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Infinitesimal deformations of restricted simple Lie algebras I

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ABSTRACT

We compute the infinitesimal deformations of two families of restricted simple modular Lie algebras of Cartan-type: the Witt–Jacobson and the Special Lie algebras.

1. Introduction

Simple Lie algebras over an algebraically closed field of characteristic zero were classified at the beginning of the XIX century by Killing and Cartan. They used the non-degeneracy of the Killing form to describe the simple Lie algebras in terms of root systems which are then classified by Dynkin diagrams.

This method breaks down in positive characteristic because the Killing form may degenerate. Indeed the classification problem remained open for a long time until it was recently solved, if the characteristic of the base field is greater than 3, by Block and Wilson (see [BW88]), Strade and Wilson (see [SW91]), Strade (see [STR89,STR92,STR91,STR93,STR94,STR98]) and Premet and Strade (see [PS97, PS99,PS01]). The classification remains still open in characteristic 2 and 3 (see [STR04, p. 209]).

According to this classification, *simple modular* (that is over a field of positive characteristic) Lie algebras are divided into two big families, called classical-type and Cartan-type algebras. The algebras of classical-type are obtained by the simple Lie algebras in characteristic zero by first taking a model over the integers (via Chevalley bases) and then reducing modulo p (see [SEL67]). The algebras of Cartan-type were constructed by Kostrikin and Shafarevich in 1966 (see [KS66]) as finite-dimensional analogues of the infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification of Lie pseudogroups, and are divided into four families, called Witt–Jacobson,

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Special, Hamiltonian and Contact algebras. The Witt–Jacobson Lie algebras are derivation algebras of truncated divided power algebras and the remaining three families are the subalgebras of derivations fixing a volume form, a Hamiltonian form and a contact form, respectively. Moreover in characteristic 5 there is one exceptional simple modular Lie algebra called the Melikian algebra (introduced in [MEL80]).

We are interested in a particular class of modular Lie algebras called *restricted*. These can be characterized as those modular Lie algebras such that the p-power of an inner derivation (which in characteristic p is a derivation) is still inner. Important examples of restricted Lie algebras are the ones coming from groups schemes. Indeed there is a one-to-one correspondence between restricted Lie algebras and finite group schemes whose Frobenius vanishes (see [DG70, Chapter 2]).

By standard facts of deformation theory, the *infinitesimal deformations* of a Lie algebra are parametrized by the second cohomology of the Lie algebra with values in the adjoint representation (see for example [GER64]).

It is a classical result (see [HS97]) that for a simple Lie algebra $\mathfrak{g}$ over a field of characteristic 0 it holds that $H^i(\mathfrak{g},\mathfrak{g})=0$ for every $i \geqslant 0$, which implies in particular that such Lie algebras are rigid. The proof of this fact relies on the non-degeneracy of the Killing form and the non-vanishing of the trace of the Casimir element, which is equal to the dimension of the Lie algebra. Therefore the same proof works also for the simple modular Lie algebras of classical type over a field of characteristic not dividing the determinant of the Killing form and the dimension of the Lie algebra. Actually Rudakov (see [RUD71]) showed that such Lie algebras are rigid if the characteristic of the base field is greater than or equal to 5 while in characteristic 2 and 3 there are non-rigid classical Lie algebras (see [CHE05,CK00,CKK00]).

The purpose of this article is to compute the infinitesimal deformations of the first two families of restricted simple Lie algebras of Cartan type: the Witt–Jacobson algebras $W(n)$ and the Special algebras $S(n)$. Unlike the classical-type simple algebras, it turns out that these two families are not rigid. More precisely we get the following two theorems (we refer to Sections 3.1 and 4.1 for the standard notations concerning $W(n)$ and $S(n)$ and to Section 2.3 for the definition of the squaring operators Sq).

Theorem 1.1. *Assume that the characteristic p of the base field F is different from 2. Then we have*

$$H^2\big(W(n),W(n)\big)=\bigoplus_{i=1}^{n} F\cdot\big\langle \mathrm{Sq}(D_i)\big\rangle$$

with the exception of the case $n=1$ and $p=3$ when it is 0.

Theorem 1.2. *Assume that the characteristic of the base field F is different from 2 and moreover it is different from 3 if $n=3$. Then we have*

$$H^2\big(S(n),S(n)\big)=\bigoplus_{i=1}^{n} F\cdot\big\langle \mathrm{Sq}(D_i)\big\rangle \bigoplus F\cdot\langle\Theta\rangle$$

where Θ is defined by $\Theta(D_i,D_j)=D_{ij}(x^{\tau})$ and extended by 0 outside $S(n)_{-1}\times S(n)_{-1}$.

In the two forthcoming papers [VIV2,VIV3], we compute the infinitesimal deformations of the remaining restricted simple Lie algebras of Cartan-type, namely the Hamiltonian, the Contact and the exceptional Melikian algebras. Moreover, in the forthcoming paper [VIV4], we apply these results to the study of the infinitesimal deformations of the simple finite group schemes corresponding to the simple restricted Lie algebras of Cartan type.

Let us mention that the infinitesimal deformations of simple Lie algebras of Cartan-type (in the general non-restricted case) have been considered already by Džumadildaev in [DZU80,DZU81, DZU89] and Džumadildaev and Kostrikin in [DK78] but a complete picture as well as detailed proofs were missing. More precisely: in [DK78] the authors compute the infinitesimal deformations of the Jacobson–Witt algebras of rank 1, in [DZU80, Theorem 4] the author describes the infinitesimal deformation of the Jacobson–Witt algebras of any rank but without a proof, in [DZU81] a general strategy for the Jacobson–Witt and Hamiltonian algebras is outlined (without proofs) and finally in [DZU89] the author clarifies this strategy and then applies it to the Jacobson–Witt algebras but with a half-page sketch of the proof.

Our approach works for all the restricted simple Lie algebras of Cartan-type and is different from the approach of Džumadildaev although we took from him the idea to consider relative cohomology with respect to the subalgebra of negative degree elements. As a byproduct of our proof, we recover the results of Celousov (see [CEL70]) on the first cohomology group of the adjoint representation (Theorems 3.3 and 4.5).

2. Some preliminaries results on the cohomology of Lie algebras

2.1. Review of general theory

In this subsection we review, in order to fix notations, the classical theory of cohomology of Lie algebras (see for example [HS53]).

If $\mathfrak{g}$ is a Lie algebra over a field F and M is a $\mathfrak{g}$-module, then the cohomology groups $H^*(\mathfrak{g}, M)$ can be computed from the complex of n-dimensional cochains $C^n(\mathfrak{g}, M)$ ($n \geqslant 0$), that are alternating n-linear functions $f : \Lambda^n(\mathfrak{g}) \to M$, with differential $d : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ defined by

$$\begin{aligned} \mathrm{d}f(\sigma_0, \ldots, \sigma_n) = & \sum_{i=0}^{n} (-1)^i \sigma_i \cdot f(\sigma_0, \ldots, \widehat{\sigma_i}, \ldots, \sigma_n) \\ & + \sum_{p<q} (-1)^{p+q} f\big([\sigma_p, \sigma_q], \sigma_0, \ldots, \widehat{\sigma_p}, \ldots, \widehat{\sigma_q}, \ldots, \sigma_n\big), \end{aligned} \tag{2.1}$$

where the sign $\widehat{\ }$ means that the argument below must be omitted. Given $f \in C^n(\mathfrak{g}, M)$ and $\gamma \in \mathfrak{g}$, we denote with f_γ the restriction of f to $\gamma \in \mathfrak{g}$, that is the element of $C^{n-1}(\mathfrak{g}, M)$ given by

$$f_\gamma(\sigma_0, \ldots, \sigma_{n-1}) := f(\gamma, \sigma_0, \ldots, \sigma_{n-1}).$$

With this notation, the above differential satisfies the following useful formula (for any $\gamma \in \mathfrak{g}$ and $f \in C^n(\mathfrak{g}, M)$):

$$\mathrm{d}(\gamma \cdot f) = \gamma \cdot (\mathrm{d}f), \tag{2.2}$$

$$(\mathrm{d}f)_\gamma = \gamma \cdot f - \mathrm{d}(f_\gamma), \tag{2.3}$$

where each $C^n(\mathfrak{g}, M)$ is a $\mathfrak{g}$-module by means of the action

$$(\gamma \cdot f)(\sigma_1, \ldots, \sigma_n) = \gamma \cdot f(\sigma_1, \ldots, \sigma_n) - \sum_{i=1}^{n} f\big(\sigma_1, \ldots, [\gamma, \sigma_i], \ldots, \sigma_n\big). \tag{2.4}$$

As usual we indicate with $Z^n(\mathfrak{g}, M)$ the subspace of n-cocycles and with $B^n(\mathfrak{g}, M)$ the subspace of n-coboundaries. Therefore $H^n(\mathfrak{g}, M) := Z^n(\mathfrak{g}, M)/\ B^n(\mathfrak{g}, M)$.

A useful tool to compute cohomology of Lie algebras is the following Hochschild–Serre spectral sequence relative to a subalgebra $\mathfrak{h} < \mathfrak{g}$:

$$E_1^{p,q} = H^q\big(\mathfrak{h}, C^p(\mathfrak{g}/\mathfrak{h}, M)\big) \quad \Rightarrow \quad H^{p+q}(\mathfrak{g}, M), \tag{2.5}$$

which in the case where $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ (which we indicate as $\mathfrak{h} \lhd \mathfrak{g}$) becomes

$$E_2^{p,q} = H^p\big(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)\big) \quad \Rightarrow \quad H^{p+q}(\mathfrak{g}, M). \tag{2.6}$$

Moreover for the second page of the first spectral sequence (2.5), we have the equality

$$E_2^{p,0} = H^p(\mathfrak{g}, \mathfrak{h}; M), \tag{2.7}$$

where $H^*(\mathfrak{g}, \mathfrak{h}; M)$ are the relative cohomology groups defined (by Chevalley and Eilenberg [CE48]) from the sub-complex $C^p(\mathfrak{g}, \mathfrak{h}; M) \subset C^p(\mathfrak{g}, M)$ consisting of cochains orthogonal to $\mathfrak{h}$, that is cochains satisfying the two conditions:

$$f_{|\mathfrak{h}} = 0, \tag{2.8}$$

$$\mathrm{d}f_{|\mathfrak{h}} = 0 \quad \text{or equivalently} \quad \gamma \cdot f = 0 \quad \text{for every } \gamma \in \mathfrak{h}. \tag{2.9}$$

Note that in the case where $\mathfrak{h} \lhd \mathfrak{g}$, the equality (2.7) is consistent with the second spectral sequence (2.6) because in that case we have $H^p(\mathfrak{g}, \mathfrak{h}, M) = H^p(\mathfrak{g}/\mathfrak{h}, M^{\mathfrak{h}})$.

2.2. Torus actions and gradings

The Lie algebras that we consider in this paper, namely the Witt–Jacobson Lie algebra $W(n)$ and the Special algebra $S(n)$, are graded algebras which admit a root space decomposition with respect to a maximal torus contained in the 0-graded piece. Under these hypothesis, the cohomology groups admit a very useful decomposition that we are going to review in this subsection.

Suppose that a torus T acts on both $\mathfrak{g}$ and M in a way that is compatible with the action of $\mathfrak{g}$ on M, which means that $t \cdot (g \cdot m) = (t \cdot g) \cdot m + t \cdot (g \cdot m)$ for every $t \in T$, $g \in \mathfrak{g}$ and $m \in M$. Then the action of T can be extended to the space of n-cochains by

$$(t \cdot f)(\sigma_1, \dots, \sigma_n) = t \cdot f(\sigma_1, \dots, \sigma_n) - \sum_{i=1}^{n} f(\sigma_1, \dots, t \cdot \sigma_i, \dots, \sigma_n).$$

It follows easily from the compatibility of the action of T and formula (2.3), that the action of T on the cochains commutes with the differential d. Therefore, since the action of a torus is always completely reducible, we get a decomposition in eigenspaces

$$H^n(\mathfrak{g}, M) = \bigoplus_{\phi \in \Phi} H^n(\mathfrak{g}, M)_\phi, \tag{2.10}$$

where $\Phi = \mathrm{Hom}_F(T, F)$ and $H^n(\mathfrak{g}, M)_\phi = \{[f] \in H^n(\mathfrak{g}, M) \mid t \cdot [f] = \phi(t)[f] \text{ if } t \in T\}$. A particular case of this situation occurs when $T \subset \mathfrak{g}$ and T acts on $\mathfrak{g}$ via the adjoint action and on M via restriction of the action of $\mathfrak{g}$. It is clear that this action is compatible and moreover the above decomposition reduces to

$$H^n(\mathfrak{g}, M) = H^n(\mathfrak{g}, M)_{\underline{0}},$$

where $\underline{0}$ is the trivial homomorphism (in this situation we say that the cohomology reduces to *homogeneous* cohomology). Indeed, if we consider an element $f \in Z^n(\mathfrak{g}, M)_\phi$, then by applying formula (2.3) with $\gamma = t \in T$ we get

$$0 = (\mathrm{d}f)_t = t \cdot f - \mathrm{d}(f_t) = \phi(t) f - \mathrm{d}(f_t),$$

from which we see that the existence of a $t \in T$ such that $\phi(t) \neq 0$ forces f to be a coboundary.

Now suppose that $\mathfrak{g}$ and M are graded and that the action of $\mathfrak{g}$ respects these gradings, which means that $\mathfrak{g}_d \cdot M_e \subset M_{d+e}$ for all $e, d \geqslant 0$. Then the space of cochains can also be graded: a homogeneous cochain f of degree d is a cochain such that $f(\mathfrak{g}_{e_1} \times \cdots \times \mathfrak{g}_{e_n}) \subset M_{\sum e_i + d}$. With this definition, the differential becomes of degree 0 and therefore we get a degree decomposition

$$H^n(\mathfrak{g}, M) = \bigoplus_{d \in \mathbb{Z}} H^n(\mathfrak{g}, M)_d. \tag{2.11}$$

Finally, if the action of T is compatible with the grading, in the sense that T acts via degree 0 operators both on $\mathfrak{g}$ and on M, then the above two decompositions (2.10) and (2.11) are compatible and give rise to the refined weight-degree decomposition

$$H^n(\mathfrak{g}, M) = \bigoplus_{\phi \in \Phi} \bigoplus_{d \in \mathbb{Z}} H^n(\mathfrak{g}, M)_{\phi, d}. \tag{2.12}$$

2.3. Squaring operation

There is a canonical way to produce 2-cocycles in $Z^2(\mathfrak{g}, \mathfrak{g})$ over a field of characteristic $p > 0$, namely the squaring operation (see [GER64]). Given a derivation $\gamma \in Z^1(\mathfrak{g}, \mathfrak{g})$ (inner or not), one defines the squaring of γ to be

$$\mathrm{Sq}(\gamma)(x, y) = \sum_{i=1}^{p-1} \frac{[\gamma^i(x), \gamma^{p-i}(y)]}{i!(p-i)!} \in Z^2(\mathfrak{g}, \mathfrak{g}), \tag{2.13}$$

where γ^i is the ith iteration of γ. In [GER64] it is shown that $[\mathrm{Sq}(\gamma)] \in H^2(\mathfrak{g}, \mathfrak{g})$ is an obstruction to integrability of the derivation γ, that is to the possibility of finding an automorphism of $\mathfrak{g}$ extending the infinitesimal automorphism given by γ.

3. The Witt–Jacobson algebra

3.1. Definition and basic properties

We first introduce some useful notations. Inside the set $\mathbb{Z}^n$ of n-tuples of integers, we consider the order relation defined by $a = (a_1, \ldots, a_n) < b = (b_1, \ldots, b_n)$ if $a_i < b_i$ for every $i = 1, \ldots, n$. We call degree of $a \in \mathbb{Z}^n$ the number $|a| = \sum_{i=1}^n a_i$. For every integer $0 \leqslant l < p$, we define $\underline{l} := (l, \ldots, l)$ and we set $\tau := \underline{p-1}$ (this n-tuple will appear often in what follows and hence it deserves a special notation). Moreover, for every $j \in \{1, \ldots, n\}$ we call ϵ_j the n-tuple having 1 at the jth place and 0 otherwise.

