Physics Letters B 723 (2013) 182–189

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

$d \geqslant 5$ magnetized static, balanced black holes with $S^2 \times S^{d-4}$ event horizon topology

Burkhard Kleihaus, Jutta Kunz, Eugen Radu *

Institut für Physik, Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, Germany

ARTICLE INFO

Article history:
Received 20 March 2013
Accepted 24 April 2013
Available online 29 April 2013
Editor: M. Cvetič

ABSTRACT

We construct static, nonextremal black hole solutions of the Einstein–Maxwell equations in $d = 6, 7$ spacetime dimensions, with an event horizon of $S^2 \times S^{d-4}$ topology. These configurations are asymptotically flat, the $U(1)$ field being purely magnetic, with a spherical distribution of monopole charges but no net charge measured at infinity. They can be viewed as generalizations of the $d = 5$ static dipole black ring, sharing its basic properties, in particular the presence of a conical singularity. The magnetized version of these solutions is constructed by applying a Harrison transformation, which puts them into an external magnetic field. For $d = 5, 6, 7$, balanced configurations approaching asymptotically a Melvin universe background are found for a critical value of the background magnetic field.

1. Introduction

A remarkable property of black rings is the existence of regular configurations with gauge dipoles that are independent of all conserved charges. This strongly contrasts with the picture valid in $d = 4$ black hole physics, and implies a violation of the 'no hair' conjecture and of the black hole uniqueness. These aspects are clearly illustrated by the $d = 5$ black ring found by Emparan in [1], which was the first example of a black object that is asymptotically flat, possesses a regular horizon and is the source of a dipolar gauge field. This exact solution of the Einstein–Maxwell dilaton equations has an event horizon of $S^2 \times S^1$ topology. The $U(1)$ field is purely magnetic, being produced by a circular distribution of magnetic monopoles.[1] Then the ring creates a dipole field only, with no net charge measured at infinity.[2] Similar to the vacuum case [2], the generic dipole rings (in particular the static ones) are plagued by conical singularities. The balance is achieved for a critical (nonzero) value of the angular momentum only.

It is clear that the dipole ring solution in [1] should have generalizations in more than five dimensions. However, the analytic construction of these solutions seems to be intractable within a nonperturbative approach. Some progress in this direction has been achieved by using the blackfold approach. There the central assumption is that some black objects, in certain ultra-spinning regimes, can be approximated by very thin black strings or branes curved into a given shape, see [5–7]. Ref. [8] has found in this way generalizations of the dipole black ring for several topologies of the horizon, in particular for the ring case, $S^1 \times S^{d-3}$. However, the blackfold approach has some limitations; for example, black holes with no black membrane behaviour cannot be described within this framework.

A different approach for the construction of $d \geqslant 5$ black objects with a nonspherical topology of the horizon has been proposed in Refs. [9,10]. The solutions are found in this case nonperturbatively, by solving numerically the Einstein equations with suitable boundary conditions. A number of new solutions have been constructed in this manner, in particular recently Ref. [11] has given numerical evidence for the existence of balanced spinning vacuum black rings in $d \geqslant 6$ dimensions beyond the blackfold limit, and analyzed their basic properties.

In this work we propose to construct new static nonextremal black objects with an $S^2 \times S^{d-4}$ topology of the event horizon in $d = 6$ and 7 dimensions, by extending the results in [9] to the case of Einstein–Maxwell theory. These solutions can be viewed as higher dimensional generalizations of the $d = 5$ static dipole ring in [1], the magnetic field being analogous to a dipole, with no net charge measured at infinity. However, in the absence of rotation, these configurations have a conical singularity which provides the force balance that allows for their existence for any $d \geqslant 5$.

However, as discussed in [12], the conical singularity of the $d = 5$ static dipole ring can be removed by "immersing" it in a background gauge field. In this work we show that this holds for $d > 5$ solutions as well. By applying a magnetic Harrison

* Corresponding author.
E-mail address: eugen.radu@uni-oldenburg.de (E. Radu).

[1] The electric dual of these solutions can be considered as well, the ring being sourced in this case by an electric two-form potential.

[2] Note that, as discussed in [3,4], the dipole moment enters the first law of thermodynamics.

0370-2693/$ – see front matter © 2013 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.physletb.2013.04.053

transformation, the conical singularities disappear for a critical value of the background magnetic field. The resulting configurations describe $d>5$ balanced black holes with a horizon of $S^2\times S^{d-4}$ topology, in a Melvin universe background.

2. The model and general relations

2.1. The ansatz and equations

We consider the Einstein–Maxwell theory in d spacetime dimensions, defined by the following action

$$S=\frac{1}{16\pi}\int d^dx\sqrt{-g}\left(\mathcal{R}-\frac{1}{4}F^2\right), \qquad (2.1)$$

the corresponding equations of motion being

$$E_i^j=R_i^j-\frac{1}{2}\delta_i^jR-\frac{1}{2}\left(F_{ik}F^{jk}-\frac{1}{4}\delta_i^jF^2\right)=0,$$
$$\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}F^{ij}\right)=0. \qquad (2.2)$$

The solutions in this work are static and axisymmetric configurations, with a symmetry group $R_t\times U(1)\times SO(d-3)$ (where R_t denotes the time translation). Following Appendix C of [10], we take the following metric ansatz:

$$ds^2=f_1(r,\theta)\left(dr^2+r^2\,d\theta^2\right)+f_2(r,\theta)\,d\psi^2+f_3(r,\theta)\,d\Omega_{d-4}^2$$
$$-f_0(r,\theta)\,dt^2, \qquad (2.3)$$

where $d\Omega_{d-4}^2$ is the unit metric on S^{d-4}, the range of θ is $0\leqslant\theta\leqslant\pi/2$ and ψ is an angular coordinate, with $0\leqslant\psi\leqslant2\pi$. Also, r and t correspond to the radial and time coordinates, respectively. We shall see that for the solutions in this work, the range of r is $0<r_H\leqslant r<\infty$; thus the (r,θ)-coordinates have a rectangular boundary well suited for numerics.

For any value of d, the $U(1)$ potential has a single component,

$$A=A_\psi(r,\theta)\,d\psi. \qquad (2.4)$$

It is of interest to mention that the model admits a dual formulation, with an 'electric' version of (2.1), with

$$S=\frac{1}{16\pi}\int d^dx\sqrt{-g}\left(R-\frac{1}{2(d-2)!}\tilde{F}_{(d-2)}^2\right), \qquad (2.5)$$

where $\tilde{F}=\star F=dB$ is a $(d-2)$-form field strength (then the only nonvanishing components of the $(d-3)$-form potential B are $B_{\Omega t}$). However, in this work we shall restrict to the magnetic description within the Einstein–Maxwell theory.

