rightarrow E^d_P \rightarrow C^{d+1}_P \rightarrow 0$ splits and hence $C^d_P$ is injective. Therefore, $P$ is regular.

Now by Lemma 2.10, $\text{Ext}^R_p(C^{d+1}, C^d) \cong \text{Ext}^R_p(C^{d+1}, C^{d+4})$. So $\text{Ext}^R_p(C^{d+1}, C^{d+4}) \cong \text{Ext}^R_p(C^{d+1}, C^{d+4})_P \cong \text{Ext}^R_p(C^{d+1}, C^{d+4})_P$. But $\text{Ext}^R_p(C^{d+1}, C^{d+4})_P = 0$ for all $P \neq \mathcal{M}$ by assumption. Hence $R_P$ is regular as above. □

References