Let $A(n) = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ be the ring of p-truncated polynomial in n variables over a field F of positive characteristic $p > 0$. Note that $A(n)$ is a finite F-algebra of dimension p^n with a basis given by the elements $\{x^a := x_1^{a_1} \cdots x_n^{a_n} \mid a \in \mathbb{Z}^n,\ \underline{0} \leqslant a \leqslant \tau\}$. Moreover it has a natural graduation $A(n) = \bigoplus_{i=0}^{n(p-1)} A(n)_i$, obtained by assigning to the monomial x^a the degree $|a|$.

Definition 3.1. The Witt–Jacobson algebra $W(n)$ is the restricted Lie algebra $\mathrm{Der}_F A(n)$ of derivations of $A(n) = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.

For every $j \in \{1, \ldots, n\}$, we put $D_j := \frac{\partial}{\partial x_j}$. The Witt–Jacobson algebra $W(n)$ is a free $A(n)$-module with basis $\{D_1, \ldots, D_n\}$. Hence $\dim_F(W(n)) = np^n$ with a basis over F given by $\{x^a D_j \mid 1 \leqslant j \leqslant n,\ \underline{0} \leqslant a \leqslant \tau\}$.

Moreover $W(n)$ is a graded Lie algebra with the $\mathbb{Z}$-gradation defined by $W(n)_i := \sum_{j=1}^n A(n)_{i+1} D_j$ where $i = -1, \ldots, n(p-1) - 1$. Note that the unique summand of negative degree is $W(n)_{-1} =$

$\bigoplus_{i=1}^n F \cdot \langle D_i \rangle$ while the summand of degree 0 is $W(n)_0 = \bigoplus_{1 \leqslant i,j \leqslant n} F \cdot \langle x_i D_j \rangle$ and its adjoint action on $W(n)_{-1}$ induces an isomorphism $W(n)_0 \cong \mathfrak{gl}(n, F)$.

The algebra $W(n)$ is simple unless $p = 2$ and $n = 1$ (see [FS88, Chapter 4, Theorem 2.4]) and it admits a root space decomposition with respect to a canonical Cartan subalgebra.

Proposition 3.2. *For each $i \in \{1, \dots, n\}$, let $h_i = x_i D_i$.*

(a) *$T := \sum_{i=1}^n F h_i$ is a maximal torus of $W(n)$ (called the canonical maximal torus).*
(b) *The centralizer of T inside $W(n)$ is T itself, which is hence a Cartan subalgebra of $W(n)$.*
(c) *Let $\Phi := \mathrm{Hom}_{\mathbb{F}_p}(\bigoplus_{i=1}^n \mathbb{F}_p \cdot h_i, \mathbb{F}_p)$, where $\mathbb{F}_p$ is the prime field of F. In the Cartan decomposition $W(n) = \bigoplus_{\phi \in \Phi} W(n)_\phi$, every direct summand $W(n)_\phi$ has dimension n. Moreover $x^a D_i \in W(n)_{a-\epsilon_i}$, where $a - \epsilon_i$ is viewed as an element of Φ by reduction modulo p.*

Proof. See [FS88, Chapter 4, Theorem 2.5]. □

3.2. Strategy of the proof of the Main Theorem

In this subsection we outline the strategy of the proof of Theorem 1.1 from the Introduction. In particular, from now on, we assume that the base field F has characteristic $p \geqslant 3$. Note that in the exceptional case $n = 1$ and $p = 3$, one has the isomorphism $W(1) \cong \mathfrak{sl}_2$ and hence we recover the known vanishing result for the simple algebras of classical-type.

We first observe that the 2-cocycles $\mathrm{Sq}(D_i)$ appearing in Theorem 1.1 are independent modulo coboundaries unless $n = 1$ and $p = 3$, in which case it is easily seen that $\mathrm{Sq}(D_1) = 0$. Indeed, on one hand, for every $g \in C^1(W(n), W(n))$ and $1 \leqslant r, s \leqslant n$, the following element

$$\mathrm{d}g\big(x_r^2 D_s, x_r^{p-2} x_s D_s\big) = \big[x_r^2 D_s, g\big(x_r^{p-2} x_s D_s\big)\big] - \big[x_r^{p-2} x_s D_s, g\big(x_r^2 D_s\big)\big]$$

cannot contain terms of negative degree. On the other hand, we get that

$$\mathrm{Sq}(D_i)\big(x_r^2 D_s, x_r^{p-2} x_s D_s\big) = \begin{cases} D_s & \text{if } i = r \neq s, \\ -3D_i & \text{if } i = r = s, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

which shows the independence of the $\mathrm{Sq}(D_i)$ modulo coboundaries, using the first case if $n \geqslant 2$ and the second if $p \geqslant 5$.

The proof that these 2-cocycles generate the whole second cohomology group is divided into three steps.

Step I. We prove that we can reduce to relative cohomology (see Section 2.1) with respect to the subalgebra $W(n)_{-1}$ of negative terms:

$$H^2\big(W(n), W(n)\big) = H^2\big(W(n), W(n)_{-1}; W(n)\big).$$

This is achieved by first observing that the second cohomology groups reduces to homogeneous cohomology with respect to the maximal torus $T < W(n)$ (see Section 2.2) and then by considering the homogeneous Hochschild–Serre spectral sequence associated to the subalgebra $W(n)_{-1} < W(n)$ (see (2.5)):

$$\big(E_1^{r,s}\big)_{\underline{0}} = H^s\big(W(n)_{-1}, C^r\big(W(n)/W(n)_{-1}, W(n)\big)\big)_{\underline{0}} \quad \Rightarrow \quad H^{r+s}\big(W(n), W(n)\big)_{\underline{0}}. \tag{3.2}$$

We prove that $(E_1^{0,1})_{\underline{0}} = (E_1^{0,2})_{\underline{0}} = 0$ (Corollary 3.5) and $(E_2^{1,1})_{\underline{0}} = 0$ (Proposition 3.6) which gives the conclusion by (2.7).

Step II. Using orthogonality with respect to $W(n)_{-1}$ (see (2.8) and (2.9)), we prove in Proposition 3.7 that

$$H^2\big(W(n), W(n)_{-1}; W(n)\big) = H^2\big(W(n)_{\geqslant 0}, W(n)_{-1}\big)$$

where $W(n)_{\geqslant 0}$ acts on $W(n)_{-1}$ by the projection onto $W(n)_{\geqslant 0}/W(n)_{\geqslant 1} = W(n)_0$ followed by the adjoint representation of $W(n)_0 = \mathfrak{gl}(n, F)$ on $W(n)_{-1}$.

Then, by using the Hochschild–Serre spectral sequence with respect to the ideal $W(n)_{\geqslant 1} \lhd W(n)_{\geqslant 0}$ (see (2.6)), we prove in Proposition 3.8 that

$$H^2\big(W(n)_{\geqslant 0}, W(n)_{-1}\big) = H^2\big(W(n)_{\geqslant 1}, W(n)_{-1}\big)^{W(n)_0}$$

where $W(n)_{-1}$ is considered as a trivial $W(n)_{\geqslant 1}$-module.

Step III. We compute the invariant second cohomology group $H^2(W(n)_{\geqslant 1}, W(n)_{-1})^{W(n)_0}$ showing that (unless $p = 3$ and $n = 1$) it is generated by the projection onto $W(n)_{-1}$ of the cocycles $\mathrm{Sq}(D_i)$ (Proposition 3.10). The idea of the proof is to approximate this cohomology group by the truncated cohomology groups

$$H^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, W(n)_{-1}\right)^{W(n)_0}$$

which for large d are equal to our cohomology group. The computation proceeds by induction on d using the Hochschild–Serre spectral sequence with respect to the ideal

$$\frac{W(n)_{\geqslant d}}{W(n)_{\geqslant d+1}} \lhd \frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d+1}}.$$

In the course of the proof of the Main Theorem, we obtain a new proof of the following result.

Theorem 3.3 *(Celousov). $H^1(W(n), W(n)) = 0$.*

Proof. The proof follows the same steps as in the proof of the Main Theorem. The spectral sequence (3.2), in view of the Corollary 3.5 and formula (2.7), gives that

$$H^1\big(W(n), W(n)\big) = H^1\big(W(n), W(n)_{-1}; W(n)\big).$$

Then the required vanishing follows from Propositions 3.7 and 3.8. □

3.3. *Reduction to $W(n)_{-1}$-relative cohomology*

This subsection is devoted to the first step of the proof (see Section 3.2), namely the reduction to the relative cohomology with respect to the subalgebra $W(n)_{-1} < W(n)$. First of all we want to prove the vanishing of the homogeneous cohomology groups $H^s(W(n)_{-1}, W(n))_{\underline{0}}$ appearing in the first column of the spectral sequence (3.2). For that purpose, we need the following proposition, in which the action of $W(n)_{-1}$ on $A(n)$ is the natural one.

Proposition 3.4. *For every $i = 1, \ldots, n$, we denote with $x_i^{p-1} D_i^*$ the linear function from $W(n)_{-1}$ to $A(n)$ which sends D_i to x_i^{p-1} and D_j to 0 for $j \neq i$. Then we have $H^s(W(n)_{-1}, A(n)) = \bigwedge^s \bigoplus_{i=1}^n F \cdot \langle x_i^{p-1} D_i^* \rangle$.*

Proof. Clearly the cochains appearing in the statement are cocycles and they are independent modulo coboundaries since it follows easily, from formula (2.1), that if $g \in C^{s-1}(W(n)_{-1}, A(n))$ then $\mathrm{d}g(D_{i_1}, \dots, D_{i_s}) \in A(n)$ cannot contain the monomial $x_{i_1}^{p-1} \cdots x_{i_s}^{p-1}$.

In order to prove that the above cocycles generate the whole cohomology group, we proceed by double induction on s and n, the case $s = 0$ being true since $A(n)^{W(n)_{-1}} = F \cdot 1$. We view $A(n-1)$ inside $A(n)$ as the subalgebra of polynomials in the variables $x_2, \dots, x_n$ and $W(n-1)_{-1}$ inside $W(n)_{-1}$ as the subalgebra generated by $D_2, \dots, D_n$. Thus the action of $W(n)$ on $A(n)$ restricts to the natural action of $W(n-1)$ on $A(n-1)$.

Consider $f \in Z^s(W(n)_{-1}, A(n))$. By adding a coboundary $\mathrm{d}g$ and using formula (2.3) for $\mathrm{d}g$ and $\gamma = D_1$, we can suppose that

$$f_{|D_1} : W(n-1)_{-1}^{s-1} \to x_1^{p-1} A(n-1).$$

Moreover, since f is a cocycle, the same formula (2.3) gives

$$0 = (\mathrm{d}f)_{D_1} = \big[D_1, f(-)\big] - \mathrm{d}(f_{|D_1}).$$

Now observe that, by the condition above, $\mathrm{d}(f_{|D_1})$ takes values in $x_1^{p-1}A(n-1)$ while obviously $[D_1, f(-)]$ cannot contain monomials with the x_1 erased to the $(p-1)$th power. Hence it follows that

$$\begin{cases} [D_1, f_{|W(n-1)_{-1}^s}] = 0, \\ \mathrm{d}(f_{|D_1}) = 0. \end{cases}$$

The first equation says that $f_{|W(n-1)_{-1}^s}$ takes values in $A(n-1)$ and hence belongs to $Z^s(W(n-1)_{-1}, A(n-1))$. The second equation says that $f_{|D_1} \in Z^{s-1}(W(n-1)_{-1}, x_1^{p-1}A(n-1)) = Z^{s-1}(W(n-1)_{-1}, A(n-1)) \otimes \langle x_1^{p-1}\rangle$. In both cases, by induction, we get that $f \in B^s(W(n)_{-1}, A(n)) + \bigwedge^s \bigoplus_{i=1}^n F \cdot \langle x_i^{p-1} D_i^*\rangle$ and this concludes the proof. □

Corollary 3.5. *We have* $H^s(W(n)_{-1}, W(n)) \cong H^s(W(n)_{-1}, A(n)) \otimes W(n)_{-1}$. *Therefore* $(E_1^{0,s})_{\underline{0}} = H^s(W(n)_{-1}, W(n))_{\underline{0}} = 0$ *for every* $s \geqslant 0$.

Proof. The first claim follows from the $W(n)_{-1}$-decomposition $W(n) = A(n) \otimes W(n)_{-1}$ and the fact that $W(n)_{-1}$ is an abelian Lie algebra. The second claim follows from the first and the fact that $H^s(W(n)_{-1}, A(n)) = H^s(W(n)_{-1}, A(n))_{\underline{0}}$ (by Proposition 3.4) while $(W(n)_{-1})_{\underline{0}} = 0$. □

Now we deal with the term in position $(1,1)$ of the above spectral sequence. We prove that it vanishes starting from the second level.

Proposition 3.6. *In the spectral sequence* (3.2), *we have that* $(E_2^{1,1})_{\underline{0}} = 0$.

Proof. We have to show the injectivity of the level 1 differential map

$$d : \big(E_1^{1,1}\big)_{\underline{0}} \to \big(E_1^{2,1}\big)_{\underline{0}}.$$

In the course of this proof, we adopt the following convention: given an element $f \in C^1(W(n)_{-1}, C^s(W(n)/W(n)_{-1}, W(n)))$, we write its value on $D_i \in W(n)_{-1}$ as $f_{D_i} \in C^s(W(n)/W(n)_{-1}, W(n))$.

We want to show, by induction on the degree of $E \in W(n)/W(n)_{-1}$, that if $[df] = 0 \in H^1(W(n)_{-1}, C^2(W(n)/W(n)_{-1}, W(n)))$ then we can choose a representative $\tilde{f}$ of $[f] \in H^1(W(n)_{-1},$

$C^1(W(n)/W(n)_{-1}, W(n)))$ such that $\tilde{f}_{D_i}(E) = 0$ for every $i = 1, \ldots, n$. So suppose that we have already found a representative f such that $f_{D_i}(F) = 0$ for every $F \in W(n)/W(n)_{-1}$ of degree less than d and for every i. First of all, we can find a representative $\tilde{f}$ of $[f]$ such that

$$\tilde{f}_{D_i}(E) \in \langle x_i^{p-1} \rangle \otimes W(n)_{-1} \tag{$*$}$$

for every i and for every $E \in W(n)$ of degree d. Indeed, by the induction hypothesis, the cocycle condition for f is $\partial f_{D_i,D_j}(E) = [D_i, f_{D_j}(E)] - [D_j, f_{D_i}(E)]$. On the other hand, by choosing an element $h \in C^1(W(n)/\ W(n)_{-1}, W(n))$ that vanishes on the elements of degree less than d, we can add to f (without changing its cohomological class neither affecting the inductive assumption) the coboundary ∂h whose value on E is $\partial h_{D_i}(E) = [D_i, h(E)]$. Hence, for a fixed element E of degree d, the map $D_i \mapsto f_{D_i}(E)$ gives rise to an element of $H^1(W(n)_{-1}, W(n))$ and, by Proposition 3.4, we can chose an element $h(E)$ as above such that the new cochain $\tilde{f} = f + \partial h$ verifies the condition $(*)$ as above.

Note that, by the homogeneity of our cocycles, the functions $\tilde{f}_{D_i}$ can assume non-zero values only on the elements E of weights $-\epsilon_k$, for a certain k, which are the form $E = x_k^{p-1} x_h D_h$ for some $k \neq h$ (note that we have already done in the case $n = 1$). Hence, from now on, we can assume that $d = p - 1 \geqslant 2$ and pay attention only to the elements of the above form.

