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Graph-like spaces: An introduction

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ABSTRACT

Thomassen and Vella (Graph-like continua, augmenting arcs, and Menger's Theorem, Combinatorica, doi:10.1007/s00493-008-2342-9) have recently introduced the notion of a graph-like space, simultaneously generalizing infinite graphs and many of the compact spaces recently used by Diestel or Richter (and their coauthors) to study cycle spaces and related problems in infinite graphs. This work is a survey to introduce graph-like spaces and shows how many of these works on compact spaces can be generalized to compact graph-like spaces.

1. Introduction

In an on-going project, Diestel and his students have been studying compact topological spaces associated with certain infinite graphs, especially locally finite ones. This study has proved fruitful for generalizing many theorems about finite graphs to the infinite context; an essential point is that cycles are generalized to include infinite cycles. This has necessitated an inherently topological point of view: a cycle is now an embedding of a circle into the compact space. Particular examples of such works are [2–4,8].

The original motivation for some of these questions arose from Bonnington and Richter [1], who proved that the cycle space of a locally finite planar graph is generated by the face boundaries of a planar embedding, together with certain 2-way infinite paths joining the different accumulation points. However, the definition of cycle space here is different from that employed by Diestel. In the next section, we shall give a unified definition of cycle space; this section is devoted to providing historical context.

Motivated in part by a desire to unify the two definitions, Vella and Richter [23] introduced the notion of *edge space* and showed that the two notions of cycle space above are both instances of the cycle space of a compact edge space. The only difference is the embedding of the graph into a larger compact space. For locally finite graphs, Diestel uses the Freudenthal compactification, while Bonnington and Richter use the 1-point compactification.

However, edge spaces ignore many inherently graph-theoretic properties. Its name was chosen to reflect the emphasis on edges rather than vertices. Continuing this line of reasoning, Vella introduced the notion of *graph-like space* (appearing for the first time in [22]), which is the main focus of this introductory article.

A *graph-like space* is a metric space X with a 0-dimensional subspace V – the set of *vertices* – of X so that every component of $X - V$ is an open subset of X homeomorphic to an open arc and whose closure in X has only one or two additional points — these components of $X - V$ are the *edges*. (The meaning of 0-dimensional is: for any two points $u, w \in V$, there is a separation (U, W) of V so that $u \in U$ and $w \in W$. In particular, V is totally disconnected. If X is compact, then so is V and then totally disconnected is equivalent to being 0-dimensional.)

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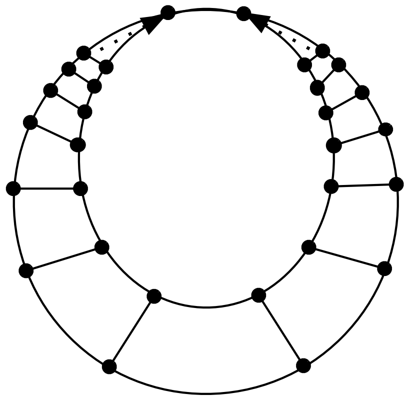

Fig. 1. 2-way infinite ladder plus an edge joining its ends.

From this definition, we see that any graph G is a graph-like space. (Give G the usual topology of a 1-dimensional cell complex; in this case, every subset of $V(G)$ is open in $V(G)$, so $V(G)$ has the discrete topology.) But many other topological spaces are graph-like: the Freudenthal and 1-point compactifications of an infinite, locally finite graph are also graph-like. Note that the added "points-at-infinity" are vertices of the graph-like space. We will be principally concerned with compact graph-like spaces.

In their more general results, Diestel et al. work with infinite graphs in which every two vertices are separated by some finite edge cut. For such a graph G, an identification space $\widetilde{G}$ obtained from the Freudenthal compactification of G is considered; it is known that $\widetilde{G}$ is another example of a compact graph-like space [23].

With the realization that the two notions of cycle space come from different compactifications of the same locally finite graph, suddenly a much broader landscape appears before us. Although the two compactifications considered above are the most natural, it is apparent that many others are possible. Moreover, one's horizons are no longer limited to compactifications of graphs. For example, in Fig. 1 we see the 2-way infinite ladder, plus its two ends, plus an edge joining the two ends. This is a compact graph-like space. Another compact graph-like space can be obtained from the 1-point compactification of the 2-way infinite path by joining the limit point to every vertex of the path.