An appropriate combination of the Einstein equations, $E_t^t=0$, $E_r^r+E_\theta^\theta=0$, $E_\psi^\psi=0$, and $E_\Omega^\Omega=0$, yields the following set of equations for the functions f_1, f_2, f_3 and f_0:

$$\nabla^2f_1-\frac{1}{f_1}(\nabla f_1)^2-(d-4)(d-5)\frac{f_1}{4f_3^2}(\nabla f_3)^2$$
$$-\frac{f_1}{2f_0f_2}(\nabla f_0)\cdot(\nabla f_2)-\frac{(d-4)f_1}{2f_0f_3}(\nabla f_0)\cdot(\nabla f_3)$$
$$-\frac{(d-4)f_1}{2f_2f_3}(\nabla f_2)\cdot(\nabla f_3)+\frac{(d-4)(d-5)f_1^2}{f_3}$$
$$+\frac{(d-4)f_1}{2(d-2)f_2}(\nabla A_\psi)^2=0,$$

$$\nabla^2f_2-\frac{1}{2f_2}(\nabla f_2)^2+\frac{1}{2f_0}(\nabla f_0)\cdot(\nabla f_2)+\frac{(d-4)}{2f_3}(\nabla f_2)\cdot(\nabla f_3)$$
$$+\frac{d-3}{d-2}(\nabla A_\psi)^2=0,$$
$$\nabla^2f_3+\frac{(d-6)}{2f_3}(\nabla f_3)^2+\frac{1}{2f_0}(\nabla f_0)\cdot(\nabla f_3)+\frac{1}{2f_2}(\nabla f_2)\cdot(\nabla f_3)$$
$$-2(d-5)f_1-\frac{f_3}{(d-2)f_2}(\nabla A_\psi)^2=0,$$
$$\nabla^2f_0-\frac{1}{2f_0}(\nabla f_0)^2+\frac{1}{2f_2}(\nabla f_0)\cdot(\nabla f_2)+\frac{(d-4)}{2f_3}(\nabla f_0)\cdot(\nabla f_3)$$
$$-\frac{f_0}{(d-2)f_2}(\nabla A_\psi)^2=0. \qquad (2.6)$$

From the Maxwell equations, it follows that the magnetic potential A_ψ is a solution of the equation

$$\nabla^2A_\psi+\frac{1}{2f_0}(\nabla f_0)\cdot(\nabla A_\psi)+\frac{1}{2f_2}(\nabla f_2)\cdot(\nabla A_\psi)$$
$$+\frac{(d-4)}{2f_3}(\nabla f_3)\cdot(\nabla A_\psi)=0. \qquad (2.7)$$

In the above relations, we have defined $(\nabla U)\cdot(\nabla V)=\partial_rU\partial_rV+\frac{1}{r^2}\partial_\theta U\partial_\theta V$, and $\nabla^2U=\partial_r^2U+\frac{1}{r^2}\partial_\theta^2U+\frac{1}{r}\partial_rU$.

The remaining Einstein equations $E_\theta^r=0$, $E_r^r-E_\theta^\theta=0$ yield two constraints. Following [13], we note that setting $E_t^t=E_\varphi^\varphi=E_r^r+E_\theta^\theta=0$ in the identities $\nabla_\mu E^{\mu r}=0$ and $\nabla_\mu E^{\mu\theta}=0$, we obtain the Cauchy–Riemann relations $\partial_\theta(\sqrt{-g}E_\theta^r)+\partial_{\bar r}(\sqrt{-g}\frac{1}{2}(E_r^r-E_\theta^\theta))=0$, $\partial_{\bar r}(\sqrt{-g}E_\theta^r)-\partial_\theta(\sqrt{-g}\frac{1}{2}(E_r^r-E_\theta^\theta))=0$ (with $r^2\partial/\partial r=\partial/\partial\bar r$). Thus the weighted constraints satisfy Laplace equations, and the constraints are fulfilled, when one of them is satisfied on the boundary and the other at a single point [13].

We close this part by remarking that the solutions in this work can also be studied by using Weyl-like coordinates, with $ds^2=\bar f_1(\rho,z)(d\rho^2+dz^2)+f_2(\rho,z)\,d\psi^2+f_3(\rho,z)\,d\Omega_{d-4}^2-f_0(\rho,z)\,dt^2$, and $A=A_\psi(\rho,z)\,d\psi$. The general transformation between (ρ,z)- and (r,θ)-coordinates is given in Ref. [10]. Indeed, the vacuum limit of the solutions in this work ($A_\psi\equiv0$) was studied in Ref. [9] by employing the (ρ,z)-coordinates. The metric ansatz (2.3) in terms of (r,θ) allows, however, for a better numerical accuracy.

2.2. Black holes with $S^2\times S^{d-4}$ topology of the event horizon

2.2.1. Boundary conditions

Eqs. (2.6) are solved subject to a set of boundary conditions which results from the requirement that the solutions describe asymptotically flat black objects with a regular horizon of $S^2\times S^{d-4}$ topology.[3] We assume that as $r\to\infty$, the Minkowski spacetime background (with $ds^2=dr^2+r^2(d\theta^2+\cos^2\theta\,d\psi^2+\sin^2\theta\,d\Omega_{d-4}^2)-dt^2$) is recovered, while the gauge potential vanishes. This implies

$$f_0|_{r=\infty}=1,\qquad f_1|_{r=\infty}=1,\qquad \lim_{r\to\infty}\frac{f_2}{r^2}=\cos^2\theta,$$
$$\lim_{r\to\infty}\frac{f_3}{r^2}=\sin^2\theta,\qquad A_\psi|_{r=\infty}=0. \qquad (2.8)$$

Also, we impose the existence of a nonextremal event horizon, which is located at a constant value of the radial coordinate, $r=r_H>0$. There we require

[3] In obtaining these conditions we are also guided by the $d=5$ exact solution discussed below.

$$f_0|_{r=r_H}=0,\qquad \partial_r f_1|_{r=r_H}=\partial_r f_2|_{r=r_H}=\partial_r f_3|_{r=r_H}=0,$$
$$\partial_r A_\psi|_{r=r_H}=0. \tag{2.9}$$

The boundary conditions at $\theta=\pi/2$ are

$$\partial_\theta f_0|_{\theta=\pi/2}=\partial_\theta f_1|_{\theta=\pi/2}=f_2|_{\theta=\pi/2}=\partial_\theta f_3|_{\theta=\pi/2}=0,$$
$$A_\psi|_{\theta=\pi/2}=0. \tag{2.10}$$

The absence of conical singularities requires also $r^2 f_1=f_2$ on that boundary.