Now we are going to use the condition that $[d\tilde{f}] = 0 \in (E_1^{2,1})_{\underline{0}}$, that is $d\tilde{f} = \partial g$ for some $g \in C^2(W(n)/W(n)_{-1}, W(n))_{\underline{0}}$. Explicitly, for $A, B \in W(n)/W(n)_{-1}$ we have that

$$\partial g_{D_i}(A, B) = \left[D_i, g(A, B)\right] - g\big([D_i, A], B\big) - g\big(A, [D_i, B]\big), \tag{3.3}$$

$$\begin{aligned} d\tilde{f}_{D_i}(A, B) = {} & \tilde{f}_{D_i}\big([A, B]\big) - \left[A, \tilde{f}_{D_i}(B)\right] + \left[B, \tilde{f}_{D_i}(A)\right] \\ & - \delta_{\deg(A),0} \tilde{f}_{[D_i,A]}(B) + \delta_{\deg(B),0} \tilde{f}_{[D_i,B]}(A), \end{aligned} \tag{3.4}$$

where the last two terms in the second formula are non-zero only if $\deg(A) = 0$ and $\deg(B) = 0$ respectively. We apply the above formulas for the elements $A = x_k^{p-2} x_h^2 D_h$ and $B = x_k D_h$. Taking into account the inductive hypothesis on the degree and the homogeneity assumptions, formula (3.4) becomes

$$d\tilde{f}_{D_i}\big(x_k^{p-2} x_h^2 D_h, x_k D_h\big) = -2\tilde{f}_{D_i}\big(x_k^{p-1} x_h D_h\big) = \alpha x_i^{p-1} D_k$$

for a certain $\alpha \in F$, while formula (3.3) gives

$$\partial g_{D_i}\big(x_k^{p-2} x_h^2 D_h, x_k D_h\big) = \left[D_i, g\big(x_k^{p-2} x_h^2 D_h, x_k D_h\big)\right] - g\big(\left[D_i, x_k^{p-2} x_h^2 D_h\right], x_k D_h\big).$$

Observe that if $\deg(B) = 0$ and $\deg(A) < p - 1$, then $\deg_{x_i}(g(A, B)) \leqslant \deg_{x_i}(A)$ (where $\deg_{x_i}(-)$ indicate the largest power of x_i which appears in the argument). Indeed, by the inductive hypothesis, formula (3.4) gives that $d\tilde{f}_{D_i}(A, B) = 0$ and hence the conclusion follows by repeatedly applying formula (3.3): $0 = \partial g_{D_i}(A, B) = [D_i, g(A, B)] - g([D_i, A], B)$.

From this observation, it follows that $g([D_i, x_k^{p-2} x_h^2 D_h], x_k D_h)$ cannot contain a monomial of the form $x_i^{p-1} D_k$ and hence neither can the element $\partial g_{D_i}(x_k^{p-2} x_h^2 D_h, x_k D_h)$, since in the above formula the first element is a derivation with respect to D_i. Therefore by imposing $d\tilde{f}_{D_i} = \partial g_{D_i}$, we obtain that $\tilde{f}_{D_i}(x_k^{p-1} x_h D_h) = 0$ which completes the inductive step. $\square$

3.4. *Reduction to $W(n)_0$-invariant cohomology*

This subsection is devoted to prove the second step of the strategy that was outlined in Section 3.2. We consider the action of $W(n)_{\geqslant 0}$ on $W(n)_{-1}$ obtained by the projection onto $W(n)_0 = W(n)_{\geqslant 0}/W(n)_{\geqslant 1}$ followed by the adjoint representation of $W(n)_0 = \mathfrak{gl}(n, F)$ on $W(n)_{-1}$.

Proposition 3.7. *For every $s \in \mathbb{Z}_{\geqslant 0}$, we have*

$$H^s\big(W(n), W(n)_{-1}; W(n)\big) = H^s\big(W(n)_{\geqslant 0}, W(n)_{-1}\big).$$

Proof. For every $s \in \mathbb{Z}_{\geqslant 0}$, consider the map

$$\phi_s : C^s\big(W(n), W(n)_{-1}; W(n)\big) \to C^s\big(W(n)_{\geqslant 0}, W(n)_{-1}\big)$$

induced by the restriction to the subalgebra $W(n)_{\geqslant 0} \subset W(n)$ and by the projection $W(n) \twoheadrightarrow W(n)/W(n)_{\geqslant 0} = W(n)_{-1}$. It is straightforward to check that the maps ϕ_s commute with the differentials and hence they define a map of complexes. Moreover the orthogonality conditions with respect to the subalgebra $W(n)_{-1}$ give the injectivity of the maps ϕ_s. Indeed, on one hand, the condition (2.8) says that an element $f \in C^s(W(n), W(n)_{-1}; W(n))$ is determined by its restriction to $\bigwedge^s W(n)_{\geqslant 0}$. On the other hand, condition (2.9) implies that the values of f on an s-tuple are determined, up to elements of $W(n)^{W(n)_{-1}} = W(n)_{-1}$, by induction on the total degree of the s-tuple.

Therefore, to conclude the proof, it is enough to prove that the maps ϕ_s are surjective. Explicitly, if $f \in C^s(W(n)_{\geqslant 0}, W(n)_{-1})$, consider the cochain $\tilde{f} \in C^s(W(n), W(n))$ defined by

$$\tilde{f}\big(x^{a^1}, \dots, x^{a^n}\big) = \sum_{i=1}^{n} \sum_{\underline{0} \leqslant b^i < a^i} \prod_{i=1}^{n} \binom{a^i}{b^i} f\big(x^{a^1-b^1}, \dots, x^{a^n-b^n}\big) x^{b^1+\dots+b^n},$$

where if $a, b \in \mathbb{N}^n$ then $\binom{a}{b} := \prod_{i=1}^n \binom{a_i}{b_i}$.

We are done if we show that $\tilde{f} \in C^s(W(n), W(n)_{-1}; W(n))$ since it is clear that $\phi_s(\tilde{f}) = f$. The first orthogonality condition (2.8) follows easily from the definition. Consider the following expression

$$\begin{aligned}\tilde{f}\big(x^{a^1}, \dots, D_j\big(x^{a^k}\big), \dots, x^{a^n}\big) &= \sum_{\underline{0} \leqslant b'^k < a^k - \epsilon_j} \sum_{\substack{i \neq k \\ \underline{0} \leqslant b^i < a^i}} \left[\big(a^k\big)_j \binom{a^k - \epsilon_j}{b'^k} \prod_{i \neq k} \binom{a^i}{b^i}\right] \\ &\quad \times f\big(x^{a^1-b^1}, \dots, x^{a^k-\epsilon_j-b^k}, \dots, x^{a^n-b^n}\big) x^{b^1+\dots+b'^k+\dots+b^n} \\ &= \sum_{i=1}^{n} \sum_{\underline{0} \leqslant b^i < a^i} \big(b^k\big)_j \prod_{i=1}^{n} \binom{a^i}{b^i} f\big(x^{a^1-b^1}, \dots, x^{a^n-b^n}\big) x^{b^1+\dots+b^n-\epsilon_j},\end{aligned}$$

where we used the substitution $b^k = b'^k + \epsilon_j$ together with the equality $(b^k)_j \binom{a^k}{b^k} = (a^k)_j \binom{a^k - \epsilon_j}{b^k - \epsilon_j}$. Summing the above expression as k varies from 1 to n, we get $[D_j, \tilde{f}(x^{a^1}, \dots, x^{a^n})]$ which proves the second orthogonality condition (2.9). $\square$

Proposition 3.8. *Consider $W(n)_{-1}$ as a trivial $W(n)_{\geqslant 1}$-module. Then*

$$\begin{cases} H^1\big(W(n)_{\geqslant 0}, W(n)_{-1}\big) = 0, \\ H^2\big(W(n)_{\geqslant 0}, W(n)_{-1}\big) = H^2\big(W(n)_{\geqslant 1}, W(n)_{-1}\big)^{W(n)_0}. \end{cases}$$

Proof. Consider the Hochschild–Serre spectral sequence relative to the ideal $W(n)_{\geqslant 1} \lhd W(n)_{\geqslant 0}$:

$$E_2^{r,s} = H^r\big(W(n)_0, H^s\big(W(n)_{\geqslant 1}, W(n)_{-1}\big)\big) \quad \Rightarrow \quad H^{r+s}\big(W(n)_{\geqslant 0}, W(n)_{-1}\big).$$

Note that since $T \subset W(n)_{\geqslant 0}$, we can restrict to homogeneous cohomology (see Section 2.2). Directly from homogeneity, it follows that the first line $E_2^{*,0} = H^*(W(n)_0, W(n)_{-1})$ vanishes. Indeed the weights that occur in $W(n)_{-1}$ are $-\epsilon_i$ while the weights that occur in $W(n)_0$ are $\underline{0}$ and $\epsilon_i - \epsilon_j$. Therefore the weights that occur in $W(n)_0^{\otimes k}$ have degree congruent to 0 modulo p and hence they cannot be equal to $-\epsilon_i$.

On the other hand, since $W(n)_{-1}$ is a trivial $W(n)_{\geqslant 1}$-module, we have that

$$H^1\big(W(n)_{\geqslant 1}, W(n)_{-1}\big) = \big\{f : W(n)_{\geqslant 1} \to W(n)_{-1} \bigm| f\big(\big[W(n)_{\geqslant 1}, W(n)_{\geqslant 1}\big]\big) = 0\big\}.$$

Therefore Lemma 3.9 gives that

$$H^1\big(W(n)_{\geqslant 1}, W(n)_{-1}\big) = \begin{cases} C^1(W(1)_1 \oplus W(1)_2, W(n)_{-1}) & \text{if } n = 1 \text{ and } p \geqslant 5, \\ C^1(W(n)_1, W(n)_{-1}) & \text{if } n \geqslant 2 \text{ or } n = 1 \text{ and } p = 3. \end{cases}$$

From this it follows that the second line $E_2^{*,1} = H^*(W(n)_{\geqslant 0}, H^1(W(n)_{\geqslant 1}, W(n)_{-1}))$ vanishes again for homogeneity reasons. Indeed, on one hand, the weights that appear on $H^1(W(n)_{\geqslant 1}, W(n)_{-1})$ have degree congruent to 2 or 3 modulo p (the last one can occur only for $n = 1$ and $p \geqslant 5$). On the other hand the weights that appear on $W(n)_0$ (that are $\underline{0}$ or $\epsilon_i - \epsilon_j$) are congruent to 0 modulo p and the same is true for $W(n)^{\otimes k}$. $\square$

Lemma 3.9. *Let $d \geqslant -1$ be an integer and suppose that it is different from 1 if $n = 1$. Then*

$$\big[W(n)_1, W(n)_d\big] = W(n)_{d+1}.$$

Proof. Clearly $[W(n)_1, W(n)_d] \subset W(n)_{d+1}$ by definition of graded algebras. Consider formulas

$$\big[x_i^2 D_i, x^b D_r\big] = \begin{cases} b_i x^{b+\epsilon_i} D_r & \text{if } i \neq r, \\ (b_r - 2)x^{b+\epsilon_r} D_r & \text{if } i = r. \end{cases}$$

Take an element $x^a D_r \in W(n)_{d+1}$. If $a_r \neq 0, 3$ the second formula above with $i = r$ and $b = a - \epsilon_r$ shows that $x^a D_r \in [W(n)_1, W(n)_d]$. On the other hand, if there exists some $i \neq r$ such that $a_i \neq 0, 1$ then the first formula above with $b = a - \epsilon_i$ gives that $x^a D_r \in [W(n)_1, W(n)_d]$. Moreover if there is an index $s \neq r$ such that $a_s = 1$, then we use the formula

$$\big[x_s^2 D_r, x^{a-\epsilon_s} D_s\big] = a_r x^{a-\epsilon_r+\epsilon_s} D_s - 2x^a D_r$$

since the first term on the right-hand side belongs to $[W(n)_1, W(n)_d]$ by what proved above. Therefore, in virtue of our hypothesis on d, it remains to consider the elements $x_r^3 D_r$ for $n \geqslant 2$. Choosing an $s \neq r$ we conclude by

$$\big[x_r^2 D_s, x_r x_s D_r\big] = x_r^3 D_r - 2x_r^2 x_s D_s. \qquad \square$$

3.5. *Computation of $W(n)_0$-invariant cohomology*

The aim of this subsection is to prove the following proposition that concludes the third and last step of the proof.

Proposition 3.10. *Denote with $\overline{\mathrm{Sq}(D_i)}$ the projection of $\mathrm{Sq}(D_i)$ onto $W(n)_{-1}$. Then*

$$H^2\big(W(n)_{\geqslant 1}, W(n)_{-1}\big)^{W(n)_0} = \bigoplus_{i=1}^{n} F \cdot \big\langle \overline{\mathrm{Sq}(D_i)} \big\rangle,$$

with the exception of the case $n = 1$ and $p = 3$ when it is 0.

Proof. First of all observe that if $n=1$ and $p=3$, then $W(n)_{\geqslant 1} = \langle x_1^2 D_1\rangle$ and hence the second cohomology group vanishes. Hence we assume that $p \geqslant 5$ if $n=1$. It is easy to see that the above cocycles $\overline{\mathrm{Sq}(D_i)}$ are $W(n)_0$-invariant and independent modulo coboundaries (same argument as in Section 3.2). So we have to prove that they generate the second cohomology group.

Consider the truncated cohomology groups

$$H^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, W(n)_{-1}\right)^{W(n)_0}$$

as d increases. Observe that if $d \geqslant np-(n+1)$ then $W(n)_{\geqslant d+1}=0$ and hence we get the cohomology we are interested in. Moreover if $n \geqslant 2$ then Lemma 3.12 below gives

$$H^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant 2}}, W(n)_{-1}\right)^{W(n)_0} = C^2\big(W(n)_1, W(n)_{-1}\big)^{W(n)_0} = 0,$$

while if $n=1$ (and $p \geqslant 5$) then by homogeneity we have that

$$H^2\left(\frac{W(1)_{\geqslant 1}}{W(1)_{\geqslant 3}}, W(1)_{-1}\right)^{W(1)_0} = C^1\big(W(1)_1 \times W(1)_2, W(1)_{-1}\big)_{\underline{0}} = 0.$$

The algebra $W(n)_{\geqslant 1}$ has a decreasing filtration $\{W(n)_{\geqslant d}\}_{d=1,\dots,n(p-1)-1}$ and the adjoint action of $W(n)_0$ respects this filtration. We consider one step of this filtration

$$W(n)_d = \frac{W(n)_{\geqslant d}}{W(n)_{\geqslant d+1}} \lhd \frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d+1}}$$

and the related Hochschild–Serre spectral sequence

$$E_2^{r,s} = H^r\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, H^s\big(W(n)_d, W(n)_{-1}\big)\right) \quad \Rightarrow \quad H^{r+s}\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d+1}}, W(n)_{-1}\right). \tag{3.5}$$

We fix a certain degree d and we study, via the above spectral sequence, how the truncated cohomology groups change if we pass from d to $d+1$. By what was said above, we can assume that $d>1$ if $n \geqslant 2$ and $d>2$ if $n=1$.