The graph in the figure has many infinite cycles; for example, the two facial cycles bounding the infinite and the innermost faces both go through the edge containing the ends of the 2-way infinite ladder. Another feature of this space is that it is 3-connected (deleting any two vertices and their incident edges does not result in a disconnected subspace), whereas the 2-way infinite ladder by itself is not 3-connected (neither is its Freudenthal compactification).

Thomassen and Vella [22] proved that:

1. if H is a closed subspace of a compact graph-like space, then H is a graph-like space;
2. a compact graph-like space is locally connected; and
3. graph-like spaces satisfy Menger's Theorem: if k is a non-negative integer and u, v are vertices of G, then either there are $k+1$ internally disjoint uv-arcs in G, or there is a set S of k points, different from u and v, so that $G-S$ has no uv-arc.

The relevance of the first two (each given a relatively short proof in [22]) is that every closed connected subspace of a compact graph-like space is arcwise connected. For the Freudenthal compactification, or more generally for the space $\widetilde{G}$, Diestel and Kühn gave a difficult proof [8].

Many cycle-based theorems about finite graphs extend to compact graph-like spaces. This is, however, not the most interesting possibility. Mirroring Thomas' work in infinite graphs [17], many aspects of the Graph Minors Project of Robertson and Seymour "lift" to graph-like spaces. We will have more to say on this point later in this work.

Furthermore, there is small, but increasing, evidence, that results can be proved for quite general topological spaces. Three examples, all related to planarity of compact, locally connected metric spaces, are [13,15,20]. In the discussion on embedding graph-like spaces into surfaces, we will point out one example of a theorem that we have proved for graph-like spaces that might hold for compact, locally connected metric spaces, but has not yet been proven in that more general context.

2. From finite graphs to compact graph-like spaces

In this section, we review some basic matters concerning cycle spaces and embeddings in surfaces of compact graph-like spaces. These results show that in many basic respects, graph-like spaces have the same properties as graphs.

We denote by 2^E the set of all subsets of edges of the compact graph-like space G. A subset A of 2^E is *thin* if every edge is in only finitely many members of A. In the case A is thin, the *thin sum of* A is the set $\sum_{a\in A} a$ of edges that are in an odd number of elements of A. For a subset A of 2^E, the *subspace generated by* A is the smallest subset of 2^E containing A and closed under thin sums. It is an easy consequence of Zorn's Lemma that every subset of 2^E generates a unique subspace.

A *circle* in a graph-like space G is a homeomorph C of the unit circle contained in G. The *cycle space* $\mathcal{Z}(G)$ of G is the subspace of 2^E generated by the (edge-sets of) circles. The basic theorems about cycles spaces can be found (in slightly greater generality) in [23].

Theorem 2.1. *Let G be a compact, connected graph-like space with vertex set V and edge set E. Then:*

1. *if (U, W) is a separation of V, then there are only finitely many edges with one end in U and one end in W;*
2. *there is a minimal connected subspace T containing V;*
3. *for each edge e not in T, $T \cup e$ contains a unique circle γ_e (a fundamental cycle);*
4. *the set of fundamental cycles is a thin set;*
5. *each element z of $\mathcal{Z}(G)$ is the thin sum $\sum_{e\in z\setminus T}\gamma_e$.*

We next consider embeddings of graph-like spaces in surfaces. Here, embedding is used with its usual topological meaning of a continuous map $f : G \to X$ so that $f : G \to f(G)$ is a homeomorphism. A surface is a connected, compact 2-manifold. The following results are in [12]. Here B and $\overline{B}$ denote, respectively, the open and closed unit discs in $\mathbb{R}^2$.