The boundary conditions for $\theta=0$ are more complicated, since they encode the nontrivial topology of the horizon. The idea here is that for some interval $r_H\leqslant r<R$, we have for the metric the same conditions as for $\theta=\pi/2$, the asymptotic behaviour $f_2\sim\cos^2\theta$, $f_3\sim\sin^2\theta$ being recovered for $r>R$ (with $R>r_H$ an input parameter). Therefore, for $r_H<r<R$, we impose

$$\partial_\theta f_0|_{\theta=0}=\partial_\theta f_1|_{\theta=0}=f_2|_{\theta=0}=\partial_\theta f_3|_{\theta=0}=0,$$
$$A_\psi|_{\theta=0}=\Psi. \tag{2.11}$$

For $r>R$ we require instead

$$\partial_\theta f_0|_{\theta=0}=\partial_\theta f_1|_{\theta=0}=\partial_\theta f_2|_{\theta=0}=f_3|_{\theta=0}=0,$$
$$\partial_\theta A_\psi|_{\theta=0}=0. \tag{2.12}$$

Although the constants R, r_H which enter the above relations have no invariant meaning, they provide a rough measure for the radii of the S^{d-4} and S^2 spheres, respectively, on the horizon. Also, we shall see that the parameter Ψ fixes the local charge of the solutions.

2.2.2. Global quantities

The metric of a spatial cross section of the horizon is

$$d\sigma^2=f_1(r_H,\theta)r_H^2\,d\theta^2+f_2(r_H,\theta)\,d\psi^2$$
$$\qquad +f_3(r_H,\theta)\,d\Omega_{d-4}^2. \tag{2.13}$$

Since, from the above boundary conditions, the orbits of ψ shrink to zero at $\theta=0$ and $\theta=\pi/2$ while the area of S^{d-4} does not vanish anywhere, the topology of the horizon is $S^2\times S^{d-4}$ (in fact, for all nonextremal solutions in this work, $f_2(r_H,\theta)\sim\sin^2 2\theta$ while $f_1(r_H,\theta)$ and $f_3(r_H,\theta)$ are strictly positive and finite functions). The event horizon area is given by

$$A_H=2\pi r_H V_{d-4}\int_0^{\pi/2}d\theta\,\sqrt{f_1 f_2 f_3^{d-4}}\Big|_{r=r_H}, \tag{2.14}$$

where V_{d-4} is the area of the unit sphere S^{d-4}.

The Hawking temperature as computed from the surface gravity or by requiring regularity on the Euclidean section, is

$$T_H=\frac{1}{2\pi}\lim_{r\to r_H}\sqrt{\frac{f_0}{(r-r_H)^2 f_1}}=\frac{1}{\beta}, \tag{2.15}$$

where the constraint equation $E_r^\theta=0$ guarantees that the Hawking temperature is constant on the event horizon.

At infinity, the Minkowski background is approached. The total mass of the solutions is given by [14] (where the integral is taken over the $(d-2)$-sphere at spatial infinity and $k=\partial/\partial t$)

$$M=-\frac{(d-2)}{(d-3)}\frac{1}{16\pi}\oint_\infty dS_{ij}\,\nabla^i k^j, \tag{2.16}$$

and can be read from the asymptotic expression for f_0,

$$-g_{tt}=f_0\sim 1-\frac{16\pi GM}{(d-2)V_{d-2}}\frac{1}{r^{d-3}}+\cdots. \tag{2.17}$$

Using Gauss' theorem, the Einstein equations and the boundary conditions (2.8)–(2.12), one finds from (2.16) the following Smarr-type relation

$$(d-3)M=(d-2)\frac{1}{4}T_H A_H+\Phi\mathcal{Q}. \tag{2.18}$$

Here $\mathcal{Q}$ is the 'local' magnetic charge which enters the thermodynamics[4] as defined by evaluating the magnetic flux over the S^2 sphere around the horizon,

$$\mathcal{Q}=\frac{1}{4\pi}\int_{S^2}F_{\theta\psi}\,d\theta\,d\psi=-\frac{\Psi}{2}, \tag{2.19}$$

and Φ is the thermodynamical conjugate variable to $\mathcal{Q}$,

$$\Phi=\frac{1}{8\pi}\int_0^{2\pi}d\psi\int d\Omega_{d-4}\int_{r_H}^{R}dr\,\sqrt{-g}F^{\theta\psi}\Big|_{\theta=0}$$
$$=\frac{1}{4}V_{d-4}\int_{r_H}^{R}\frac{dr}{r}\sqrt{\frac{f_0 f_3^{d-4}}{f_2}}\partial_\theta A_\psi\Big|_{\theta=0}, \tag{2.20}$$

such that $\frac{1}{16\pi}\int F^2\sqrt{-g}\,d^{d-1}x=2\Phi\mathcal{Q}$. Therefore, following [15], we interpret the solutions as describing a spherical S^{d-4} distribution of monopole charges, though with a zero net charge (see also [16]).

As expected, in the absence of rotation, all these black objects with $S^2\times S^{d-4}$ horizon topology are plagued by conical singularities. As one can see from the boundary conditions, in this work we have chosen[5] to locate the conical singularity at $\theta=0$, $r_H<r<R$, where we find a conical excess

$$\delta=2\pi\left(1-\lim_{\theta\to 0}\frac{f_2}{\theta^2 r^2 f_1}\right)<0. \tag{2.21}$$

This can be interpreted as the higher dimensional analogue of a 'strut' (*e.g.* a membrane for $d=5$), preventing the collapse of the configurations. Although the presence of a conical singularity is an undesirable feature, it has been argued in [17,18], that such asymptotically flat black objects still admit a thermodynamical description. Moreover, when working with the appropriate set of thermodynamical variables, the Bekenstein–Hawking law still holds, while the parameter δ enters the first law of thermodynamics. Without going into details, we mention that the conjugate extensive variable to δ is

$$\mathcal{A}\equiv\frac{Area}{\beta}, \tag{2.22}$$

where *Area* is the spacetime area of the conical singularity's worldvolume. For the line element (2.3), the line element of the two dimensional surface spanned by the conical singularity is

$$d\sigma^2=-f_0\,dt^2+f_1\,dr^2+f_3\,d\Omega_{d-4}^2, \tag{2.23}$$

which implies

$$\mathcal{A}=V_{d-4}\int_{r_H}^{R}dr\,\sqrt{f_0 f_1 f_3^{d-4}}\Big|_{\theta=0}. \tag{2.24}$$

[4] The asymptotic behaviour of the magnetic potential is $A_\psi\to\mathcal{Q}_{(\infty)}\cos^2\theta/r^{d-3}$. However, $\mathcal{Q}_{(\infty)}$ does *not* enter any global law (this holds also for the $d=5$ balanced solution in [1]).