Observe that, since $W(n)_d$ is in the center of $W(n)_{\geqslant 1}/W(n)_{\geqslant d+1}$ and $W(n)_{-1}$ is a trivial module, then $H^s(W(n)_d, W(n)_{-1}) = C^s(W(n)_d, W(n)_{-1})$ and $W(n)_{\geqslant 1}/W(n)_{\geqslant d}$ acts trivially on it. Since $E_\infty^{0,2}=0$ by Lemma 3.11 below, the above spectral sequence gives us the two following exact sequences

$$\begin{tikzcd}
C^1(W(n)_d, W(n)_{-1}) \arrow[d, hook, "\alpha"'] & & \\
H^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, W(n)_{-1}\right) \arrow[d, two heads] & & \\
E_\infty^{2,0} \arrow[r, hook] & H^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d+1}}, W(n)_{-1}\right) \arrow[r, two heads] & E_\infty^{1,1}
\end{tikzcd}$$

where the injectivity of the map α follows from the exactness of the sequence

$$E_\infty^{1,0} = H^1\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, W(n)_{-1}\right) \hookrightarrow H^1\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d+1}}, W(n)_{-1}\right) \twoheadrightarrow E_\infty^{0,1} = \mathrm{Ker}(\alpha)$$

together with Lemma 3.9 which says that the first two terms are both equal to $C^1(W(n)_1, W(n)_{-1})$. Moreover, Lemma 3.9 gives that

$$E_\infty^{1,1} \subset E_2^{1,1} = \begin{cases} C^1(W(n)_1 \times W(n)_d, W(n)_{-1}) & \text{if } n \geqslant 2, \\ C^1([W(1)_1 \oplus W(1)_2] \times W(1)_d, W(1)_{-1}) & \text{if } n = 1. \end{cases} \tag{3.6}$$

By taking cohomology with respect to $W(n)_0$ and using Lemmas 3.12, 3.13, 3.14, 3.15 below, we see that the only terms responsible for the growth of the invariant truncated cohomology groups are $H^1(W(n)_0, C^1(W(n)_d, W(n)_{-1}))$ if $n \geqslant 2$ and $d = p-1$ (see Lemma 3.15) and $(E_\infty^{1,1})^{W(n)_0}$ if $n = 1$ and $d = p-2$ (see Lemma 3.13). In both cases, we get the desired statement. $\square$

Lemma 3.11. *In the above spectral sequence* (3.5), *we have* $E_3^{0,2} = 0$.

Proof. By definition, $E_3^{0,2}$ is the kernel of the map

$$\mathrm{d} : C^2\big(W(n)_d, W(n)_{-1}\big) = E_2^{0,2} \to E_2^{2,1} = H^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, C^1\big(W(n)_d, W(n)_{-1}\big)\right)$$

that sends a 2-cochain f to the element $\mathrm{d}f$ given by $\mathrm{d}f_{(E,F)}(G) = -f([E,F],G)$ whenever $\deg(E) + \deg(F) = d$ and 0 otherwise.

The subspace of coboundaries $B^2(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, C^1(W(n)_d, W(n)_{-1}))$ is the image of the map

$$\partial : C^1\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, C^1\big(W(n)_d, W(n)_{-1}\big)\right) \to C^2\left(\frac{W(n)_{\geqslant 1}}{W(n)_{\geqslant d}}, C^1\big(W(n)_d, W(n)_{-1}\big)\right)$$

that sends the element g to the element ∂g given by $\partial g_{(E,F)}(G) = -g_{[E,F]}(G)$. Hence ∂g vanishes on the pairs (E,F) for which $\deg(E) + \deg(F) = d$.

Therefore, if an element $f \in C^2(W(n)_d, W(n)_{-1})$ is in the kernel of d, that is $\mathrm{d}f = \partial g$ for some g as before, then it should satisfy $f([E,F],G) = 0$ for every E, F, G such that $\deg(G) = d$ and $\deg(E) + \deg(F) = d$. By letting E vary in $W(n)_1$ and F in $W(n)_{d-1}$, the bracket $[E,F]$ varies in all $W(n)_d$ by Lemma 3.9 (note that we are assuming $d \geqslant 3$ if $n = 1$). Hence the preceding condition implies that $f = 0$. $\square$

Lemma 3.12. *If* $n \geqslant 2$ *and* $d \geqslant 1$, *then*

$$C^1\big(W(n)_1 \times W(n)_d, W(n)_{-1}\big)^{W(n)_0} = 0.$$

Proof. Note that invariance with respect to $T \subset W(n)_0$ is the same as homogeneity, hence we can limit ourselves to considering homogeneous cochains. In particular this implies the vanishing if $d \not\equiv p-2 \mod p$.

Consider a homogeneous cochain $f \in C^1(W(n)_1 \times W(n)_d, W(n)_{-1})^{W(n)_0}$. Since the action of $W(n)_0$ on $W(n)_1$ is transitive, the result will follow if we prove that $f(x_1^2 D_2, -) = 0$. Indeed, assuming this is the case, imposing invariance respect to an element $x_i D_j \in W(n)_0$, we get

$$\begin{aligned} 0 &= (x_i D_j \circ f)\big(x_1^2 D_2, -\big) = -f\big(\big[x_i D_j, x_1^2 D_2\big], -\big) - f\big(x_1^2 D_2, [x_i D_j, -]\big) + \big[x_i D_j, f\big(x_1^2 D_2, -\big)\big] \\ &= -f\big(\big[x_i D_j, x_1^2 D_2\big], -\big), \end{aligned}$$

which shows the vanishing for f when restricted to $[x_i D_j, x_1^2 D_2]$. Continuing in this way one gets the vanishing of f on every element of $W(n)_1$ and hence the vanishing of f. So it is enough to prove that for every element $x^a D_r \in W(n)_d$ one has $f(x_1^2 D_2, x^a D_r) = 0$.

Suppose that $p \geqslant 5$. Then by the homogeneity assumption on f, we have the required vanishing as soon as $a_1 = 0$ or $a_2 = p-1$ (because $p \geqslant 5$!). If $a_1 \geqslant 1$ and $a_2 < p-1$, we proceed by induction on a_1. Suppose that we have proved the vanishing for all the elements $x^b D_s$ such that $b_1 < a_1$. Then, using the induction hypothesis, the following invariance condition

$$0 = (x_1 D_2 \circ f)\big(x_1^2 D_2, x^{a-\epsilon_1+\epsilon_2} D_r\big) = -f\big(x_1^2 D_2, (a_2+1) x^a D_r\big)$$

gives the required vanishing.

Finally, in the case $p = 3$, we can apply the same inductive argument, provided that we first prove the vanishing in the case when $a_1 = 0$ or $a_2 = p-1 = 2$. This vanishing is provided by the homogeneity of f unless $x^a D_r$ is equal to $x_2^2 D_2$, $x_2 x_j D_j$ or $x_1 x_2^2 x_j^2 D_2$ (with $3 \leqslant j \leqslant n$). In this three exceptional cases one proves the vanishing using the following invariance conditions:

$$\begin{cases} 0 = (x_1 D_2 \circ f)\big(x_1 x_2 D_2, x_2^2 D_2\big) = -f\big(x_1^2 D_2, x_2^2 D_2\big) - f(x_1 x_2 D_2, 2 x_1 x_2 D_2), \\ 0 = (x_j D_2 \circ f)\big(x_1^2 D_2, x_2^2 D_j\big) = -f\big(x_1^2 D_2, 2 x_2 x_j D_j - x_2^2 D_2\big), \\ 0 = (x_j D_2 \circ f)\big(x_1^2 D_2, x_1 x_2^2 x_j^2 D_2\big) = \big[x_j D_2, f\big(x_1^2 D_2, x_1 x_2^2 x_j^2 D_2\big)\big]. \end{cases} \qquad \square$$

Lemma 3.13. *Consider the above spectral sequence* (3.5)*. If $n = 1$ then*

$$\big(E_3^{1,1}\big)_{\underline{0}} = \big(E_\infty^{1,1}\big)_{\underline{0}} = \begin{cases} \langle \overline{\mathrm{Sq}(D_1)} \rangle & \textit{if } d = p-2, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\overline{\mathrm{Sq}(D_1)}$ denotes the restriction of $\mathrm{Sq}(D_1)$ to $W(1)_1 \times W(1)_{p-2}$.

Proof. For $n = 1$ we have that $T = W(1)$ and therefore the $W(n)_0$-invariance is the same as homogeneity. By formula (3.6) and homogeneity, we get

$$\big(E_2^{1,1}\big)_{\underline{0}} = \begin{cases} \langle x_1^3 D_1 \times x_1^{p-2} D_1 \to D_1 \rangle & \text{if } d = p-3, \\ \langle x_1^2 D_1 \times x_1^{p-1} D_1 \to D_1 \rangle & \text{if } d = p-2, \\ 0 & \text{otherwise.} \end{cases}$$

The term $(E_\infty^{1,1})_{\underline{0}} = (E_3^{1,1})_{\underline{0}}$ is the kernel of the differential map $\mathrm{d} : (E_2^{1,1})_{\underline{0}} \to (E_2^{3,0})_{\underline{0}} = H^3(\frac{W(1)_{\geqslant 1}}{W(1)_{\geqslant d}}, W(1)_{-1})_{\underline{0}}$. In view of the explicit description of $(E_2^{1,1})_{\underline{0}}$ as above, it is enough to show that the map d is different from 0 if $d = p-3$, since if $d = p-2$ then the cocycle $\overline{\mathrm{Sq}(D_1)}$ belongs to $(E_\infty^{1,1})_{\underline{0}}$ and is different from 0 because $\mathrm{Sq}(D_1)(x_1^2 D_1, x_1^{p-1} D_1) = -3 D_1$ ($\neq 0$ for $p \geqslant 5$!).

So let $d = p-3$ (and hence $p \geqslant 7$) and suppose that $\mathrm{d}\langle x_1^3 D_1 \times x_1^{p-2} D_1 \to D_1 \rangle = \partial g$ for $g \in C^2(\frac{W(1)_{\geqslant 1}}{W(1)_{\geqslant p-3}}, W(1)_{-1})_{\underline{0}}$. If $p = 7$ then $g = 0$ for homogeneity reasons. Otherwise (if $p > 7$) then note that the cocycle $\mathrm{d}\langle x_1^3 D_1 \times x_1^{p-2} D_1 \to D_1 \rangle$ vanishes on the triples $(x_1^2 D_1, x_1^{j+1} D_1, x_1^{p-1-j} D_1)$ for $3 \leqslant j \leqslant (p-3)/2$ and hence we get the following conditions on g:

$$\begin{aligned} 0 &= \partial g\big(x_1^2 D_1, x_1^{j+1} D_1, x_1^{p-1-j} D_1\big) \\ &= -(j-1) g\big(x_1^{j+2} D_1, x_1^{p-1-j} D_1\big) + (p-3-j) g\big(x_1^{p-j} D_1, x_1^{j+1} D_1\big) \end{aligned}$$

from which, by decreasing induction on j, we deduce that $g(x_1^{j+1}D_1, x_1^{p-j}D_1) = 0$ and hence that $g = 0$. But this is absurd since

$$\mathrm{d}\langle x_1^3 D_1 \times x_1^{p-2} D_1 \to D_1\rangle\big(x_1^2 D_1, x_1^3 D_1, x_1^{p-3} D_1\big) = -(p-5)D_1 \neq 0. \qquad \square$$

Lemma 3.14. *Let $d \in \mathbb{Z}_{\geqslant 0}$. Then we have $C^1(W(n)_d, W(n)_{-1})^{W(n)_0} = 0$.*

Proof. Observe that $C^1(W(n)_d, W(n)_{-1})^{W(n)_0} \subset C^1(W(n)_d, W(n)_{-1})_{\underline{0}}$ and the last term is non-vanishing only if $d = p-1$ and $n \geqslant 2$, in which case we have the homogeneous cochains $g(x_i^{p-1}x_j D_j) = a_j^i D_i$, $a_j^i \in F$ (for $i \neq j$). We get the vanishing of g by means of the following cocycle condition

$$0 = \mathrm{d}g_{x_i D_j}\big(x_i^{p-2}x_j^2 D_j\big) = -2g\big(x_i^{p-1}x_j D_j\big) = -2a_j^i D_i. \qquad \square \tag{3.7}$$

Lemma 3.15. *Let $d \in \mathbb{Z}_{\geqslant 0}$. Then*

$$H^1\big(W(n)_0, C^1\big(W(n)_d, W(n)_{-1}\big)\big) = \begin{cases} \bigoplus_{i=1}^n \langle \overline{\mathrm{Sq}(D_i)}\rangle & \text{if } n \geqslant 2 \text{ and } d = p-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{Sq}(D_i)}$ denotes the restriction of $\mathrm{Sq}(D_i)$ to $W(n)_0 \times W(n)_{p-1}$.

Proof. Observe that, since the maximal torus T is contained in $W(n)_0$, the cohomology with respect to $W(n)_0$ reduces to homogeneous cohomology. Hence the required group can be non-zero only if $d \equiv p-1 \mod p$ (and hence only if $n \geqslant 2$). More precisely, since the weights appearing on $W(n)_{-1}$ are $-\epsilon_k$ and the weights appearing on $W(n)_0$ are $\epsilon_i - \epsilon_j$ (possibly with $i = j$), the weights appearing on $W(n)_d$ can be $-\epsilon_i + \epsilon_j - \epsilon_k$ (for every $1 \leqslant i, j, k \leqslant n$). Hence the required group can be non-zero only if $d = p-1$ or $d = 2p-1$ (this last case only if $n \geqslant 3$).

Consider first the case $d = 2p-1$ ($n \geqslant 3$). A homogeneous cochain $f \in C^1(W(n)_0, C^1(W(n)_{2p-1}, W(n)_{-1}))_{\underline{0}}$ takes the following non-zero values

$$f_{x_i D_j}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big) = \alpha_{ijk}^h D_k$$

for every i, j, k mutually distinct and $h \neq i, k$. From the vanishing of $\mathrm{d}f$, we get

$$\begin{aligned} 0 &= \mathrm{d}f_{(x_i D_j, x_k D_i)}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big) \\ &= -\big[x_k D_i, f_{x_i D_j}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big)\big] + f_{x_k D_j}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big) = \alpha_{ijk}^h D_i + \alpha_{kji}^h D_i, \\ 0 &= \mathrm{d}f_{(x_i D_j x_k D_j)}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big) \\ &= \big[x_i D_j, f_{x_k D_j}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big)\big] - \big[x_k D_j, f_{x_i D_j}\big(x_i^{p-1}x_k^{p-1}x_j x_h D_h\big)\big] = -\alpha_{kji}^h D_j + \alpha_{ijk}^h D_j. \end{aligned}$$

Adding these two equations, it follows that $2\alpha_{ijk}^h = 0$ and hence $f = 0$.