Theorem 2.2. *Let G be a compact, connected graph-like space with vertex set V and edge set E, let Σ be a surface, and let $f : G \to \Sigma$ be an embedding so that every face is homeomorphic to an open disc. Then:*

1. *for each face F of $f(G)$ (that is, F is a component of $\Sigma \setminus f(G)$), there is a continuous map $g_F : \overline{B} \to \mathrm{cl}(F)$ so that $g_F : B \to F$ is a homeomorphism;*
2. *for each face F of $f(G)$, the set $\partial(F)$ defined to be*
$$\{e \in E \mid g_F^{-1}(f(e)) \text{ has one component}\}$$
is in $\mathcal{Z}(G)$;
3. *if $\mathcal{B}$ is the subspace generated by $\{\partial(F) \mid F \text{ is a face of } f(G)\}$, then every element of $\mathcal{B}$ is the thin sum of $\partial(F)$'s;*
4. $\dim(\mathcal{Z}(B)/\mathcal{B}) = \xi(\Sigma)$, *where, for Σ being the sphere with h handles and k crosscaps, $\xi(\Sigma) = 2h + k$.*

This theorem asserts that compact graph-like spaces behave in certain respects, relative to embeddings in surfaces, in precisely the same way as finite graphs. The last item in Theorem 2.2 is proved for finite graphs in [14]. In fact, more is proved there, and also holds true for compact graph-like spaces. If T is any spanning tree of the graph-like space G, then there is a spanning tree T^* of the geometric dual (which is a graph, not a graph-like space; this is discussed more fully in the next section) so that no edge of T has its dual in T^*. It is then relatively easy to see that there are only finitely many edges not in either T or T^*. The fundamental cycles of these finitely many edges are a basis for the quotient space and the general case reduces to the finite case to show that there are $\xi(\Sigma)$ of them.

One expects that the cycle spaces of different compactifications of the same graph will sometimes have relations between them. Something slightly more general than the following is proved in [5] (1 and 2) and [11] (3).

Theorem 2.3. *Let G and H be compact, connected, graph-like spaces with edge sets E and F, respectively. Let $f : G \to H$ be a continuous surjection so that $f : E \to F$ is a bijection and $f(G \setminus E) = H \setminus F$. Let T_H be a spanning tree of H with fundamental cycles γ_e, for $e \in F \setminus T_H$. Then:*

1. *there is a spanning tree T_G of G for which $f^{-1}(T_H) \subseteq T_G$;*
2. *identifying $e \in E$ with $f(e) \in F$, $\mathcal{Z}(G) \subseteq \mathcal{Z}(H)$; and*
3. *for z in $\mathcal{Z}(H)$, there exist a unique $z' \in \mathcal{Z}(G)$ and a unique subset A of $E \cap T_G \setminus T_H$ so that $z = f(z') + \sum_{e\in A}\gamma_{f(e)}$.*

As a simple example, consider the 2-way infinite ladder L. It has two ends, which we denote by a and b. Then $\mathbb{F}(L)$ is L plus these two additional points. Let v be any vertex of L and set $L_{a=v}$ to be the space obtained by identifying a with v.

Comparing $\mathbb{F}(L)$ with $L_{a=v}$, we see that any ray joining a and v in $\mathbb{F}$ is a cycle in $L_{a=v}$. In particular, if T is a spanning tree of $\mathbb{F}$, there is a unique such ray in T and we must delete one of its edges in order to obtain a spanning tree of $L_{a=v}$. Likewise, in comparing the cycle spaces of $\mathbb{F}(L)$ and $L_{a=v}$, we see that there is only one new fundamental cycle and so the cycle space of $\mathbb{F}(L)$ has index 1 in the cycle space of $L_{a=v}$.

A main result in [1], although not expressed in these terms, is that if G is a locally finite graph embedded in the sphere with k accumulation points, then the quotient of the cycle space of the 1-point compactification of G modulo the cycle space of the closure of G in the sphere (that is, some k-point compactification of G) has dimension $k - 1$. Theorem 2.3 is applied in [11] to generalize this result from [1]. In particular, if f is a homeomorphism except that it identifies two vertices of G, then the dimension of the quotient space is 1 and it is generated by an arc in G joining the two identified points; this arc is mapped to a circle in H. This is precisely the situation described in the example in the preceding paragraph.

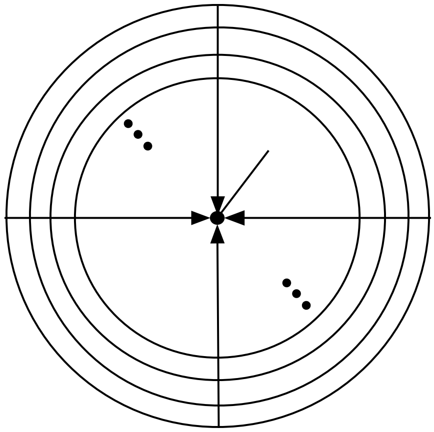

Fig. 2. A generalized thumbtack.