[5] It is also possible to work with the conical singularity stretching towards the boundary. However, in that case the spacetime will not be asymptotically flat.

3. The solutions

3.1. The $d = 5$ static dipole black ring

The static dipole black ring is usually written in ring or in Weyl coordinates, where it takes a relatively simple form. In what follows we shall write it within the ansatz (2.3), (2.4), which results in rather complicated expressions. However, this helps us to make contact with the numerical solutions found for $d > 5$.

In the (r, θ)-coordinates, the metric functions f_i in the line element (2.3) are given by (note that for $d = 5$, the sphere Ω_{d-4} reduces to a circle):

$$f_1(r,\theta) = c_2(r,\theta) f_1^{(0)}(r,\theta), \qquad f_2(r,\theta) = \frac{f_2^{(0)}(r,\theta)}{c_1^2(r,\theta)},$$
$$f_3(r,\theta) = c_1(r,\theta) f_3^{(0)}(r,\theta), \qquad f_0(r,\theta) = c_1(r,\theta) f_0^{(0)}(r,\theta), \tag{3.1}$$

where

$$c_1 = 1 - \frac{2(R^4 - r_H^4)^2}{R^4 r_H^2} \frac{w}{1+w} \frac{1}{P_{(-)}},$$
$$c_2 = \frac{1}{(1+w)^3}\left(1 + w\frac{R^4 + r_H^4}{2R^2 r_H^2}\frac{Q_{(-)}}{S_1}\right)\left(1 + w\frac{R^4 + r_H^4}{2R^2 r_H^2}\frac{Q_{(+)}}{S_1}\right)^2, \tag{3.2}$$

the magnetic potential (written in a gauge such that $A_\psi(\theta = \pi/2) = 0$) being

$$A_\psi = \sqrt{6}\sqrt{1 + \frac{R^2 - r_H^2}{2R^2 r_H^2}\frac{w}{1+w}}\,\frac{R^2 - r_H^2}{R^2 r_H}\frac{\sqrt{(1-w)w}}{1+w}\frac{S_{(+)}}{P_{(+)}}. \tag{3.3}$$

In the above relations we note

$$S_{(\pm)} = r^2 + \frac{r_H^4}{r^2} \pm \frac{(R^2 + r_H^2)^2}{R^2} + 2r_H^2 \cos 2\theta - R_4,$$
$$P_{(\pm)} = \frac{(r^2 \pm r_H^2)^2 (R^2 + r_H^2)^2}{r^2 r_H^2 R^2} - 2\left(1 + \frac{R^2 - r_H^2}{2R^2 r_H^2}\frac{w}{1+w}\right) S_{(\pm)},$$
$$Q_{(\mp)} = \frac{(r^2 \pm r_H^2)^2 (R^2 + r_H^2)^2}{r^2 (R^4 + r_H^4)} - S_{(\pm)},$$
$$S_1 = \frac{(r^2 - r_H^2)^2 (R^2 + r_H^2)^2}{2r^2 R^2 r_H^2} - S_{(-)}, \tag{3.4}$$

with $R_4 = \sqrt{(\frac{r_H^4 + R^4}{R^2} - \frac{r^4 + r_H^4}{r^2}\cos 2\theta)^2 + \frac{(r^4 - r_H^4)^2}{r^4}\sin^2 2\theta}$. Also, $f_i^{(0)}$ are the functions which enter the line element of the $d = 5$ static vacuum black ring, with

$$f_1^{(0)}(r,\theta) = \frac{1}{F_1(r,\theta)}, \qquad f_2^{(0)}(r,\theta) = r^2\frac{F_2(r,\theta)}{F_3(r,\theta)},$$
$$f_3^{(0)}(r,\theta) = r^2 F_3(r,\theta), \qquad f_0^{(0)}(r,\theta) = F_0(r), \tag{3.5}$$

and

$$F_0 = \left(\frac{r^2 - r_H^2}{r^2 + r_H^2}\right)^2,$$
$$F_1 = \frac{R_3}{(1 - \frac{r_H^2}{R^2})^2 (1 - \frac{r_H^2}{r^2})(1 + \frac{r_H^2}{r^2})^4} \times \left[\left(1 + \frac{r_H^4}{r^4}\right)\left(1 + \frac{r_H^4}{R^4}\right) - \frac{4r_H^4}{r^2 R^2}\cos 2\theta - \frac{2r_H^2}{R^2}R_3\right],$$
$$F_2 = \left(1 + \frac{r_H^2}{r^2}\right)^4 \sin^2\theta \cos^2\theta,$$
$$F_3 = \frac{1}{2}\left[R_3 + \frac{R^2}{r^2}\left(1 + \frac{r_H^4}{R^4} - \frac{r_H^2}{R^2}\left(\frac{r_H^2}{r^2} + \frac{r^2}{r_H^2}\right)\cos 2\theta\right)\right], \tag{3.6}$$

where $R_3 = \sqrt{(1 + \frac{R^4}{r^4} - \frac{2R^2}{r^2}\cos 2\theta)(1 + \frac{r_H^8}{r^4 R^4} - \frac{2r_H^4}{r^2 R^2}\cos 2\theta)}$.

This solution has three parameters, r_H, R (which were introduced in the previous section) and w, which is fixed by the value of the magnetic potential at $\theta = 0$, $r_H < r < R$ via (note that $0 \leqslant w < 1$):

$$\Psi = \frac{2}{R}\sqrt{\frac{w}{1 - w^2}}\sqrt{2R^2 r_H^2 + (R^4 + r_H^4)w}. \tag{3.7}$$

A direct computation shows that this is indeed a solution of the Einstein–Maxwell equations. Also, one can see that $c_1 \to 1$, $c_2 \to 1$ and $A_\psi \to 0$ as $w \to 0$, this corresponding to the vacuum black ring limit.

