A complete monotonicity property of the gamma function✩

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Received 20 July 2003
Available online 2 July 2004
Submitted by H.M. Srivastava

Abstract

A logarithmically completely monotonic function is completely monotonic. The function $1 - \ln x + \frac{1}{x} \ln \Gamma(x + 1)$ is strictly completely monotonic on $(0, \infty)$. The function $\sqrt{\Gamma(x + 1)/x}$ is strictly logarithmically completely monotonic on $(0, \infty)$.

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Keywords: Gamma function; Psi function; Logarithmically completely monotonic function

1. Introduction

The classical gamma function is usually defined for Re $z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt. \quad (1)$$
The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [6, p. 16] as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_{0}^{\infty} \frac{e^{-xt} - e^{-x}}{1 - e^{-x}} dt,$$

$$\psi^{(m)}(x) = (-1)^{m+1} \int_{0}^{\infty} \frac{t^{m}}{1 - e^{-x}} e^{-xt} dt$$

for \(x > 0\) and \(m \in \mathbb{N}\), where \(\gamma = 0.57721566490153286\ldots\) is the Euler–Mascheroni constant.

In 1985, D. Kershaw and A. Laforgia [5] showed that the function \(x[\Gamma(1 + 1/x)]^s\) is strictly increasing on \((0, \infty)\), which is equivalent to the function \([\Gamma(x+1)]^{1/s}/x\) being strictly decreasing on \((0, \infty)\). In addition, it was proved that the function \(x^{1-\gamma}[\Gamma(1 + 1/x)]^s\) decreases for \(0 < x < 1\), which is equivalent to \([\Gamma(1 + x)]^{1/s}/x^{1-\gamma}\) being increasing on \((1, \infty)\).

In [2,10], it is proved that the function \(f(x) = [\Gamma(x+1)]^{1/s}/(x+1)\) is strictly decreasing and strictly logarithmically convex in \((0, \infty)\) and the function \(g(x) = [\Gamma(x+1)]^{1/s}/\sqrt{x+1}\) is strictly increasing and strictly logarithmically concave in \((0, \infty)\). Some new proofs for the monotonicity of the function \(x^r[\Gamma(x+1)]^{1/s}\) on \((0, \infty)\) are given for \(r \notin (0, 1)\). In addition, if \(s\) is a positive real number, then for all real numbers \(x > 0\), \(f(x) = e^{-\gamma}\) and \(\lim_{x \to \infty} f(x) = e^{-1}\).

Using monotonicity and inequalities of the generalized weighted mean values \([1, 7,8,12]\), the first author proved [9] that the functions \([\Gamma(s)/\Gamma(r)]^{1/(r-s)}\), \([\Gamma(s,x)/\Gamma(r,x)]^{1/(r-s)}\), and \([\gamma(s,x)/\gamma(r,x)]^{1/(r-s)}\) are increasing in \(r > 0\), \(s > 0\), and \(x > 0\). For any given \(x > 0\), the function \((s\gamma(s,x))/x^s\) is decreasing in \(s > 0\).

In [3], N. Elezović, C. Giordana, and J. Pečarić, among others, verified the convexity with respect to variable \(x\) of the function \([\Gamma(x+t)/\Gamma(x+s)]^{1/(t-s)}\) for \(|t-s| < 1\).

Recall that a function \(f\) is said to be completely monotonic on an interval \(I\) if \(f\) has derivatives of all orders on \(I\) which alternate successively in sign, that is

$$(-1)^{n} f^{(n)}(x) > 0$$  \(\text{for } x \in I \text{ and } n > 0\)  \(\text{if inequality (5) is strict for all } x \in I \text{ and for all } n \geq 0\), then \(f\) is said to be strictly completely monotonic.

Similarly, we give the following definition.

**Definition 1.** A function \(f\) is said to be logarithmically completely monotonic on an interval \(I\) if its logarithm \(\ln f\) satisfies

$$(-1)^{k} \left[\ln f(x)\right]^{(k)} > 0$$  \(\text{for } k \in \mathbb{N} \text{ on } I\). If inequality (6) is strict for all \(x \in I\) and for all \(k \geq 1\), then \(f\) is said to be strictly logarithmically completely monotonic.
In [13] it was established that the function \(1 + \frac{1}{x} \ln \Gamma(x + 1) - \ln(x + 1)\) is strictly completely monotone in \((-1, \infty)\) and tends to 1 as \(x \to -1\) and to 0 as \(x \to \infty\). This property is derived from the following integral representation:

\[
\ln \Gamma(x + 1) = x \ln(x + 1) - x + \int_0^\infty \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) e^{-t} \frac{1 - e^{-xt}}{t} dt.
\]  

In this short note, we are about to prove a complete monotonicity result of a function involving the gamma function as follows.