3. Planar embeddings

There are many characterizations of planarity for finite graphs. In this section we shall discuss three of them (the theorems of Kuratowski, MacLane, and Whitney) and how they generalize to graph-like spaces. In [20], Thomassen considers the planarity of compact, locally connected metric spaces. The main result in that work is the following very deep form of Kuratowski's Theorem. (It seems that this result was also proved by Claytor in 1934 [7].)

Theorem 3.1. *A 2-connected, compact, locally connected metric space M is homeomorphic to a subspace of the sphere if and only if M contains no homeomorph of either K_5 or $K_{3,3}$.*

A metric space M is *2-connected* if it is connected and, for each point x of M, $M \setminus \{x\}$ is connected. Thomassen points out that the *thumbtack space* (a closed disc D with a line segment L disjoint from D except that one end of L is the centre of D) is an example of a non-planar, connected, compact, locally connected metric space that does not contain either K_5 or $K_{3,3}$. Thus, the assumption that M be 2-connected cannot simply be deleted.

In [13], a family of graph-like spaces related to the thumbtack is introduced; these are called *generalized thumbtacks*. An example is given in Fig. 2. The main point is that each circle C has the property that it has two bridges, one containing the circles with diameter larger than C, the other containing the circles with diameter smaller than C together with the line segment L, and these two bridges overlap.

The main result of [13] is the following extension of Theorem 3.1.

Theorem 3.2. *A compact, locally connected metric space M is homeomorphic to a subspace of the sphere if and only if M does not contain any of:*

1. *K_5 and $K_{3,3}$;*
2. *a generalized thumbtack; and*
3. *the disjoint union of a sphere and a point.*

In particular, this theorem applies to compact graph-like spaces.

MacLane [16] proved that a finite graph is planar if and only if it has a 2-basis for its cycle space. (A *2-basis* is a basis in which each edge occurs in at most two elements of the basis.) Bruhn and Stein [4] extended this to the Freudenthal compactification of a locally finite graph. In [6], this is extended to compact graph-like spaces in the following sense.

Theorem 3.3. *A compact graph-like space has a 2-basis for its cycle space if and only if it does not have a homeomorph of either K_5 or $K_{3,3}$.*

It is easy to see that a compact graph-like space G has a 2-basis if and only if each of its blocks has a 2-basis. Likewise, G has no $K_{3,3}$ or K_5 if and only if each block has no $K_{3,3}$ or K_5. By Theorem 3.1, we see that G has a 2-basis if and only if each block of G is planar. This is not, however, the same thing as G being planar, since the union of two planar blocks may contain a generalized thumbtack. We point out that to show that if G has a 2-basis for its cycle space, then it suffices to show that every finite minor has a 2-basis for its cycle space. The argument in [6] is precisely that of Bruhn and Stein; the only change required is to convert "Freudenthal compactification" to "compact graph-like space".

The other planarity criterion we consider is Whitney's Theorem [24], that a graph is planar if and only if its matroid dual is graphic. In the infinite case, there are three previous generalizations of Whitney's Theorem. Thomassen [19] showed that a 2-connected graph G has a graph dual if and only if G is planar and any two vertices of G are separated by a finite edge cut.

This extended his locally finite result in [18]. Bruhn and Diestel [3] went one step further to show that, if any two vertices of G are separated by a finite edge cut, then the identification space $\widetilde{G}$ is planar if and only if there is another identification space $\widetilde{H}$ dual to G. This seemed like a satisfying conclusion to Thomassen's work. However, we show that it can be taken further, into the context of compact graph-like spaces.

Whitney's Theorem involves duality and matroids, so we discuss the appropriate generalizations of these notions. As briefly mentioned in the preceding section, the *geometric dual* G^* of an embedding of a compact graph-like space G in a surface is a graph: the faces of G are the vertices of G^* and two vertices of G^* are adjacent by the edge of G^* dual to the edge e of G if they are the two faces on each side of e. It is not completely trivial to see that the dual is connected.