The computation of the quantities of interest for this solution is a straightforward application of the general formalism in Section 2.2.2. In the nonextremal case, one can write the following suggestive expressions:

$$M = M^{(0)}(1 + U), \qquad T_H = \frac{T_H^{(0)}}{(1 + U)^{3/2}},$$
$$A_H = A_H^{(0)}(1 + U)^{3/2}, \tag{3.8}$$

where

$$M^{(0)} = \frac{3\pi r_H^2}{4}, \qquad T_H^{(0)} = \frac{R^2 + r_H^2}{8\pi R r_H^2},$$
$$A_H^{(0)} = 4\pi^2 \frac{R r_H^4}{R^2 + r_H^2} \tag{3.9}$$

are the mass, temperature and area of the vacuum static black ring solution, and

$$\mathcal{Q} = \frac{2\sqrt{3} R r_H^2 \sqrt{U(1+U)}}{\sqrt{R^4 + r_H^4 - 2R^2 r_H^2 (1 + 2U)}},$$
$$\Phi = \frac{\sqrt{3}\pi}{2R}\frac{\sqrt{U(R^4 + r_H^4 - 2R^2 r_H^2(1 + 2U))}}{\sqrt{1 + U}}, \tag{3.10}$$

with $0 \leqslant U < (R^2 - r_H^2)^2/(4R^2 r_H^2)$ a free parameter.[6] The static dipole rings have a conical excess

$$\delta = 2\pi\left[1 - \frac{R^2 + r_H^2}{R^2 - r_H^2}\left(1 + \frac{4R^2 r_H^2 U}{R^4 + r_H^4 - 2R^2 r_H^2(1 + 2U)}\right)^{3/2}\right], \tag{3.11}$$

while the expression of the corresponding conjugate extensive variable $\mathcal{A}$ cannot be written in closed form.

The basic properties of the $d = 5$ nonextremal solution turn out to be generic and will be discussed in the following subsection. Here we mention only that the extremal solutions are found by taking the limit $r_H \to 0$ in the relations (3.1)–(3.6). They have a relatively simple form

[6] The relation between w and U is $w = 2R^2 r_H^2 U/((R^2 - r_H^2)^2 - 2R^2 r_H^2 U)$.

$$f_1=\frac{(r^2+wT_{(-)})(r^2+wT_{(+)})^2}{(1+w)^4r^4R_1},$$
$$f_2=\frac{2r^4\sin^2\theta\cos^2\theta(r^2+wT_{(+)})^2}{(T_+-r^2\cos2\theta)(r^2+wT_{(+)})^2},$$
$$f_3=\frac{(T_{(+)}-r^2\cos2\theta)(r^2+wT_{(-)})}{2(r^2+wT_{(+)})},$$
$$f_0=\frac{r^2+wT_{(-)}}{r^2+wT_{(+)}},\qquad A_\psi=Rw\sqrt{3}\sqrt{\frac{1-w}{1+w}}\frac{r^2-T_{(-)}}{r^2+wT_{(+)}},\tag{3.12}$$

with $T_{(\pm)}=R_1\pm R^2$, $R_1=\sqrt{r^4+R^4-2r^2R^2\cos2\theta}$ and $w=\mathcal{Q}/\sqrt{\mathcal{Q}^2+3R^2}$. The horizon of the extremal solutions has zero area, since the length of the S^1 direction vanishes there, $g_{\psi\psi}\to 0$. Their mass and potential are given by $M=\frac{3\pi\mathcal{Q}R^2}{4(\mathcal{Q}+\sqrt{\mathcal{Q}^2+3R^2})}$, $\Phi=\frac{3\pi R^2}{2(\mathcal{Q}+\sqrt{\mathcal{Q}^2+3R^2})}$, their conical excess is $\delta=-\frac{4\pi(4\mathcal{Q}^3+9R^2)}{(-\mathcal{Q}+\sqrt{\mathcal{Q}^2+3R^2})^3}$.

3.2. $d=6,7$ *numerical solutions*

3.2.1. *Remarks on the numerics*

Higher dimensional generalizations of the $d=5$ nonextremal solution (3.1)–(3.6) are found by replacing in the five dimensional line element the S^1 direction which is not associated with the magnetic potential, with the line element of a round $(d-4)$-sphere, while preserving at the same time the basic properties of the metric functions and of the magnetic potential.

Since no closed form solution is available in this case, the set of five coupled nonlinear elliptic partial differential equations (2.6), (2.7) is solved numerically, subject to the boundary conditions (2.8)–(2.12).

The numerical scheme we have used is identical with that described at length in [10] and thus we shall not enter into details. We mention only that in practice we have worked with a set of 'auxiliary' functions $\mathcal{F}_i$ defined via[7]

$$f_0=f_0^{(0)}e^{\mathcal{F}_0},\qquad f_1=f_1^{(0)}e^{\mathcal{F}_1},\qquad f_2=f_2^{(0)}e^{\mathcal{F}_2},$$
$$f_3=f_3^{(0)}e^{\mathcal{F}_3},\tag{3.13}$$

where $f_i^{(0)}$ are 'background' functions corresponding to the $d=5$ static vacuum black ring as given by (3.5). These 'background' functions $f_i^{(0)}$ are used to fix the topology of the horizon and to 'absorb' the coordinate divergences of the functions f_i. The 'auxiliary' functions $\mathcal{F}_i$ are smooth and finite everywhere such that they do not lead to the occurrence of new zeros of the functions f_i (therefore the rod structure of the solutions remains fixed by $f_i^{(0)}$ [10]). However, $\mathcal{F}_i$ encode the effects of changing the spacetime dimension from $d=5$ and also of introducing the local charge $\mathcal{Q}$.

In our approach, the input parameters are the value d of the spacetime dimension, the event horizon radius r_H, the radius R of the S^{d-4}-sphere, and the value of the local charge $\mathcal{Q}$ (*i.e.* the parameter Ψ in the boundary conditions (2.11)). The physical parameters are encoded in the values of the functions f_i (and their derivatives) on the boundary of the integration domain. For example, the mass parameter M is computed from the asymptotic form (2.17) of the metric function $g_{tt}=-f_0$, the Smarr relation (2.18) being used to verify the accuracy of the solutions.

[7] Note that this procedure has some similarities with the construction of distorted black holes [19]. However, in our case, the field equations do not reduce to simple Laplace equations.

3.2.2. *Properties of the solutions*

To obtain nonextremal Einstein–Maxwell solutions with $S^2\times S^{d-4}$ horizon topology, one starts with the vacuum configurations in [9] and turns on the parameter Ψ which enters the boundary conditions for the magnetic potential. The iterations converge, and, in principle, repeating the procedure it is possible to obtain solutions with arbitrary values of $\mathcal{Q}$.