Consider now the case $d = p-1$. First of all, a homogeneous cocycle f must satisfy $f_{x_i D_i} = 0$. Indeed, by formula (2.3), we have $0 = \mathrm{d}f_{|x_i D_i} = x_i D_i \circ f - \mathrm{d}(f_{|x_i D_i})$ from which, since the first term vanishes for homogeneity reasons, it follows that $f_{|x_i D_i} \in C^1(W(n)_{p-1}, W(n)_{-1})^{W(n)_0}$ which is zero by Lemma 3.14. Therefore a homogeneous cocycle can take the following non-zero values (for i, j, k mutually distinct):

$$\begin{cases} f_{x_iD_j}\big(x_i^{p-2}x_j^2D_j\big)=\alpha_{ij}D_i,\\ f_{x_iD_j}\big(x_i^{p-1}x_kD_k\big)=\alpha_{ij}^kD_j,\\ f_{x_iD_j}\big(x_i^{p-2}x_jx_kD_k\big)=\beta_{ij}^kD_i,\\ f_{x_iD_j}\big(x_k^{p-1}x_jD_i\big)=\gamma_{ij}^kD_k,\\ f_{x_iD_j}\big(x_i^{p-1}x_jD_k\big)=\delta_{ij}^kD_k,\\ f_{x_iD_j}\big(x_i^{p-1}x_jD_i\big)=\beta_{ij}D_i,\\ f_{x_iD_j}\big(x_i^{p-1}x_jD_j\big)=\gamma_{ij}D_j. \end{cases}$$

By possibly modifying f with a coboundary (see formula (3.7)), we can assume that $\underline{\alpha_{i,j}=0}$. Using this, we get the vanishing of $\underline{\alpha_{ij}^k}$, $\underline{\beta_{ij}^k}$ and $\underline{\gamma_{ij}^k}$ by means of the following three cocycle conditions:

$$\begin{aligned} 0&=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(x_i^{p-2}x_k^2D_k\big)=\big[x_iD_j,f_{x_iD_k}\big(x_i^{p-2}x_k^2D_k\big)\big]+f_{x_iD_j}\big(2x_i^{p-1}x_kD_k\big)=\big[-\alpha_{ik}+2\alpha_{ij}^k\big]D_j,\\ 0&=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(x_i^{p-3}x_jx_k^2D_k\big)=-f_{x_iD_k}\big(x_i^{p-2}x_k^2D_k\big)+f_{x_iD_j}\big(2x_i^{p-2}x_jx_kD_k\big)=\big[-\alpha_{ik}+2\beta_{ij}^k\big]D_i,\\ 0&=\mathrm{d}f_{(x_iD_j,x_kD_j)}\big(x_k^{p-2}x_j^2D_i\big)=-f_{x_kD_j}\big(\big[x_iD_j,x_k^{p-2}x_j^2D_i\big]\big)+f_{x_iD_j}\big(\big[x_kD_j,x_k^{p-2}x_j^2D_i\big]\big)\\ &=-f_{x_kD_j}\big(2x_k^{p-2}x_jx_iD_i\big)+f_{x_kD_j}\big(x_k^{p-2}x_j^2D_j\big)+f_{x_iD_j}\big(2x_k^{p-1}x_jD_i\big)=\big[-2\beta_{kj}^i+\alpha_{kj}+2\gamma_{ij}^k\big]D_k. \end{aligned}$$

The coefficients $\underline{\delta_{ij}^k}$ and $\underline{\beta_{ij}}$ are determined by the coefficients γ_{ij} by the following two cocycle conditions:

$$\begin{aligned} 0&=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(x_i^{p-2}x_jx_kD_k\big)\\ &=-f_{x_iD_k}\big(x_i^{p-1}x_kD_k\big)+f_{x_iD_j}\big(x_i^{p-1}x_jD_k\big)-\big[x_iD_k,f_{x_iD_j}\big(x_i^{p-2}x_jx_kD_k\big)\big]\\ &=\big[-\gamma_{ik}+\delta_{ij}^k+\beta_{ij}^k\big]D_k=\big[-\gamma_{ik}+\delta_{ij}^k\big]D_k, \end{aligned} \tag{$*$}$$

$$\begin{aligned} 0&=\mathrm{d}f_{(x_iD_j,x_jD_i)}\big(x_i^{p-1}x_jD_j\big)\\ &=f_{x_iD_j}\big(-x_i^{p-2}x_j^2D_j\big)+f_{x_iD_j}\big(-x_i^{p-1}x_jD_i\big)-\big[x_jD_i,f_{x_iD_j}\big(x_i^{p-1}x_jD_j\big)\big]\\ &=[-\alpha_{ij}-\beta_{ij}+\gamma_{ij}]D_i=[-\beta_{ij}+\gamma_{ij}]D_i. \end{aligned} \tag{$**$}$$

The coefficients $\underline{\gamma_{ij}}$ satisfy the relation $\gamma_{ij}=\gamma_{ik}$ (for i, j, k mutually distinct as before). Indeed from the cocycle condition

$$\begin{aligned} 0&=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(x_i^{p-1}x_jD_i\big)\\ &=-f_{x_iD_k}\big(x_i^{p-1}x_jD_j\big)-\big[x_iD_k,f_{x_iD_j}\big(x_i^{p-1}x_jD_i\big)\big]+f_{x_iD_j}\big(-x_i^{p-1}x_jD_k\big)\\ &=\big[-\alpha_{ik}^j+\beta_{ij}-\delta_{ij}^k\big]D_k=\big[\beta_{ij}-\delta_{ij}^k\big]D_k, \end{aligned}$$

and using the relations $(*)$ and $(**)$ as above, we get $\gamma_{ij}=\beta_{ij}=\delta_{ij}^k=\gamma_{ik}:=\gamma_i$.

We conclude the proof by observing that the elements $\overline{\mathrm{Sq}(D_i)}$ are independent modulo coboundaries (if $n \geqslant 2$) as it follows from

$$\mathrm{Sq}(D_i)\big(x_i D_j, x_i^{p-1} x_j D_j\big) = \frac{1}{(p-1)!}\big[D_i(x_i D_j), (D_i)^{p-1}\big(x_i^{p-1} x_j D_j\big)\big] = D_j. \qquad \square$$

4. The special algebra

4.1. Definition and basic properties

Throughout this section, we use the notations introduced in Section 3.1 and we fix an integer $n \geqslant 3$. Consider the following map, called divergence:

$$\mathrm{div}: \begin{cases} W(n) \to A(n), \\ \displaystyle\sum_{i=1}^n f_i D_i \mapsto \sum_{i=1}^n D_i(f_i). \end{cases}$$

Clearly it is a linear map of degree 0 that satisfies the following formula (see [FS88, Chapter 4, Lemma 3.1]):

$$\mathrm{div}\big([D, E]\big) = D\big(\mathrm{div}(E)\big) - E\big(\mathrm{div}(D)\big).$$

Therefore the space $S'(n) := \{E \in W(n) \mid \mathrm{div}(E) = 0\}$ is a graded subalgebra of $W(n)$ and we have an exact sequence of $S'(n)$-modules

$$0 \to S'(n) \to W(n) \xrightarrow{\mathrm{div}} A(n)_{<\tau} \to 0. \tag{4.1}$$

Definition 4.1. The Special algebra is the derived algebra of $S'(n)$:

$$S(n) := S'(n)^{(1)} = \big[S'(n), S'(n)\big].$$

In order to describe the structure of $S(n)$, we introduce the following maps (for $1 \leqslant i, j \leqslant n$)

$$D_{ij}: \begin{cases} A(n) \to W(n), \\ f \mapsto D_j(f) D_i - D_i(f) D_j. \end{cases}$$

Note that $D_{ij}(A(n)) \subset S'(n)$ and moreover if $\sum_{i=1}^n f_i D_i$ and $\sum_{j=1}^n g_j D_j$ are two elements of $S'(n)$ then we have the following formula

$$\left[\sum_{i=1}^n f_i D_i, \sum_{j=1}^n g_j D_j\right] = -\sum_{1 \leqslant i, j \leqslant n} D_{ij}(f_i g_j), \tag{4.2}$$

which in particular gives the following special case

$$\big[D_{ij}(f), D_{ij}(g)\big] = D_{ij}\big(D_{ij}(f)(g)\big). \tag{4.3}$$

Theorem 4.2. *The algebra $S(n)$ satisfies the following properties*:

(i) *$S(n)$ is generated by the elements $D_{ij}(f)$ for $f \in A(n)$ and $1 \leqslant i < j \leqslant n$.*

(ii) *We have the following exact sequence of $S(n)$-modules*

$$0 \to S(n) \to S'(n) \to \bigoplus_{i=1}^{n} F \cdot \langle x^{\tau-(p-1)\epsilon_i} D_i \rangle \to 0, \tag{4.4}$$

where the last term is a trivial $S(n)$-module.

(iii) *$S(n)$ is a restricted simple graded Lie algebra of dimension $(n-1)(p^n-1)$.*

Proof. See [FS88, Chapter 4, Proposition 3.3, Theorems 3.5 and 3.7]. □

Note that the unique term of negative degree is $S(n)_{-1} = \bigoplus_{i=1}^{n} F \cdot \langle D_i \rangle$ while the term of degree 0 is $S(n)_0 = \bigoplus_{i=2}^{n} F \cdot \langle x_i D_i - x_1 D_1 \rangle \bigoplus_{1 \leqslant j \neq k \leqslant n} F \cdot \langle x_j D_k \rangle$ and its adjoint action on $S(n)_{-1}$ induces an isomorphism $S(n)_0 \cong \mathfrak{sl}(n, F)$.

The algebra $S(n)$ admits a root space decomposition with respect to a canonical Cartan subalgebra.

Proposition 4.3. *Recall that $h_i := x_i D_i$ for every $i \in \{1, \ldots, n\}$.*

(a) *$T_S := T \cap S(n) = \bigoplus_{i=2}^{n} F \cdot \langle h_i - h_1 \rangle$ is a maximal torus of $H(n)$ (called the canonical maximal torus).*

(b) *The centralizer of T_S inside $S(n)$ is the subalgebra*

$$C_S = \bigoplus_{\substack{2 \leqslant j \leqslant n \\ 0 \leqslant a \leqslant p-2}} F \cdot \langle D_{1j}(x^{\underline{a}+\epsilon_1+\epsilon_j}) \rangle$$

which is hence a Cartan subalgebra (called the canonical Cartan subalgebra). The dimension of C_H is $(n-1)(p-1)$.

(c) *Let $\Phi_S := \mathrm{Hom}_{\mathbb{F}_p}(\bigoplus_{i=2}^{n} \mathbb{F}_p \langle h_i - h_1 \rangle, \mathbb{F}_p)$, where $\mathbb{F}_p$ is the prime field of F. In the Cartan decomposition $S(n) = C_S \bigoplus_{\phi \in \Phi_S - \underline{0}} S(n)_\phi$, the dimension of every $S(n)_\phi$, with $\phi \in \Phi_S - \underline{0}$, is $(n-1)p$.*

Proof. See [FS88, Chapter 4, Theorem 3.6]. □

4.2. Strategy of the proof of the Main Theorem

In this subsection, we outline the strategy of the proof of Theorem 1.2 from the Introduction. Hence, from now on, we assume that the characteristic p of the base field F is different from 2.

We first check that Θ is a cocycle. It is enough to verify that it is a cocycle when restricted to $S(n)_{-1}$ and that it is $S(n)_0$-invariant:

$$\begin{aligned} \mathrm{d}\Theta(D_i, D_j, D_k) &= [D_i, D_{jk}(x^\tau)] - [D_j, D_{ik}(x^\tau)] + [D_k, D_{ij}(x^\tau)] \\ &= -D_{jk}(x^{\tau-\epsilon_i}) + D_{ik}(x^{\tau-\epsilon_j}) - D_{ij}(x^{\tau-\epsilon_k}) = 0 \end{aligned}$$

and (for $h \neq k$)

$$\begin{aligned} (x_h D_k \circ \Theta)(D_i, D_j) &= [x_h D_k, D_{ij}(x^\tau)] + \delta_{ih}\Theta(D_k, D_j) + \delta_{jh}\Theta(D_i, D_k) \\ &= \delta_{hj} D_{ki}(x^\tau) - \delta_{hi} D_{kj}(x^\tau) + \delta_{ih} D_{kj}(x^\tau) + \delta_{jh} D_{ik}(x^\tau) = 0. \end{aligned}$$

Moreover the cocycles Θ and $\mathrm{Sq}(D_i)$ appearing in Theorem 1.2 are independent modulo coboundaries. Indeed, if $\gamma \in \{\mathrm{Sq}(D_1), \ldots, \mathrm{Sq}(D_n), \Theta\}$ then we have (for $i \neq j$)

$$\gamma(D_i, D_j) = \begin{cases} D_{ij}(x^\tau) & \text{if } \gamma = \Theta, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$
$$\gamma\big(x_i D_j, D_{ji}\big(x_i^{p-1}x_j^2\big)\big) = \begin{cases} -2D_i & \text{if } \gamma = \mathrm{Sq}(D_i), \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

while for every $g \in C^1(S(n), S(n))$ the coboundary $\mathrm{d}g(D_i, D_j) = [D_i, g(D_j)] - [D_j, g(D_i)]$ cannot contain the monomial $D_{ij}(x^\tau)$ for degree reasons and $\mathrm{d}g(x_i D_j, D_{ji}(x_i^{p-1}x_j^2)) = [x_i D_j, g(D_{ji}(x_i^{p-1}x_j^2))] - [D_{ji}(x_i^{p-1}x_j^2), g(x_i D_j)]$ cannot contain the monomial D_i.

Assuming the results of the next subsection, we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. From the sequence (4.1), using Proposition 4.6, we get the exact sequence

$$0 \to H^1\big(S(n), A(n)_{<\tau}\big) \xrightarrow{\partial} H^2\big(S(n), S'(n)\big) \to H^2\big(S(n), W(n)\big).$$

By Proposition 4.8, we known that $H^2(S(n), W(n))$ is generated by the cocycles $\mathrm{Sq}(D_i)$. These clearly belong to $H^2(S(n), S'(n))$ and hence the above exact sequence splits

$$H^2\big(S(n), S'(n)\big) = \bigoplus_{i=1}^{n} \big\langle \mathrm{Sq}(D_i)\big\rangle \bigoplus \partial H^1\big(S(n), A(n)_{<\tau}\big).$$

On the other hand, from the sequence (4.4), we get the exact sequence

$$0 \to H^2\big(S(n), S(n)\big) \to H^2\big(S(n), S'(n)\big) \to \bigoplus_{i=1}^{n} H^2\big(S(n), x^{\tau-(p-1)\epsilon_i} D_i\big),$$

where we used that $H^1(S(n), M) = 0$ for a trivial $S(n)$-module M. Since the cocycles $\mathrm{Sq}(D_i)$ belong to $H^2(S(n), S(n))$, we are left with verifying which of the elements of $\partial H^1(S(n), A(n)_{<\tau})$ (which we know by Proposition 4.4) belong to $H^2(S(n), S(n))$.

Consider first the cocycle $\mathrm{ad}(x^\tau) : D_i \mapsto D_i(x^\tau) = -x^{\tau-\epsilon_i}$. It lifts to the cocycle $\widetilde{\mathrm{ad}}(x^\tau) \in C^1(S(n), W(n))$ given by $\widetilde{\mathrm{ad}}(x^\tau) : D_i \mapsto x^\tau D_i$ and 0 on the other elements. Therefore the only non-zero values of $\partial(\widetilde{\mathrm{ad}}(x^\tau))$ can be (for $k \neq h$):

$$\begin{aligned} \partial\big(\widetilde{\mathrm{ad}}\big(x^\tau\big)\big)(D_i, D_j) &= \big[D_i, x^\tau D_j\big] - \big[D_j, x^\tau D_i\big] = -D_{ij}\big(x^\tau\big), \\ \partial\big(\widetilde{\mathrm{ad}}\big(x^\tau\big)\big)(D_i, x_k D_h) &= -\big[x_k D_h, x^\tau D_i\big] - \widetilde{\mathrm{ad}}\big(x^\tau\big)\big([D_i, x_k D_h]\big) \\ &= \delta_{ik} x^\tau D_h - \phi(\delta_{ik} D_h) = 0 \end{aligned}$$

and hence we have that $\partial(\mathrm{ad}(x^\tau)) = -\Theta$.