**Theorem 1.** A (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

**Theorem 2.** The function \(1 - \ln x + \frac{1}{x} \ln \Gamma(x + 1)\) is strictly completely monotonic on \((0, \infty)\) and tends to \(\infty\) as \(x \to 0\) and to 0 as \(x \to \infty\). Moreover, the function \(\sqrt{x} \frac{\Gamma(x + 1)}{x}\) is strictly logarithmically completely monotonic on \((0, \infty)\).

### 2. Proofs of theorems

**Proof of Theorem 1.** It is clear that \(\exp \phi(x) \geq 0\). Further, it is easy to see that \([\exp \phi(x)]' = \phi'(x) \exp \phi(x) \leq 0\) and \([\exp \phi(x)]'' = [\phi''(x) + [\phi'(x)]^2] \exp \phi(x) \geq 0\).

Suppose \((-1)^k[\exp \phi(x)]^{(k)} \geq 0\) for all nonnegative integers \(k \leq n\), where \(n \in \mathbb{N}\) is a given positive integer. By Leibnitz’s formula, we have

\[
(-1)^{n+1}[\exp \phi(x)]^{(n+1)} = (-1)^{n+1} \left[\left[\exp \phi(x)\right]^{(n)}\right] = (-1)^{n+1} \phi'(x) [\exp \phi(x)]^{(n)}
\]

\[
= (-1)^{n+1} \sum_{i=0}^{n} \binom{n}{i} [\phi^{(i+1)}(x)] [\exp \phi(x)]^{(n-i)}
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left[(-1)^{i+1} \phi^{(i+1)}(x)\right] \left[(-1)^{n-i} [\exp \phi(x)]^{(n-i)}\right] \geq 0.
\]  

By induction, it is proved that the function \(\exp \phi(x)\) is completely monotonic. \(\Box\)

**Proof of Theorem 2.** It has been shown in [5] that the function \([\Gamma(x + 1)]^{1/x}/x\) is strictly decreasing on \((0, \infty)\), then \(f(x) = 1 - \ln x + \frac{1}{x} \ln \Gamma(x + 1)\) is strictly decreasing on \((0, \infty)\). From the asymptotic expansion in [4]:

\[
\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \frac{1}{12x} + O(x^{-3})\quad \text{as} \quad x \to \infty,
\]

we conclude that \(\lim_{x \to \infty} f(x) = 0\) and \(\lim_{x \to 0} f(x) = \infty\). This implies \(f(x) > 0\) for \(x > 0\).
Using Leibnitz’ rule
\[
(u(x)v(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x),
\]
we obtain
\[
f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{x} \right)^{(n-k)} \left[ \ln \Gamma(x+1) \right]^{(k)} - \frac{(-1)^{n-1}(n-1)!}{x^n}
\]
\[
= \left( \frac{1}{x} \right)^{(n)} \ln \Gamma(x+1) + \sum_{k=1}^{n} \binom{n}{k} \left( \frac{1}{x} \right)^{(n-k)} \psi^{(k-1)}(x+1) + \frac{(-1)^{n}n!}{nx^n}
\]
\[
\triangleq (-1)^{n} \frac{n!}{x^{n+1}} g(x),
\]
and
\[
g'(x) = \frac{(-1)^{n}}{n!} x^n \psi^{(n)}(x+1) + \frac{1}{n}
\]
Using (3) and \((n-1)!/x^n = \int_{0}^{\infty} t^{n-1} e^{-xt} dt\) for \(x > 0\) and \(n \in \mathbb{N}\), we conclude
\[
\frac{1}{x^n} g'(x) = \frac{1}{n!} \int_{0}^{\infty} \left( 1 - \frac{t}{e^t - 1} \right) t^{n-1} e^{-xt} dt > 0,
\]
since \(0 < t/(e^t - 1) < 1\) for \(t > 0\). Thus, the function \(g\) is strictly increasing and \(g(x) > g(0) = 0\) on \((0, \infty)\), which implies \((-1)^{n} f^{(n)}(x) > 0\) for \(x > 0\) and \(n = 0, 1, 2, \ldots\)

The rest follows from Theorem 1. The proof is complete. □

**Remark.** It is worthwhile to point out that the integral form and completely monotonic property of the function \((bt - at)/t\) for \(t \in (-\infty, \infty)\) have been researched in [11]. The function \((bt - at)/t\) is important to the extended mean values [7,8].

**Acknowledgment**

The authors thank the anonymous referee for his/her many valuable suggestions to improve this manuscript.

**References**