We have started with a test of the numerical scheme, by recovering in this way the $d=5$ static dipole black rings. Afterwards, new solutions in $d=6,7$ dimensions have been studied in a systematic way. Solutions with $d>7$ should also exist; however, we did not try to find them and their study may require a different numerical method. We mention that, for all solutions, we have verified that the Kretschmann scalar stays finite everywhere.[8]

The central result in this work is that the $d=5$ static nonextremal dipole ring has higher dimensional generalizations with an $S^2\times S^{d-4}$ horizon topology. Moreover, the properties of the five dimensional solutions are generic, being recovered for $d>5$.

Let us start with a discussion of the solutions' features for a fixed value of the magnetic charge $\mathcal{Q}$. Perhaps the most important feature is that all $d\geqslant 5$ solutions have conical singularities. Thus we have found it convenient to take the relative conical excess $\delta/(\delta-2\pi)$ as the control parameter and to consider the following dimensionless quantities,[9] the scale being fixed here by M:

$$a_H=p_1\frac{A_H}{M^{\frac{d-2}{d-3}}},\qquad t_H=p_2T_HM^{\frac{1}{d-3}},\qquad a_\delta=\frac{1}{V_{d-4}}\frac{\mathcal{A}}{M},$$
$$\varphi=\frac{\Phi}{M^{\frac{d-4}{d-3}}},\tag{3.14}$$

with $p_1=((\frac{d-2}{16\pi})^{d-2}V_{d-2})^{\frac{1}{d-3}}$, $p_2=\frac{1}{d-3}(\frac{2^{2(d-1)}\pi^{d-2}}{(d-2)V_{d-2}})^{\frac{1}{d-3}}$ two coefficients which have been chosen such that $a_H=1$, $t_H=1$ corresponds to the Schwarzschild–Tangherlini black hole.

In terms of the dimensionless ratio r_H/R, the solutions interpolate between two limits (although these regions of the parameter space are difficult to approach numerically). For $R\to\infty$ and r_H, $\mathcal{Q}$ nonvanishing, the radius on the horizon of the S^{d-4}-sphere increases and asymptotically it becomes a $(d-4)$-plane, while $\delta\to 0$. After a suitable rescaling,[10] one finds the magnetically charged black brane solution

$$ds^2=H^2(r)U_1(r)\big[dr^2+r^2\big(4\,d\theta^2+\sin^2 2\theta\,d\psi^2\big)\big]$$
$$\qquad+\frac{1}{(H(r))^{\frac{2}{d-3}}}\big(dx_1^2+\cdots+dx_{d-4}^2-U_0(r)\,dt^2\big),$$
$$A=-\mathcal{Q}(1+\cos2\theta)\,d\psi,\tag{3.15}$$

where

$$H(r)=\frac{1}{(r+r_H)^2}\left(r^2+r_H^2+2rr_H\sqrt{1+\frac{d-3}{8(d-2)}\frac{\mathcal{Q}^2}{r_H^2}}\right),$$
$$U_1(r)=\left(1+\frac{r_H}{r}\right)^4,\qquad U_0(r)=\left(\frac{r-r_H}{r+r_H}\right)^2.\tag{3.16}$$

This corresponds to a magnetically charged Reissner–Nordström black hole uplifted to d dimensions, *i.e.* with $(d-4)$ flat directions.

[8] Here we ignore the δ-Dirac terms in the expression of the Riemann tensor in the presence of a conical singularity. In fact, the presence of a conical singularity has a rather neutral effect on the numerics, since the solver does not notice it directly.

[9] Note that in a numerical approach it is rather difficult to work with dimensionless 'reduced' quantities in a systematic way, since, except for $\mathcal{Q}$, it is not possible to fix any other quantity which enters the Smarr relation and the first law.

[10] For the $d=5$ exact solution, this rescaling is $r\to\sqrt{2Rr}$, $r_H\to\sqrt{2Rr_H}$, $w\to w/R$.