Consider now the element $\chi_i \in H^1(S(n), A(n)_{<\tau})$ and choose a lifting $\widetilde{\chi}_i \in C^1(S(n), W(n))$ in such a way that if $\chi_i(\gamma) = 0$ then $\widetilde{\chi}_i(\gamma) = 0$. Then (if $j \neq i$), we have

$$\partial(\chi_i)\big(D_j, x^{\tau-(p-1)(\epsilon_i+\epsilon_j)} D_i\big) = \big[D_j, \widetilde{\chi}_i\big(x^{\tau-(p-1)(\epsilon_j+\epsilon_i)}\big)\big] = x^{\tau-(p-1)\epsilon_j} D_j,$$

because the only possible lifting to $W(n)$ of the element $\chi_i(x^{\tau-(p-1)(\epsilon_i+\epsilon_j)} D_i) = x^{\tau-(p-1)\epsilon_j}$ is $x^{\tau-(p-2)\epsilon_j} D_j$. On the other hand, for every cochain $g \in C^1(S(n), x^{\tau-(p-1)\epsilon_j})$ we have $\mathrm{d}g(D_j, x^{\tau-(p-1)\epsilon_i-(p-1)\epsilon_j} D_i) = 0$ because the module is trivial and $[D_j, x^{\tau-(p-1)\epsilon_i-(p-1)\epsilon_j} D_i] = 0$. Hence the projection of $\partial(\chi_i)$ into $H^2(S(n), x^{\tau-(p-1)\epsilon_j})$ is non-zero and therefore $\partial(\chi_i) \notin H^2(S(n), S(n))$. $\square$

Proposition 4.4. *Consider the natural action of $S(n)$ on $A(n)_{<\tau}$. We have*

$$H^1\big(S(n), A(n)_{<\tau}\big) = \bigoplus_{i=1}^{n} \langle \chi_i \rangle \oplus \big\langle \mathrm{ad}\big(x^\tau\big)\big\rangle,$$

where the $\chi_i \in H^1(S(n), A(n)_{<\tau})$ are defined by

$$\chi_i\big(x^a D_k\big) = \begin{cases} x^a \cdot x_i^{p-1} & \text{if } k = i, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. First of all note that χ_i takes values in $A(n)_{<\tau}$ (and not merely on $A(n)$) since $x^{\tau-(p-1)\epsilon_i} D_i \notin S(n)$. To prove that χ_i are cocycles, it is enough to verify the following two cocycle conditions (where j, h, k are different from i)

$$\begin{aligned} \mathrm{d}(\chi_i)\big(D_{ij}\big(x^a\big), D_{hk}\big(x^b\big)\big) &= -D_{hk}\big(x^b\big)\big(D_j\big(x^a\big)x_i^{p-1}\big) - \chi_i\big(\big[D_{ij}\big(x^a\big), D_{hk}\big(x^b\big)\big]\big) \\ &= -D_{hk}\big(x^b\big)\big(D_j\big(x^a\big)x_i^{p-1}\big) + \chi_i\big(D_{hk}\big(x^b\big)\big(D_j\big(x^a\big)\big)D_i\big) = 0, \\ \mathrm{d}(\chi_i)\big(D_{ij}\big(x^a\big), D_{ih}\big(x^b\big)\big) &= D_{ij}\big(x^a\big)\big(D_h\big(x^b\big)x_i^{p-1}\big) - D_{ih}\big(x^b\big)\big(D_j\big(x^a\big)x_i^{p-1}\big) \\ &\quad - \chi_i\big(D_{ij}\big(x^a\big)\big(D_h\big(x^b\big)\big)D_i - D_{ih}\big(x^b\big)\big(D_j\big(x^a\big)\big)D_i\big) \\ &= D_h\big(x^b\big)D_{ij}\big(x^a\big)\big(x_i^{p-1}\big) - D_j\big(x^a\big)D_{ih}\big(x^b\big)\big(x_i^{p-1}\big) = 0. \end{aligned}$$

The independence of the above cocycles γ_i and $\mathrm{ad}(x^\tau)$ modulo coboundaries follows from the fact that if $\gamma \in \{\chi_1, \ldots, \chi_n, \mathrm{ad}(x^{(\tau)})\}$ then

$$\gamma(D_i) = \begin{cases} x_i^{p-1} & \text{if } \gamma = \chi_i, \\ D_i(x^\tau) = -x^{\tau-\epsilon_i} & \text{if } \gamma = \mathrm{ad}(x^\tau), \\ 0 & \text{otherwise}, \end{cases}$$

while for any $g \in A(n)_{<\tau}$ the element $\mathrm{d}g(D_i) = D_i(g)$ cannot the monomials x_i^{p-1} or $x^{\tau-\epsilon_i}$.

In order to prove that the whole cohomology group is generated by the above cocycles, we consider the exact sequence of $S(n)$-modules

$$0 \to A(n)_{<\tau} \to A(n) \to \big\langle x^\tau \big\rangle \to 0,$$

where $\langle x^\tau \rangle$ is a trivial $S(n)$-module. By taking cohomology and using the fact that $H^1(S(n), x^\tau) = 0$, we obtain

$$H^1\big(S(n), A(n)_{<\tau}\big) = \big\langle \mathrm{ad}\big(x^\tau\big)\big\rangle \oplus H^1\big(S(n), A(n)\big).$$

Finally, to compute the last cohomology group we use the Hochschild–Serre spectral sequence with respect to the subalgebra $S(n)_{-1} < S(n)$:

$$E_1^{r,s} = H^s\big(S(n)_{-1}, C^r\big(S(n)/S(n)_{-1}, A(n)\big)\big) \quad \Rightarrow \quad H^{r+s}\big(S(n), A(n)\big).$$

Note that $E_1^{0,1} = H^1(S(n)_{-1}, A(n)) = \bigoplus_{i=1}^n F \cdot \langle x_i^{p-1} D_i^* \rangle$ (by Proposition 3.4) and the χ_i are global cocycles lifting them. On the other hand, by the same argument as in Proposition 3.7, we have that $E_2^{1,0} = H^1(S(n), S(n)_{-1}; A(n)) = H^1(S(n)_{\geqslant 0}, 1)$. But this last group vanishes since $[S(n)_{\geqslant 0}, S(n)_{\geqslant 0}] = S(n)_{\geqslant 0}$ as it follows easily from Lemma 4.7 above. □

In the course of the proof of the main result, we obtain a new proof of the following result.

Theorem 4.5 *(Celousov).*

$$H^1\big(S(n), S(n)\big) = \bigoplus_{i=1}^{n} \mathrm{ad}\big(x^{\tau-(p-1)\epsilon_i} D_i\big) \oplus \mathrm{ad}(x_1 D_1).$$

Proof. From the exact sequence (4.4) of $S(n)$-modules and using the fact that $S'(n)^{S(n)} = H^1(S(n), x^{\tau-(p-1)\epsilon_i}) = 0$, we get that

$$H^1\big(S(n), S(n)\big) = \bigoplus_{i=1}^{n} \big\langle \mathrm{ad}\big(x^{\tau-(p-1)\epsilon_i} D_i\big)\big\rangle \oplus H^1\big(S(n), S'(n)\big).$$

From the exact sequence (4.1) and using the facts that $W(n)^{S(n)} = 0$ and $A(n)_{<\tau}^{S(n)} = F \cdot \langle 1 \rangle$ together with Proposition 4.6, an easy computation with the coboundary map gives $H^1(S(n), S'(n)) = \langle \mathrm{ad}(x_1 D_1)\rangle$. □

4.3. Cohomology of $W(n)$

In this section we complete the proof of the Main Theorem by computing the first and the second cohomology group of $W(n)$ as an $S(n)$-module.

Proposition 4.6. $H^1(S(n), W(n)) = 0$.

Proof. Consider the homogeneous Hochschild–Serre spectral sequence (2.5) with respect to the subalgebra $S(n)_{-1} < S(n)$:

$$\big(E_1^{r,s}\big)_{\underline{0}} = H^s\big(S(n)_{-1}, C^r\big(S(n)/S(n)_{-1}, W(n)\big)\big)_{\underline{0}} \quad \Rightarrow \quad H^{r+s}\big(S(n), W(n)\big)_{\underline{0}}. \tag{4.6}$$

Note that the vertical line $E_1^{0,*} = H^*(S(n)_{-1}, W(n))_{\underline{0}} = H^*(W(n)_{-1}, W(n))_{\underline{0}}$ vanishes by Corollary 3.5 and hence we get that

$$H^1\big(S(n); W(n)\big) = H^1\big(S(n), S(n)_{-1}; W(n)\big).$$

The same argument of Proposition 3.7, using $S(n)^{S(n)_{-1}} = S(n)_{-1}$, gives that

$$H^1\big(S(n), S(n)_{-1}; W(n)\big) = H^1\big(S(n)_{\geqslant 0}, S(n)_{-1}\big),$$

where $S(n)_{-1}$ is an $S(n)_{\geqslant 0}$-module via the projection $S(n)_{\geqslant 0} \twoheadrightarrow S(n)_0$ followed by the adjoint representation of $S(n)_0$ on $S(n)_{-1}$.

Now consider the Hochschild–Serre spectral sequence (2.6) relative to the ideal $S(n)_{\geqslant 1} \lhd S(n)_{\geqslant 0}$:

$$E_2^{r,s} = H^r\big(S(n)_0, H^s\big(S(n)_{\geqslant 1}, S(n)_{-1}\big)\big) \quad \Rightarrow \quad H^{r+s}\big(S(n)_{\geqslant 0}, S(n)_{-1}\big). \tag{4.7}$$

By direct inspection, it is easy to see that $E_2^{1,0} = H^1(S(n)_0, S(n)_{-1}) = 0$ for homogeneity reasons. On the other hand, since $S(n)_{-1}$ is a trivial $S(n)_{\geqslant 1}$-module, it follows from Lemma 4.7 above that $H^1(S(n)_{\geqslant 1}, S(n)_{-1}) = C^1(S(n)_1, S(n)_{-1})$ and hence that $E_2^{0,1} = C^1(S(n)_1, S(n)_{-1})^{S(n)_0} = 0$ by Lemma 4.10 above. □

Lemma 4.7. *Let $d \geqslant -1$ be an integer. Then*

$$\big[S(n)_1, S(n)_d\big] = S(n)_{d+1}.$$

Proof. The inclusion $[S(n)_1, S(n)_d] \subset S(n)_{d+1}$ is obvious, so we fix an element $D_{ij}(x^a) \in S(n)_{d+1}$ (that is $\deg(x^a) = d+3 \geqslant 2$) and we want to prove that it belongs to $[S(n)_1, S(n)_d]$.

Suppose first that $a_i \geqslant 2$ and $a_j < p-1$. Then we are done by formula

$$\left[x_i^2 D_j, D_{ij}\left(x^{a-2\epsilon_i+\epsilon_j}\right)\right] = D_{ij}\left(x_i^2 D_j\left(x^{a-2\epsilon_i+\epsilon_j}\right)\right) = (a_j+1)D_{ij}\left(x^a\right).$$

Therefore (by interchanging i and j) it remains to consider the elements x^a for which $a_i = a_j = p-1$ or $0 \leqslant a_i, a_j \leqslant 1$. We first consider the elements satisfying this latter possibility. If $a_i = a_j = 1$ then we use formula (see (4.3))

$$\left[D_{ij}\left(x_i^2 x_j\right), D_{ij}\left(x^{a-\epsilon_i}\right)\right] = D_{ij}\left(\left(x_i^2 D_i - 2x_i x_j D_j\right)\left(x^{a-\epsilon_i}\right)\right) = -2D_{ij}\left(x^a\right).$$

On the other hand, if $(a_i, a_j) = (1, 0)$ then, by the hypothesis $\deg(x^a) = d+3 \geqslant 2$, there should exist an index $k \neq i, j$ such that $a_k \geqslant 1$ and hence we use formula

$$\left[D_{ij}\left(x_i^2 x_k\right), D_{ij}\left(x^{a-\epsilon_i-\epsilon_k+\epsilon_j}\right)\right] = -2D_{ij}\left(x^a\right).$$

Analogously, if $a_i = a_j = 0$ then there should exist either two different indices $k, h \notin \{i, j\}$ such that $a_k, a_h \geqslant 1$ or one index $k \neq i, j$ such that $a_k \geqslant 2$. We reach the desired conclusion using formula (with $h = k$ in the second case)

$$\left[D_{ij}(x_k x_h x_j), D_{ij}\left(x^{a-\epsilon_h-\epsilon_k+\epsilon_i}\right)\right] = D_{ij}\left(x^a\right).$$

Hence we are reduced to considering the elements $D_{ij}(x^a)$ such that $a_i = a_j = p-1$. Here we have to use the hypothesis that $n \geqslant 3$. Suppose first that there exist an index $k \notin \{i, j\}$ such that $a_k \neq p-2$. Consider formula (see (4.2))

$$\left[D_{ik}\left(x_k x_i^2\right), D_{ij}\left(x^{a-\epsilon_i}\right)\right] = -2D_{ij}\left(x^a\right) + 2D_{ik}\left(x^{a+\epsilon_k-\epsilon_j}\right) + 4D_{kj}\left(x^{a-\epsilon_i+\epsilon_k}\right).$$

The last two elements have k-coefficients different from $p-1$ (by the hypothesis $a_k \neq p-2$) and therefore belong to $[S(n)_1, S(n)_d]$ by what proved above. This implies also that our element $D_{ij}(x^a)$ belongs to $[S(n)_1, S(n)_d]$.

At this point, only the elements $D_{ij}(x^a)$ with $a = \underline{p-2} + \epsilon_i + \epsilon_j$ are left. Consider the following linear system (where $k \neq i, j$):

$$\begin{cases} \left[D_{ik}\left(x_k x_i^2\right), D_{ij}\left(x^{a-\epsilon_i}\right)\right] = -2D_{ij}\left(x^a\right) + 2D_{ik}\left(x^{a-\epsilon_j+\epsilon_k}\right) - 4D_{jk}\left(x^{a-\epsilon_i+\epsilon_k}\right), \\ \left[D_{ik}\left(x_k^2 x_i\right), D_{ij}\left(x^{a-\epsilon_k}\right)\right] = -2D_{ij}\left(x^a\right) + D_{ik}\left(x^{a-\epsilon_j+\epsilon_k}\right) - D_{jk}\left(x^{a-\epsilon_i+\epsilon_k}\right), \\ \left[D_{ik}(x_i x_j x_k), D_{ij}\left(x^{a-\epsilon_j}\right)\right] = -D_{ij}\left(x^a\right) + 2D_{ik}\left(x^{a-\epsilon_j+\epsilon_k}\right) - D_{jk}\left(x^{a-\epsilon_i+\epsilon_k}\right). \end{cases}$$

Since the matrix $\begin{pmatrix} -2 & 2 & -4 \\ -2 & 1 & -1 \\ -1 & 2 & -1 \end{pmatrix}$ has determinant equal to 8 and hence is invertible over F, from the preceding system we get that $D_{ij}(x^a) \in [S(n)_1, S(n)_d]$. $\square$

Proposition 4.8. *Assume that the characteristic of the base field F is different from 3 if $n = 3$. Then*

$$H^2\left(S(n), W(n)\right) = \bigoplus_{i=1}^{n} F \cdot \left\langle \mathrm{Sq}(D_i)\right\rangle.$$

Proof. We have already proved that the above cocycles are independent modulo coboundaries so that we are left with showing that they generate the whole second cohomology group. This will be done in several steps.

Step I. $H^2(S(n), W(n)) = H^2(S(n), S(n)_{-1}; W(n))$.

Consider the homogeneous Hochschild–Serre spectral sequence (4.6) with respect to the subalgebra $S(n)_{-1} < S(n)$. Since, by Corollary 3.5, the vertical line $E_1^{0,*} = H^*(S(n)_{-1}, W(n))_{\underline{0}}$ vanishes, we will conclude this first step by showing that $(E_2^{1,1})_{\underline{0}} = 0$.