The notion of B-matroid was introduced by Higgs [9]; the basics can be found in [10]. A *B-matroid* is a pair $(X, \mathcal{I})$ consisting of a set X and a set $\mathcal{I}$ of subsets of X so that:

1. $\mathcal{I} \neq \varnothing$;
2. if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$;
3. for each $Y \subseteq X$, there is a maximal element of $\mathcal{I}$ contained in Y; and
4. for each $Y \subseteq X$, if I and J are two maximal elements of $\mathcal{I}$ contained in Y and $x \in I \setminus J$, then there is a $y \in J \setminus I$ so that $(I \cup \{y\}) \setminus \{x\} \in \mathcal{I}$.

Every matroid (on a finite set) is a B-matroid and every B-matroid has a B-matroid as its dual.

The existence of spanning trees in either an infinite graph or a compact graph-like space makes it easy to show that sets of edges that do not contain the edge set of a circle are, in both cases, the independent sets of a B-matroid. If G^* is the geometric dual of a connected, compact graph-like space G embedded in the sphere, then the dual of the B-matroid of G is the B-matroid of G^*. In [6], we prove the following form of Whitney's Theorem.

Theorem 3.4. *Let G be a compact graph-like space. Then there is a graph H so that the B-matroids of G and H are duals if and only if G does not contain a homeomorph of* K_5 *or* $K_{3,3}$.

4. Sequences of finite graphs and matroids

In this section, we describe a sequential technique that is used to prove other facts about embeddings of compact graph-like spaces in surfaces and about matroids. We start with graph-like spaces in surfaces.

It is known that, for a surface Σ, there is a finite set $\mathfrak{F}(\Sigma)$ of finite graphs so that a finite graph G embeds in Σ if and only if G does not contain a homeomorph of any of the graphs in $\mathfrak{F}(\Sigma)$. Theorem 3.2 shows that some modification must be made for compact graph-like spaces, since generalized thumbtacks do not embed in Σ.

We have recently proved the following [5]. Although this generalizes Theorem 3.1 for compact graph-like spaces, its proof uses that result.

Theorem 4.1. *Let G be a connected, compact graph-like space and let* Σ *be a surface. Then G embeds in* Σ *if and only if G contains no generalized thumbtack and G contains no element of* $\mathfrak{F}(\Sigma)$*. If G is 2-connected, then G embeds in* Σ *if and only if G contains no element of* $\mathfrak{F}(\Sigma)$.

The important point to note here is that it is not necessary to know the elements of $\mathfrak{F}(\Sigma)$. Currently, these are only known for the sphere and the projective plane.

It is a very interesting possibility that some variant of Theorem 4.1 might hold for any compact, locally connected, connected metric space. We do not yet know how to attack this question, but note that Thomassen has proved something along these lines [21].

This theorem gives us some hope that significant elements of the Graph Minors Project of Robertson and Seymour will "lift" to compact graph-like spaces. It seems quite plausible that one could obtain a structure theorem for compact graph-like spaces that do not contain a particular finite graph as a minor.

The earlier theorems are proved directly, making use of properties of the graph-like space. Theorem 4.1, however, is proved using sequences of finite graphs and taking a certain kind of limit. The argument reduces the general case of Theorem 4.1 to the planar case Theorem 3.2. This method was introduced by Christian and will appear in detail in his doctoral dissertation.

In the context of graph-like spaces, the sequences are complicated by the topology. Here we restrict our attention to matroids, where the situation is more straightforward.

Let $(M_n)_{n\geq 0}$ be a sequence of matroids so that M_n is a minor of M_{n+1} for each n. We label the elements of the M_n so that the ground set S_n of M_n is naturally contained in S_{n+1}. The limit B-matroid $[(M_n)]$ has as ground set $\bigcup_{n\geq 0} S_n$ and a set J is independent in $[(M_n)]$ if, for every finite subset I of J, there is an n so that I is independent in M_n (and, therefore, independent in M_m for $m \geq n$).

Obviously, there is another B-matroid $[(M_n^*)]$ derived from the sequence $(M_n^*)_{n\geq 0}$ of the duals of the M_n and $[(M_n^*)]^*$ is its dual B-matroid. It is easy to see that a set D dependent in $[(M_n)]$ is also dependent in $[(M_n^*)]^*$ (but not necessarily the other way around).