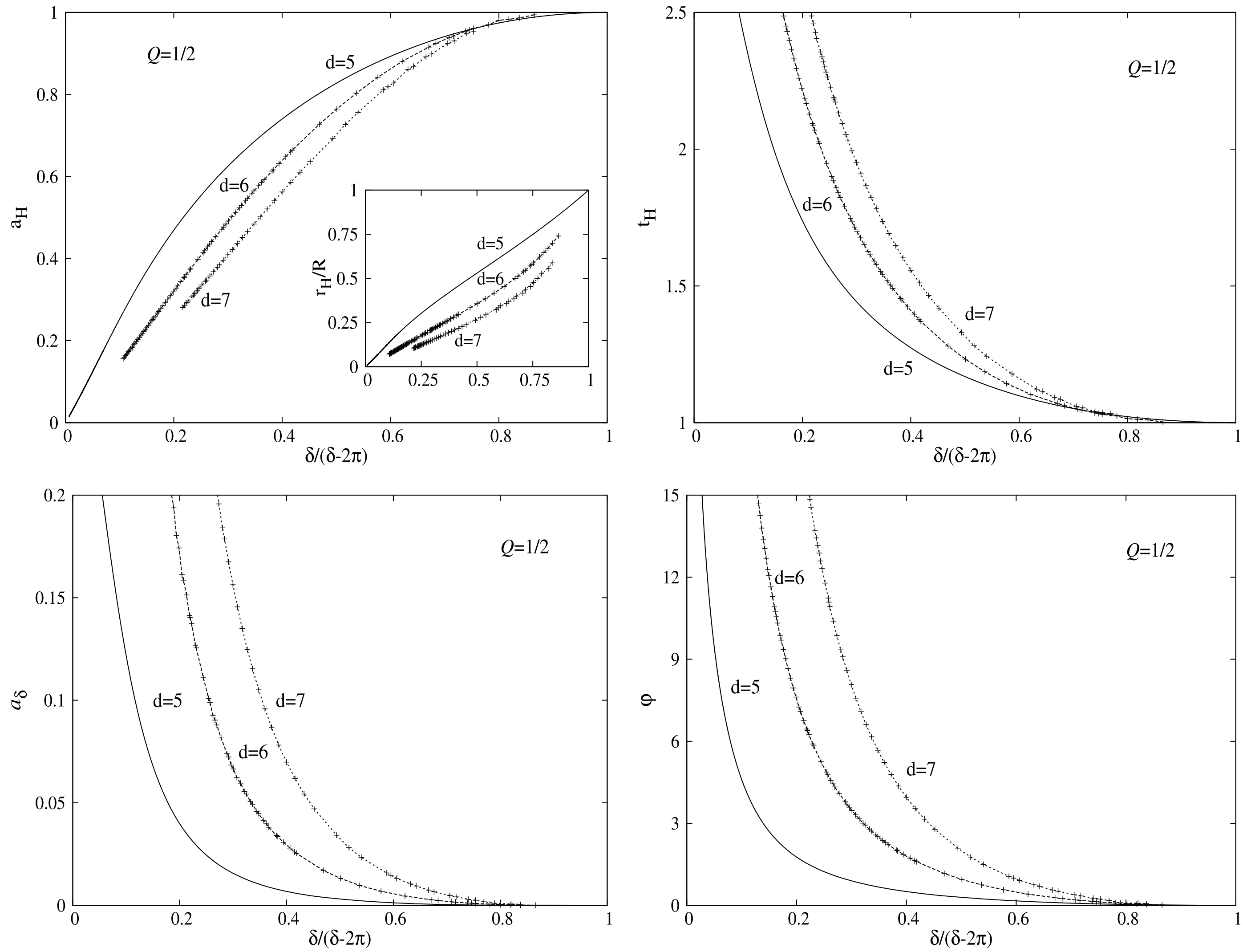

Fig. 1. A number of quantities are shown as functions of the relative angular excess $\delta/(\delta - 2\pi)$ for black hole solutions with the same local charge $\mathcal{Q}$.

The limit $r_H/R \to 1$ is somehow more subtle, since the conical excess diverges, $\delta \to -\infty$, and the magnetic field vanishes. As can be seen in Fig. 1, the Schwarzschild–Tangherlini black hole with an S^{d-2} horizon topology is recovered in this limit. This can be understood by studying the $d = 5$ exact solution. There, as $R \to r_H$ one finds $c_1 \to 1 + O(R - r_H)$, $c_2 \to 1 + O(R - r_H)$ while $A_\psi \sim O(R - r_H)^2$ (*i.e.* a vanishing charge), with the limiting expressions $f_1 = f_2/(r^2 \cos^2\theta) = f_3/(r^2 \sin^2\theta) = (1 + r_H^2/r^2)^2$, $f_0 = (r^2 - r_H^2)^2/(r^2 + r_H^2)^2$.

Some results illustrating these aspects are shown in Fig. 1 (note that we have found similar results for other values of $\mathcal{Q}$ as well).

A different situation which can be studied numerically is to keep fixed the radii r_H and R and to vary the value of the local charge $\mathcal{Q}$. Interestingly, turning on a magnetic field increases the absolute value of the conical excess, see Fig. 2 (left). For fixed r_H, R, the values of the magnetic potential, horizon area, the parameter $\mathcal{A}$ and the mass increase with $\mathcal{Q}$, while the temperature decreases.

It seems that similar to the $d = 5$ case, the extremal solutions are found in the limit $r_H \to 0$, for nonvanishing R and $\mathcal{Q}$. However, we could not approach this limit and the numerical construction of the extremal solutions would require a different numerical scheme, with another set of 'background' functions. This holds also for the $d = 5$ solutions, in which case it can be understood by noticing that the behaviour of the metric functions f_1, f_3 as $r \to r_H$ (*i.e.* $f_1 \sim 1/r^2$, $f_2 \sim r^2$) is not compatible with the boundary conditions (2.9). We conjecture that the picture found for $d = 5$ is generic and the extremal solutions will always possess a horizon with vanishing area.

4. Balanced black holes with $S^2 \times S^{d-4}$ event horizon topology in a Melvin universe background

The occurrence of conical singularities is not an unusual feature in general relativity. However, sometimes this pathology can be cured by placing the solutions in an external field (see *e.g.* [20–22]). This was the case for the $d = 5$ static dipole ring [12] and also for extremal solutions in [15], which could be balanced by "immersing" them in a background gauge field, via a magnetic Harrison transformation. Unsurprisingly, this works also for the configurations considered in this work.

The magnetic Harrison transformation can be summarized as follows (see *e.g.* [23]). Let us consider a solution of the Einstein–Maxwell equations of the form

$$ds^2 = g_{yy}\, dy^2 + d\sigma_{d-1}^2, \qquad A = A_y\, dy, \tag{4.1}$$

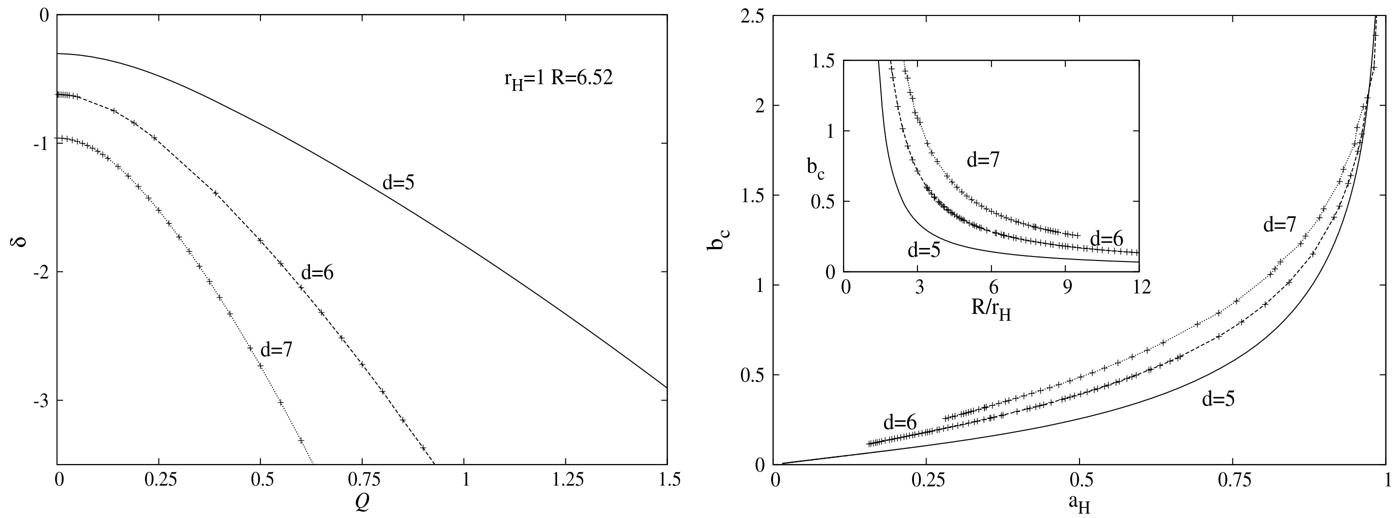

Fig. 2. *Left*: The angular excess δ is shown as a function of the local charge for solutions with the same values of r_H, R. *Right*: The dimensionless critical magnetic field $b_c = B\mathcal{Q}$ is shown as a function of the dimensionless event horizon area for balanced black holes with $S^2 \times S^{d-4}$ horizon topology in a Melvin universe background. The inset shows b_c as a function of the ratio R/r_H.