The proof of that is similar to the one of Proposition 3.6. We sketch a proof referring to that proposition for notations and details. So suppose that we have an element $[f] \in (E_1^{1,1})_{\underline{0}} = H^1(S(n)_{-1}, C^1(S(n)/S(n)_{-1}, W(n))_{\underline{0}}$ that goes to 0 under the differential map $\mathrm{d} : (E_1^{1,1})_{\underline{0}} \to (E_1^{2,1})_{\underline{0}}$. First of all, arguing by induction on degree as in Proposition 3.6, we can find a representative $\tilde{f}$ of the class $[f]$ such that for a certain d and for every $i = 1, \ldots, n$, we have that

$$\begin{cases} \tilde{f}_{D_i}(F) = 0 & \text{for every } F \in S(n)\text{: } \deg(F) < d, \\ \tilde{f}_{D_i}(E) \in \langle x_i^{p-1} \rangle \otimes W(n)_{-1} & \text{for every } E \in S(n)\text{: } \deg(E) = d. \end{cases}$$

By homogeneity, it is easy to see that $\tilde{f}_{D_i}$ can take non-zero values only on the elements E of the form (for a certain k)

$$\begin{cases} D_{kh}\big(x^{\underline{a}+\epsilon_h}\big) & \text{for } 1 \leqslant a \leqslant p-1 \text{ and } h \neq k, \quad \text{(I)} \\ x_k^{p-1}(x_r D_r - x_s D_s) & \text{for } k, r, s \text{ mutually distinct.} \quad \text{(II)} \end{cases}$$

In particular, note that the degree d of E is at least $n - 1 \geqslant 2$. Now we can conclude the proof using exactly the same argument as in Proposition 3.6: we have to find, for every E as above, two elements $A \in S(n)_0$ and $B \in S(n)_d$ such that $[A, B] = E$ and $A \notin S(n)_{-\epsilon_j}, S(n)_{\epsilon_2+\cdots+\epsilon_n}$ for any $j = 2, \ldots, n$ (which are exactly the weights appearing on $S(n)_{-1}$). Explicitly: if E is of type (II) we take $B = x_k D_r$ and $A = 1/2 \cdot D_{rs}(x_k^{p-2} x_r^2 x_s)$; if E is of type (I) with $a \neq p-2$ then we take $B = x_k D_h$ and $A = -1/(a+2) \cdot D_{kh}(x^{\underline{a}+2\epsilon_h-\epsilon_k})$. Finally if E is of type (I) with $a = p-2$, then, choosing an index j different from k and h (this is possible since $n \geqslant 3$), the same argument as above gives the vanishing of $\tilde{f}_{D_i}$ on the following two elements

$$\begin{cases} 3D_{hj}\big(x^{\underline{p-2}-\epsilon_k+\epsilon_j+\epsilon_h}\big) - D_{hk}\big(x^{\underline{p-2}+\epsilon_h}\big) = \big[x_k D_h, D_{jk}\big(x^{\underline{p-2}-\epsilon_k+\epsilon_j+\epsilon_h}\big)\big], \\ 2D_{hj}\big(x^{\underline{p-2}-\epsilon_k+\epsilon_j+\epsilon_h}\big) - 2D_{hk}\big(x^{\underline{p-2}+\epsilon_h}\big) = \big[x_j D_h, D_{jk}\big(x^{\underline{p-2}+\epsilon_h}\big)\big]. \end{cases}$$

But then, since the matrix $\left(\begin{smallmatrix} 3 & -1 \\ 2 & -2 \end{smallmatrix}\right)$ has determinant equal to -4 and hence is invertible over F, we can take an appropriate linear combination of the two elements above to get the vanishing of $\tilde{f}_{D_i}$ on the element $D_{hk}(x^{\underline{p-2}+\epsilon_h})$.

Step II. $H^2(S(n), S(n)_{-1}; W(n)) \hookrightarrow H^2(S(n)_{\geqslant 1}, S(n)_{-1})^{S(n)_0}$.

First of all, exactly as in Proposition 3.7 (using that $S(n)^{S(n)_{-1}} = S(n)_{-1}$), we get

$$H^2\big(S(n), S(n)_{-1}; W(n)\big) = H^2\big(S(n)_{\geqslant 0}, S(n)_{-1}\big)$$

where as usual $S(n)_{-1}$ is an $S(n)_{\geqslant 0}$-module via the projection $S(n)_{\geqslant 0} \twoheadrightarrow S(n)_0$ followed by the adjoint representation of $S(n)_0$ on $S(n)_{-1}$.

Finally, we consider the Hochschild–Serre spectral sequence (4.7) with respect to the ideal $S(n)_{\geqslant 1} \lhd S(n)_{\geqslant 0}$. Using that $E_2^{2,0} = H^2(S(n)_0, S(n)_{-1}) = 0$ for homogeneity reasons and $E_2^{1,1} = H^1(S(n)_0, C^1(S(n)_1, S(n)_{-1})) = 0$ by Lemmas 4.7 and 4.11, we get the inclusion

$$H^2\big(S(n)_{\geqslant 0}, S(n)_{-1}\big) \hookrightarrow H^2\big(S(n)_{\geqslant 1}, S(n)_{-1}\big)^{S(n)_0}.$$

Step III. $H^2(S(n)_{\geqslant 1}, S(n)_{-1})^{S(n)_0} = \bigoplus_{i=1}^n F \cdot \langle \mathrm{Sq}(D_i)\rangle$.

The strategy of the proof is the same as that of Proposition 3.10: to compute, step by step as d increases, the truncated invariant cohomology groups

$$H^2\left(\frac{S(n)_{\geqslant 1}}{S(n)_{\geqslant d+1}}, S(n)_{-1}\right)^{S(n)_0}.$$

By Lemma 4.9, we get that $H^2(\frac{S(n)_{\geqslant 1}}{S(n)_{\geqslant 2}}, 1)^{S(n)_0} = C^2(S(n)_1, S(n)_{-1})^{K(n)_0} = 0$. On the other hand, if $d \geqslant n(p-1)-2$ then $S(n)_{\geqslant d+1} = 0$ and hence we get the cohomology we are interested in.

Consider the Hochschild–Serre spectral sequence associated to the ideal $S(n)_d = \frac{S(n)_{\geqslant d}}{S(n)_{\geqslant d+1}} \lhd \frac{S(n)_{\geqslant 1}}{S(n)_{\geqslant d+1}}$:

$$E_2^{r,s} = H^r\left(\frac{S(n)_{\geqslant 1}}{S(n)_{\geqslant d}}, H^s\big(S(n)_d, S(n)_{-1}\big)\right) \quad\Rightarrow\quad H^{r+s}\left(\frac{S(n)_{\geqslant 1}}{S(n)_{\geqslant d+1}}, S(n)_{-1}\right). \tag{4.8}$$

We get the same diagram as in Proposition 3.10 (the vanishing of $E_3^{0,2}$ and the injectivity of the map α are proved in exactly the same way). We conclude by taking cohomology with respect to $S(n)_0$ and using Lemmas 4.9, 4.10 and 4.11 below. $\square$

Lemma 4.9. *Assume that the characteristic of F is different from 3 if $n = 3$. Then in the above spectral sequence (4.8), we have that*

$$\big(E_\infty^{1,1}\big)^{S(n)_0} = 0.$$

Proof. For the above spectral sequence (4.8), we have the inclusion

$$\big(E_\infty^{1,1}\big)^{S(n)_0} \subset \big(E_2^{1,1}\big)^{S(n)_0} = C^1\big(S(n)_1 \times S(n)_d, S(n)_{-1}\big)^{S(n)_0}.$$

Let f be a homogeneous cochain belonging to $C^1(S(n)_1 \times S(n)_d, S(n)_{-1})^{S(n)_0}$. Since the action of $S(n)_0$ on $S(n)_1$ is transitive, the cochain f is determined by its restriction $f(x_1^2 D_2, -)$ (see the proof of Lemma 3.12). Even more, f is determined by its restriction to the pairs $(x_1^2 D_2, E)$ for which $f(x_1^2 D_2, E) \in \langle D_2\rangle$, which is equivalent to $E \in S(n)_{-2\sum_{i\geqslant 2}\epsilon_i}$ by the homogeneity of f. Indeed, the values of f on the other pairs (x_1^2, F) for which $f(x_1^2 D_2, F) \in \langle D_j\rangle$ (for a certain $j \neq 2$) are determined by the invariance condition

$$0 = (x_j D_2 \circ f)\big(x_1^2 D_2, F\big) = \big[x_j D_2, f\big(x_1^2 D_2, F\big)\big] - f\big(x_1^2 D_2, [x_j D_2, F]\big).$$

A base for the space $S(n)_{-2\sum_{i\geqslant 2}\epsilon_i}$ consists of the elements

$$D_{1k}\big(x^{\underline{p-1}-\epsilon_1+\epsilon_k}\big) \quad \text{for } k \neq 1, \tag{A}$$

$$D_{3h}\big(x^{\underline{a}-2\epsilon_1+\epsilon_3+\epsilon_h}\big) \quad \text{for } 0 \leqslant a \leqslant p-2 \text{ and } h \neq 3. \tag{B}$$

For the elements of type (A) with $k \geqslant 3$, we get the vanishing as follows

$$0 = (x_1 D_k \circ f)\big(x_1^2 D_2, D_{1k}\big(x^{\underline{p-1}-2\epsilon_1+2\epsilon_k}\big)\big) = -f\big(x_1^2 D_2, D_{1k}\big(x^{\underline{p-1}-\epsilon_1+\epsilon_k}\big)\big).$$

On the other hand for the element $D_{12}(x^{\underline{p-1}-\epsilon_1+\epsilon_2})$, we first use the following invariance condition

$$
\begin{aligned}
0 &= (x_1 D_2 \circ f)\big(x_1^2 D_2, D_{12}\big(x^{\underline{p-1}-2\epsilon_1+2\epsilon_2}\big)\big) \\
&= \big[x_1 D_2, f\big(x_1^2 D_2, D_{12}\big(x^{\underline{p-1}-2\epsilon_1+2\epsilon_2}\big)\big)\big] - f\big(x_1^2 D_2, D_{12}\big(x^{\underline{p-1}-\epsilon_1+\epsilon_2}\big)\big),
\end{aligned}
$$

and then we get the vanishing by means of the following

$$0 = (x_1 D_2 \circ f)\big(x_1^2 D_2, D_{12}\big(x^{\underline{p-1}-3\epsilon_1+3\epsilon_2}\big)\big) = -2 f\big(x_1^2 D_2, D_{12}\big(x^{\underline{p-1}-2\epsilon_1+2\epsilon_2}\big)\big).$$

Consider now an element $D_{3h}(x^{\underline{a}-2\epsilon_1+\epsilon_3+\epsilon_h})$ of type (B) and suppose that $a \neq p-2$. Also in this case we get the vanishing using the following condition

$$0 = (x_1 D_3 \circ f)\big(x_1^2 D_2, D_{3h}\big(x^{\underline{a}-3\epsilon_1+2\epsilon_3+\epsilon_h}\big)\big) = -(a+2) f\big(x_1^2 D_2, D_{3h}\big(x^{\underline{a}-2\epsilon_1+\epsilon_3+\epsilon_h}\big)\big).$$

Therefore it remains to consider only the elements of type (B) with $a = p-2$. Define $f(x_1^2 D_2, D_{3h}(x^{\underline{p-2}-2\epsilon_1+\epsilon_3+\epsilon_h})) := \gamma_h D_2$ for every $h \neq 3$. Consider the following invariance conditions for $h \neq 1, 3$:

$$0 = (x_1 D_3 \circ f)\big(x_1^2 D_2, D_{1h}\big(x^{\underline{p-2}-2\epsilon_1+\epsilon_3+\epsilon_h}\big)\big) = [-\gamma_1 + 4\gamma_h] D_2 \quad \text{if } p \geqslant 5, \tag{$*$}$$

$$0 = (x_1 D_3 \circ f)\big(x_1^2 D_2, D_{1h}\big(x^{\underline{1}+\epsilon_1+\epsilon_3+\epsilon_h}\big)\big) = \gamma_h D_2 \quad \text{if } p = 3, \tag{$*'$}$$

$$0 = (x_4 D_3 \circ f)\big(x_1^2 D_2, D_{12}\big(x^{\underline{p-2}-\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4}\big)\big) = [-\gamma_1 + 3\gamma_2] D_2 \quad \text{if } n \geqslant 4. \tag{$**$}$$

If $n \geqslant 4$ and $p \geqslant 5$ then, using $(**)$ and $(*)$ with $h = 2$, we get that $\gamma_1 = \gamma_2 = 0$. Substituting $\gamma_1 = 0$ in $(*)$, we find $\gamma_h = 0$ for every h.

If $n \geqslant 4$ and $p = 3$, then from $(*')$, we get the vanishing of γ_h for all $h \neq 1$ and from $(**)$ we get the vanishing of γ_1.

Finally, if $n = 3$ (and $p \geqslant 5$ by hypothesis) then from $(*)$ we get that $\gamma_1 = 4\gamma_2$. We want to prove that if $f \in (E_\infty^{1,1})^{S(n)_0}$ then $\gamma_2 = 0$. So suppose that f can be lifted to an $S(n)_0$-invariant global cocycle (which we will continue to call f). First of all, by using the $S(n)_0$-invariance condition $0 = (x_2 D_3 \circ f)(x_1^2 D_2, D_{21}(x^{\underline{p-2}-\epsilon_1+\epsilon_3}))$, we get that $f(x_1^2 D_3, D_{21}(x^{\underline{p-2}-\epsilon_1+\epsilon_3})) = -5\gamma_2 D_2$. Using this, we find the following cocycle condition (where we use that $p \geqslant 5$)

$$
\begin{aligned}
0 &= \mathrm{d} f\big(x_1^2 D_2, x_1^3 D_3, D_{12}\big(x^{\underline{p-2}-3\epsilon_1+\epsilon_2+\epsilon_3}\big)\big) \\
&= -f\big(x_1^2 D_3, D_{21}\big(x^{\underline{p-2}-\epsilon_1+\epsilon_3}\big)\big) + f\big(x_1^2 D_2, D_{31}\big(x^{\underline{p-2}-\epsilon_1+\epsilon_3}\big) - 5 D_{32}\big(x^{\underline{p-2}-2\epsilon_1+\epsilon_2+\epsilon_3}\big)\big) \\
&= 5\gamma_2 D_2 + 4\gamma_2 D_2 - 5\gamma_2 D_2,
\end{aligned}
$$

from which we deduce that $\gamma_2 = 0$. $\square$

Lemma 4.10. *Let $d \in \mathbb{Z}_{\geqslant 0}$. Then $C^1(S(n)_d, S(n)_{-1})^{S(n)_0} = 0$.*

Proof. Obviously an $S(n)_0$-invariant cochain $g \in C^1(S(n)_d, S(n)_{-1})$ must be homogeneous. Fix $D_i \in S(n)_{-1}$ and let ϕ_i be the corresponding weight (hence $\phi_i = \epsilon_i$ if $i \geqslant 2$ while $\phi_1 = \sum_{j=2}^n \epsilon_j$). A base for the space $S(n)_{\phi_i}$ (which has dimension $(n-1)p$) consists of the following elements (plus D_i):

$$x_i^{p-1} \otimes T_S, \tag{A}$$

$$D_{ij}\big(x^{\underline{a}+\epsilon_j}\big) \quad \text{for } j \neq i \text{ and } 1 \leqslant a \leqslant p-1. \tag{B}$$

We have to show that g vanishes on the elements of the above form.