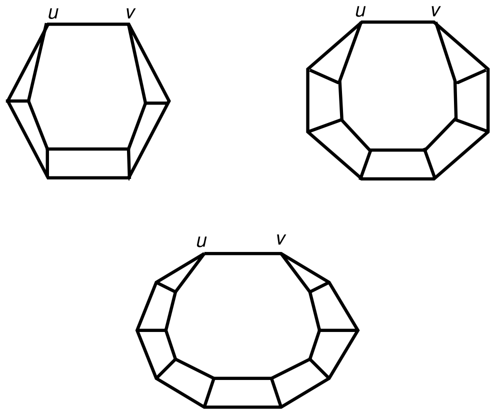

Fig. 3. Three terms of a sequence.

It turns out, and is not at all trivial, that every dependent set of $[(M_n^*)]^*$ contains the non-empty limit (that is, the elements that are infinitely often in) of a sequence (C_n), with, for n large enough, C_n being a circuit of M_n.

By way of illustration, consider the three graphs illustrated in Fig. 3. Their graphic matroids are three terms in a sequence (M_n). These graphs are obviously related to the compact graph-like space shown in Fig. 1. To get from a larger one of these three to the next smaller one involves deleting each of the "rungs" nearest u and v and then contracting the four edges other than uv incident with either u or v. A set of edges is independent in $[(M_n)]$ if and only if it eventually does not contain any cycle of any M_n. In particular, the infinite cycle C from Fig. 1 bounding the infinite face is independent in $[(M_n)]$. Note that this is naturally the limit of the cycles in the finite graphs bounding the infinite face, and so C is a circuit in $[(M_n^*)]^*$.

The definition of $[(M_n)]$ makes it evident that the circuits of $[(M_n)]$ are finite. Also $[(M_n)]$ has only countably many elements. It is easy to see that if M is a B-matroid with countably many elements and every circuit of M is finite, then $M = [(M_n)]$ for some sequence. It is now a triviality that if M has only finite cocircuits, then $M = [(M_n^*)]^*$ for some sequence.

These sequences can be used to deduce another version of Whitney's Theorem: if G is a countable graph, then the dual B-matroid of G is the B-matroid of a compact graph-like space if and only if G does not have either $K_{3,3}$ or K_5 as a minor. It is well-known that, for a countable graph G, not having a $K_{3,3}$ or K_5 minor is equivalent to being planar.

The following results will also be in Christian's thesis.

Theorem 4.2. *Let M be a B-matroid with countably many elements.*

1. *If M has only finite circuits, then:*
   (a) *M is the B-matroid of a graph if and only if M has no $U_{2,4}$, F_7, F_7^*, $K_{3,3}^*$ or K_5^* minor; and*
   (b) *the following are equivalent:*
      i. *M has no $U_{2,4}$ minor;*
      ii. *every finite symmetric difference of circuits of M is the disjoint union of circuits of M;*
      iii. *every circuit of M is the finite symmetric difference of fundamental circuits of M;*
      iv. *the intersection of any circuit and any cocircuit has even cardinality;*
      v. *M is binary.*
2. *If M has only finite cocircuits, then:*
   (a) *M is the B-matroid of a compact graph-like space if and only if M has no $U_{2,4}$, F_7, F_7^*, $K_{3,3}^*$ or K_5^* minor; and*
   (b) *the following are equivalent:*
      i. *M has no $U_{2,4}$ minor;*
      ii. *every thin sum of circuits of M is the disjoint union of circuits of M;*
      iii. *every circuit of M is the thin sum of fundamental circuits of M;*
      iv. *the intersection of any circuit and any cocircuit has even cardinality;*
      v. *M is binary.*

Unfortunately, it is not clear what it means for a B-matroid to be binary. Two different definitions are used in the statement of Theorem 4.2. If all circuits of M are finite, then M is binary means there is a map $f : E(M) \to V$, where V is some binary vector space, so that $I \subseteq E(M)$ is independent in M if and only if $f(I)$ is independent in V. If all the cocircuits of M are finite, then M is binary means there is a map f taking the elements of M to elements of $2^{E(M)}$ so that I is independent in M if and only if $f(I)$ does not have any non-empty thin set with an empty thin sum.

5. Open questions

There are many open questions concerning graph-like spaces. While we know that connected graphs and compact, connected graph-like spaces have spanning trees, we do not know this for connected graph-like spaces having, say, countably many edges. We expect that if spanning trees exist, then they would be the bases of a B-matroid. This is a major question relating to Theorem 4.2.