An element of type (A) must be of the form $x_i^{p-1}D_{jk}(x_jx_k)=x_i^{p-1}(x_jD_j-x_kD_k)$ for some $j,k\neq i$. The vanishing of g on such an element follows from

$$0=(x_iD_j\circ g)\big(D_{jk}\big(x_i^{p-2}x_j^2x_k\big)\big)=-2g\big(x_i^{p-1}D_{jk}(x_jx_k)\big). \tag{$*$}$$

Consider now an element $D_{ij}(x^{\underline{a}+\epsilon_j})$ of type (B) and suppose that $a\neq p-2$. Then we get the vanishing by means of

$$0=(x_iD_j\circ g)\big(D_{ji}\big(x^{\underline{a}+2\epsilon_j-\epsilon_i}\big)\big)=(a+2)g\big(D_{ij}\big(x^{\underline{a}+\epsilon_j}\big)\big). \tag{$**$}$$

Therefore it remains to prove the vanishing for the elements $D_{ij}(x^{\underline{p-2}+\epsilon_j})$. Put $g(D_{ij}(x^{\underline{p-2}+\epsilon_j})):=\alpha_j^iD_i$ for $i\neq j$. Chose three indices i, j, k mutually distinct (which is possible since $n\geqslant 3$) and consider the following cocycle condition

$$\begin{aligned}
0&=(2x_iD_j\circ g)\big(D_{jk}\big(x^{\underline{p-2}+\epsilon_j+\epsilon_k-\epsilon_i}\big)\big)\\
&=\big[x_iD_j,g\big(D_{ik}\big(x^{\underline{p-2}+\epsilon_k}\big)-D_{ij}\big(x^{\underline{p-2}+\epsilon_j}\big)\big)\big]+2g\big(D_{jk}\big(x^{\underline{p-2}+\epsilon_k}\big)\big)\\
&=\big(\alpha_j^i-\alpha_k^i+2\alpha_k^j\big)D_j,
\end{aligned} \tag{$***$}$$

where in the first equality we used the relation $D_{ik}(x^{\underline{p-2}+\epsilon_k})-D_{ij}(x^{\underline{p-2}+\epsilon_j})=2D_{jk}(x^{\underline{p-2}+\epsilon_j+\epsilon_k-\epsilon_i})$. Summing Eq. $(***)$ with the one obtained interchanging k with j, we get

$$\alpha_k^j+\alpha_j^k=0. \tag{$***1$}$$

Moreover, summing Eq. $(***)$ with the analogous one obtained by interchanging i with j and using the antisymmetric property $(***1)$, we obtain

$$\alpha_k^i+\alpha_k^j=0. \tag{$***2$}$$

Finally, using Eqs. $(***1)$ and $(***2)$, we get $\alpha_j^i=-\alpha_j^k=\alpha_k^j$ and $\alpha_k^i=-\alpha_k^j$. Substituting into Eq. $(***)$, we find $4\alpha_k^j=0$. $\square$

Lemma 4.11. *Let $d\in\mathbb{Z}_{\geqslant 0}$. Then*

$$H^1\big(S(n)_0,C^1\big(S(n)_d,S(n)_{-1}\big)\big)=\begin{cases}\bigoplus_{i=1}^n F\cdot\langle\overline{\mathrm{Sq}(D_i)}\rangle & \text{if } d=p-1,\\ 0 & \text{otherwise,}\end{cases}$$

where $\overline{\mathrm{Sq}(D_i)}$ denotes the restriction of $\mathrm{Sq}(D_i)$ to $S(n)_0\times S(n)_{p-1}$.

Proof. First of all, observe that the computations made at the beginning of Section 4.2 show that the above cocycles $\overline{\mathrm{Sq}(D_i)}$ are independent modulo coboundaries. Consider a cocycle $f\in\bigoplus_{d\geqslant 0}Z^1(S(n)_0,C^1(S(n)_d,S(n)_{-1}))$. Since the maximal torus T_S is contained in $S(n)_0$, we can assume that f is homogeneous. Exactly as in the proof of Lemma 3.15, one can show, using the above Lemma 4.10, that the restriction of f to the maximal torus T_S is zero. Therefore, by homogeneity, the cocycle f can take only the following non-zero values (with $1\leqslant i,j,k\leqslant n$ mutually distinct):

$$f_{x_iD_j}(E)\subset\langle D_j\rangle\quad\text{if } E=\begin{cases} x_i^{p-1}D_{kh}(x_kx_h) & \text{for } h\neq i,k, & (1A)\\ D_{ih}(x^{\underline{a}+\epsilon_h}) & \text{for } a\neq 0,p-2 \text{ and } h\neq i, & (1B)\\ D_{jh}(x^{\underline{p-2}-\epsilon_i+\epsilon_j+\epsilon_h}) & \text{for } h\neq j, & (1C)\end{cases}$$

$$f_{x_iD_j}(E)\subset\langle D_i\rangle\quad\text{if } E=\begin{cases} D_{jh}(x_i^{p-2}x_j^2x_h) & \text{for } h\neq j, & (2A)\\ D_{jh}(x^{\underline{a}+2\epsilon_j-2\epsilon_i+\epsilon_h}) & \text{for } a\neq 0,p-2,\ h\neq j, & (2B)\\ D_{kh}(x^{\underline{p-2}-2\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_h}) & \text{for } h\neq k, & (2C)\end{cases}$$

$$f_{x_iD_j}(E)\subset\langle D_k\rangle\quad\text{if } E=\begin{cases} D_{jh}(x^{-\epsilon_i+2\epsilon_j-\epsilon_k+\epsilon_h}) & \text{for } h\neq j, & (3A)\\ D_{jh}(x^{\underline{a}-\epsilon_i+2\epsilon_j-\epsilon_k+\epsilon_h}) & \text{for } a\neq 0,p-2,\ h\neq j, & (3B)\\ D_{kh}(x^{\underline{p-2}-\epsilon_i+\epsilon_j+\epsilon_h}) & \text{for } h\neq k. & (3C)\end{cases}$$

We want to show that we can modify f, by adding coboundaries and the cocycles $\overline{\mathrm{Sq}(D_i)}$, in such a way that it vanishes on the above elements. We divide the proof in several steps according to the elements of the above list.

$\boxed{(2A)}$ For every index i, we choose an index $j\neq i$ and we modify f, by adding a multiple of $\overline{\mathrm{Sq}(D_i)}$, in such a way that $f_{x_iD_j}(D_{ji}(x_i^{p-1}x_j^2))=0$ (see Eq. (4.5)). Moreover, by adding a coboundary $\mathrm{d}g$, we can further modify f in such a way that $f_{x_iD_j}(D_{jk}(x_i^{p-2}x_j^2x_k))=0$ for every $k\neq i,j$ (see Eq. $(*)$ of Lemma 4.10). Therefore we get the required vanishing for the chosen index j. Using this, we obtain the following cocycle condition (for every $k\neq i,j$ and $h\neq j$):

$$0=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(D_{jh}\big(x^{-3\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_h}\big)\big)=-2f_{x_iD_k}\big(D_{jh}\big(x^{-2\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_h}\big)\big),$$

from which we get the required vanishing, using (for $h\neq k$) the transformation rule $D_{kh}(x_i^{p-2}x_k^2x_h)=2D_{jh}(x^{-2\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_h})-D_{jk}(x^{-2\epsilon_i+\epsilon_j+2\epsilon_k})$.

$\boxed{(3A)}$ If $p\geqslant 5$ then we get the required vanishing by means of the following condition, where we used the vanishing of the elements of type (2A):

$$0=\mathrm{d}f_{(x_kD_j,x_iD_j)}\big(D_{jh}\big(x^{-2\epsilon_k+3\epsilon_j-\epsilon_k+\epsilon_h}\big)\big)=-3f_{x_iD_j}\big(D_{jh}\big(x^{-\epsilon_i+2\epsilon_j-\epsilon_k+\epsilon_h}\big)\big).$$

If $p=3$ a little extra-work is necessary and we have to consider the following three conditions according to the three cases $h\neq i,k$, $h=i$ and $h=k$ respectively:

$$\begin{aligned}
0&=\mathrm{d}f_{(x_kD_h,x_iD_j)}\big(x_i^2x_k^2D_{jh}\big(x_j^2x_h\big)\big)=\big[x_kD_h,f_{x_iD_j}\big(x_i^2x_k^2D_{jh}\big(x_j^2x_h\big)\big)\big],\\
0&=\mathrm{d}f_{x_iD_j,x_kD_j)}\big(x_kx_j^2D_i\big)=-f_{x_iD_j}\big(2x_k^2x_jD_i\big)=f_{x_iD_j}\big(x_k^2D_{ji}\big(x_j^2\big)\big),\\
0&=\mathrm{d}f_{(x_iD_j,x_kD_i)}\big(x_i^2D_{jk}\big(x_j^2\big)\big)=-\big[x_kD_i,f_{x_iD_j}\big(x_i^2D_{jk}\big(x_j^2\big)\big)\big],
\end{aligned}$$

where in the last condition we used the first two vanishing.

$\boxed{(1A)}$ Using the vanishing of (2A) and (3A), we get

$$0=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(x_i^{p-2}D_{kh}\big(x_k^2x_h\big)\big)=2f_{x_iD_j}\big(x_i^{p-1}D_{kh}(x_kx_h)\big).$$

$\boxed{(2B)}$ Fix an integer $1\leqslant a\leqslant p-1$ different from $p-2$ and define $f_{x_iD_j}(D_{jh}(x^{\underline{a}+2\epsilon_j-2\epsilon_i+\epsilon_h}))=\gamma_{jh}^iD_i$ for every $j\neq i,h$. By adding a coboundary $\mathrm{d}g$, we can modify f in such a way that $\gamma_{ji}^i=0$ for every $j\neq i$ (see Eq. $(**)$ of Lemma 4.10). Consider first the cocycle condition (for i,j,k mutually distinct)

$$\begin{aligned}
0&=\mathrm{d}f_{(x_iD_j,x_iD_k)}\big(D_{ij}\big(x^{\underline{a}-2\epsilon_i+2\epsilon_j+\epsilon_k}\big)\big)\\
&=(a+2)f_{x_iD_j}\big(D_{ji}\big(x^{\underline{a}-\epsilon_i+\epsilon_j+\epsilon_k}\big)\big)-(a+2)f_{x_iD_j}\big(D_{ki}\big(x^{\underline{a}-\epsilon_i+\epsilon_j+\epsilon_k}\big)\big)-(a-2)\gamma_{jk}^iD_i.
\end{aligned}$$

By considering the analogous condition obtained by interchanging j with k together with the transformation rule

$$(a+2)D_{ji}\left(x^{\underline{a}-\epsilon_i+\epsilon_j+\epsilon_k}\right)=(a+1)D_{ki}\left(x^{\underline{a}-\epsilon_i+2\epsilon_k}\right)-(a-1)D_{kj}\left(x^{\underline{a}-2\epsilon_i+2\epsilon_k+\epsilon_j}\right),$$

we get the relation $(1-a)\gamma^i_{kj}+\gamma^i_{jk}=0$. Next consider the other cocycle condition

$$0=\mathrm{d}f_{(x_iD_j,x_iD_k)}\left(D_{jk}\left(x^{\underline{a}-3\epsilon_i+2\epsilon_j+2\epsilon_k}\right)\right)=(a+2)\left(\gamma^i_{kj}+\gamma^i_{jk}\right)D_i.$$

Since $\det\begin{pmatrix}1-a & 1\\ a+2 & a+2\end{pmatrix}=-a(a+2)\neq 0$, putting together these two relations we get that $\gamma^i_{jk}=0$.

$\boxed{(3B)}$ If $a\neq p-3$ then, using the vanishing of the elements of type (2B), we get

$$0=\mathrm{d}f_{(x_kD_j,x_lD_j)}\left(D_{jh}\left(x^{\underline{a}-2\epsilon_k+3\epsilon_j-\epsilon_i+\epsilon_h}\right)\right)=-(a+3)f_{x_iD_j}\left(D_{jh}\left(x^{\underline{a}-\epsilon_i+2\epsilon_j-\epsilon_k+\epsilon_h}\right)\right).$$

If $a=p-3$ (and hence $p\geqslant 5$) we use the following condition (again by the vanishing of (2B))

$$0=\mathrm{d}f_{(x_kD_i,x_iD_j)}\left(D_{ih}\left(x^{\underline{p-3}-2\epsilon_k+\epsilon_i+\epsilon_j+\epsilon_h}\right)\right)=2f_{x_iD_j}\left(D_{ih}\left(x^{\underline{p-3}-\epsilon_k+\epsilon_j+\epsilon_h}\right)\right),$$

together with the transformation rule (if $h\neq i, j$)

$$3D_{jh}\left(x^{\underline{p-3}-\epsilon_i+2\epsilon_j-\epsilon_k+\epsilon_h}\right)=-(2+\delta_{hk})D_{ij}\left(x^{\underline{p-3}-\epsilon_k+2\epsilon_j}\right)+D_{ih}\left(x^{\underline{p-3}-\epsilon_k+\epsilon_j+\epsilon_h}\right).$$

$\boxed{(1B)}$ Take indices $r\neq i, j$ and $s\neq r$ and consider the following condition (using the vanishing of (2B) and (3B))

$$0=\mathrm{d}f_{(x_iD_j,x_iD_r)}\left(D_{rs}\left(x^{\underline{a}+\epsilon_r-\epsilon_i+\epsilon_s}\right)\right)=(a+2)f_{x_iD_j}\left(D_{rs}\left(x^{\underline{a}-\epsilon_i+\epsilon_r+\epsilon_s}\right)\right).$$

By taking $r=h$ and $s=i$, we get the required vanishing if $h\neq j$. If $h=j$ and $a\neq p-1$, we use the transformation rule

$$(a+1)D_{ij}\left(x^{\underline{a}+\epsilon_j}\right)=(a+1)^2D_{ri}\left(x^{\underline{a}+\epsilon_r}\right)-a(a+1)D_{rj}\left(x^{\underline{a}-\epsilon_i+\epsilon_r+\epsilon_j}\right).$$

If $h=j$ and $a=p-1$ we use the following condition (by the vanishing of (2B) and (3B))

$$0=\mathrm{d}f_{(x_kD_j,x_iD_j)}\left(D_{ji}\left(x^{\underline{p-1}-\epsilon_k+2\epsilon_j}\right)\right)=f_{x_iD_j}\left(D_{ij}\left(x^{\underline{p-1}+\epsilon_j}\right)\right).$$

$\boxed{(1C)}$ We define $f_{x_iD_j}(D_{jk}(x^{\underline{p-2}-\epsilon_i+\epsilon_j+\epsilon_k}))=\beta^i_{jk}D_j$ for every $j\neq i,k$ (but possibly $i=k$). The space of all such cochains has dimension $n(n-1)^2$. Using the notations of Lemma 4.10, the subspace of coboundaries is formed by the β^i_{jk} such that there exist $\{\alpha^i_j\colon\ i\neq j\}$ with the property that $2\beta^i_{jk}=\alpha^i_j-\alpha^i_k+2\alpha^j_k$ (see Eq. $(***)$ of Lemma 4.10). Moreover in the above quoted lemma, we prove that different values of α^i_j give rise to different values of β^i_{jk}. Hence the dimension of the subspace of coboundaries is $n(n-1)$. Therefore, in order to prove the vanishing of the elements of type $(1C)$, it will be enough to exhibit $n(n-1)(n-2)$ linearly independent relations among the coefficients β^i_{jk}.

Fix three integers i, j, k mutually distinct and consider the following cocycle condition

$$0=\mathrm{d}f_{(x_iD_j,x_jD_k)}\left(D_{jk}\left(x^{\underline{p-2}+\epsilon_j+\epsilon_k-\epsilon_i}\right)\right)=\left(-\beta^j_{kj}+\beta^i_{jk}+\beta^i_{kj}\right)D_k.$$

We get first of all that the β's with two coincident indices are determined by those with three different indices and this give $n(n-1)$ linearly independent relations. Moreover we deduce also that for any $k\neq j$ the value of the sum $\beta^i_{jk}+\beta^i_{kj}$ is independent of i and this give $n(n-1)(n-3)$ linearly independent relations. Since the two types of relations are also independent one of the other, the total number of independent relations we get is $n(n-1)(n-2)$, as required.

$\boxed{(3C)}$ Using the vanishing of (1C), we get

$$0 = \mathrm{d}f_{(x_i D_j, x_j D_k)}\big(D_{kh}\big(x^{\underline{p-2}+\epsilon_k+\epsilon_h-\epsilon_i}\big)\big) = -f_{x_i D_j}\big(D_{kh}\big(x^{\underline{p-2}+\epsilon_h-\epsilon_i+\epsilon_j}\big)\big).$$

$\boxed{(2C)}$ Using the vanishing of (1C) and (3C), we compute

$$\begin{aligned} 0 &= \mathrm{d}f_{(x_i D_k, x_i D_j)}\big(D_{kh}\big(x^{\underline{p-2}-2\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_h}\big)\big) \\ &= \big[x_i D_k, f_{x_i D_j}\big(D_{kh}\big(x^{\underline{p-2}-2\epsilon_i+\epsilon_j+\epsilon_k+\epsilon_h}\big)\big)\big]. \qquad \square \end{aligned}$$

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