We are currently working on two separate projects. Bruhn showed that the peripheral cycles of the Freudenthal compactification of a locally finite 3-connected graph generate the cycle space [2]. This generalizes Tutte's theorem for finite graphs and we hope to extend it to compact graph-like spaces.

The other project is to prove theorems about branch-width or tree-width of compact graph-like spaces. The aim here, of course, is to extend to compact graph-like spaces the graph minor theorems of Robertson and Seymour. A natural first step is to describe the structure of a compact graph-like space that does not have a given finite graph as a minor.

It would be valuable to be able to apply logical compactness to compact graph-like spaces. What are the kinds of properties for which this will work?

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References

[1] C.P. Bonnington, R.B. Richter, Graphs embedded in the plane with a bounded number of accumulation points, J. Graph Theory 44 (2) (2003) 132–147.
[2] H. Bruhn, The cycle space of a 3-connected locally finite graph is generated by its finite and infinite peripheral circuits, J. Combin. Theory Ser. B 92 (2) (2004) 235–256.
[3] H. Bruhn, R. Diestel, Duality in infinite graphs, Combin. Probab. Comput. 15 (1–2) (2006) 75–90.
[4] H. Bruhn, M. Stein, MacLane's planarity criterion for locally finite graphs, J. Combin. Theory Ser. B 96 (2) (2006) 225–239.
[5] K. Casteels, R.B. Richter, The bond and cycle spaces of an infinite graph, J. Graph Theory 59 (2) (2008) 162–176.
[6] R. Christian, R.B. Richter, B. Rooney, The theorems of MacLane and Whitney for graph-like spaces, Electron. J. Combin. (in press).
[7] S. Claytor, Topological immersion of Peanian continua in a spherical surface, Ann. Math. 35 (4) (1934) 809–835.
[8] R. Diestel, D. Kühn, Topological paths, cycles and spanning trees in infinite graphs, European J. Combin. 25 (6) (2004) 835–862.
[9] D.A. Higgs, Matroids and duality, Colloq. Math. 20 (1969) 215–220.
[10] J. Oxley, Infinite matroids, in: N. White (Ed.), Matroid Applications, in: Encyclopedia Math. Appl., 40, Cambridge Univ. Press, Cambridge, 1992, pp. 73–90.
[11] R.B. Richter, B. Rooney, Relating different cycle spaces of the same infinite graph (submitted for publication).
[12] R.B. Richter, B. Rooney, Embedding graph-like spaces in surfaces, in preparation.
[13] R.B. Richter, B. Rooney, C. Thomassen, On planarity of compact, locally connected, metric spaces (submitted for publication).
[14] B. Richter, H. Shank, The cycle space of an embedded graph, J. Graph Theory 8 (3) (1984) 365–369.
[15] R.B. Richter, C. Thomassen, 3-connected planar spaces uniquely embed in the sphere, Trans. Amer. Math. Soc. 354 (11) (2002) 4585–4595.
[16] S. MacLane, A combinatorial condition for planar graphs, Fund. Math. 28 (1937) 22–32.
[17] R. Thomas, Well-quasi-ordering infinite graphs with forbidden finite planar minor, Trans. Amer. Math. Soc. 312 (1) (1989) 279–313.
[18] C. Thomassen, Planarity and duality of finite and infinite graphs, J. Combin. Theory Ser. B 29 (2) (1980) 244–271.
[19] C. Thomassen, Duality of infinite graphs, J. Combin. Theory Ser. B 33 (2) (1982) 137–160.
[20] C. Thomassen, The locally connected compact metric spaces embeddable in the plane, Combinatorica 24 (4) (2004) 699–718.
[21] C. Thomassen, Classification of locally 2-connected compact metric spaces, Combinatorica 25 (1) (2005) 85–103.
[22] C. Thomassen, A. Vella, Graph-like continua, augmenting arcs, and Menger's Theorem, Combinatorica (doi:10.1007/s00493-008-2342-9).
[23] A. Vella, R.B. Richter, Cycle spaces in topological spaces, J. Graph Theory 59 (2) (2008) 115–144.
[24] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc. 34 (1932) 339–362.