with $\partial/\partial y$ a Killing vector. Then the configuration

$$ds^2 = \frac{1}{\Lambda^2} g_{yy}\, dy^2 + \Lambda^{\frac{2}{d-3}}\, d\sigma^2_{d-1},$$
$$A = \frac{1}{\Lambda}\left[A_y + B\left(g_{yy} + \frac{d-3}{2(d-2)} A_y^2\right)\right] dy, \tag{4.2}$$

with

$$\Lambda = \left(1 + \frac{d-3}{2(d-2)} B A_y\right)^2 + \frac{d-3}{2(d-2)} B^2 g_{yy}, \tag{4.3}$$

solves also the Einstein–Maxwell equations (with B an arbitrary parameter).

The Harrison transformation (4.2) applied with respect to the Killing vector $\partial/\partial\psi$ results in the following line element

$$ds^2 = \Lambda^{\frac{2}{d-3}}\left(f_1\left(dr^2 + r^2\, d\theta^2\right) + f_3\, d\Omega^2_{d-4} - f_0\, dt^2\right) + \frac{1}{\Lambda^2} f_2\, d\psi^2,$$
$$\text{with } \Lambda = \left(1 + \frac{d-3}{2(d-2)} B A_\psi\right)^2 + \frac{d-3}{2(d-2)} B^2 f_2, \tag{4.4}$$

and the new magnetic potential

$$A'_\psi = \frac{1}{\Lambda}\left[A_\psi + B\left(f_2 + \frac{d-3}{2(d-2)} A_\psi^2\right)\right]. \tag{4.5}$$

One can see that the new line element preserves some of the basic properties of the $B = 0$ seed configuration. The horizon is still located at $r = r_H$ and has an $S^2 \times S^{d-4}$ topology, since the qualitative behaviour of the metric functions at $\theta = 0, \pi/2$ remains unchanged (note that $\Lambda > 0$ everywhere). However, the geometry is distorted and the asymptotic behaviour is very different. As $r \to \infty$, the solution becomes

$$ds^2 = \Lambda^{\frac{2}{d-3}}\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\Omega^2_{d-4}\right) - dt^2\right) + \frac{r^2\cos^2\theta}{\Lambda^2}\, d\psi^2,$$
$$A_\psi = \frac{Br^2\cos^2\theta}{\Lambda}, \quad \text{with } \Lambda = 1 + \frac{d-3}{2(d-2)} B^2 r^2 \cos^2\theta,$$

which is a higher dimensional generalization of the $d = 4$ Melvin magnetic universe [24]. A direct calculation shows that the horizon area and the temperature of the new solutions (4.4), (4.5) are not affected by the external magnetic field, coinciding with the corresponding quantities of the $B = 0$ seed configurations.

Moreover, by employing the same approach as in [25,12], it is straightforward to show that the mass of the new solutions, as defined with respect to the Melvin universe background, still preserves the expression found in the asymptotically flat case.[11]

The configurations with generic values of B possess again a conical singularity at $\theta = 0$, $r_H < r < R$. However, this conical singularity vanishes for a critical value of the magnetic field,

$$B_c = \frac{1}{\mathcal{Q}} \frac{4(d-2)}{(d-3)}\left(1 - \left(1 - \frac{\delta}{2\pi}\right)^{\frac{d-3}{2(d-2)}}\right). \tag{4.6}$$

The dimensionless quantity $b_c = B_c\mathcal{Q}$ is shown in Fig. 2 (right) as a function of the parameters a_H and R/r_H for $d = 5, 6, 7$ solutions. One can see that b_c diverges as the Schwarzschild limit is approached.

5. Conclusions

In this work we have shown numerical evidence that the vacuum static black holes with $S^2 \times S^{d-4}$ horizon topology discussed in [9] admit nonextremal generalizations in Einstein–Maxwell theory. These new solutions have a dipolar magnetic field, which is created by a spherical S^{d-4} distribution of monopoles. They also share the basic properties of the $d = 5$ static dipole ring and possess conical singularities, which, in the absence of rotation, prevent the black objects to collapse. Of course, on general grounds, one expects the $d > 5$ new solutions in this work to possess rotating generalizations and thus to achieve balance for a critical value of the angular momentum. Unfortunately, the explicit construction of such solutions proves a very difficult numerical problem, see the discussion in [10].

However, as discussed in the second part of this work, these static black objects with an $S^2 \times S^{d-4}$ topology of the horizon can be held in equilibrium by switching on a magnetic field with an appropriate strength. To the best of our knowledge, this is the

[11] The fact that the thermodynamics of a magnetized static black hole is not affected by the presence of the background magnetic field has also been noticed for other solutions, see *e.g.* [25,12,22].

first explicit construction of $d > 5$ static and balanced black objects which are regular on and outside an event horizon of nonspherical topology.[12] However, the magnetic field does not vanish asymptotically, such that the background spacetime corresponds in this case to a d dimensional Melvin universe. Therefore the construction of asymptotically flat, static balanced black objects with a nonspherical horizon topology remains an open problem.

Our preliminary results indicate that the solutions in this work can be generalized to include a dilaton. In this ways, they could be uplifted to higher dimensions and interpreted in a string theory context. Moreover, we expect that all static configurations with a nonspherical horizon topology discussed in [10] would admit generalizations with a dipolar magnetic field. Although the asymptotically flat static solutions will possess conical singularities, the interaction with an external magnetic field would balance them.

Acknowledgements

We gratefully acknowledge support by the DFG, in particular within the DFG Research Training Group 1620 "Models of Gravity".

References

[1] R. Emparan, JHEP 0403 (2004) 064, arXiv:hep-th/0402149.
[2] R. Emparan, H.S. Reall, Phys. Rev. Lett. 88 (2002) 101101, arXiv:hep-th/0110260.
[3] K. Copsey, G.T. Horowitz, Phys. Rev. D 73 (2006) 024015, arXiv:hep-th/0505278.
[4] D. Astefanesei, E. Radu, Phys. Rev. D 73 (2006) 044014, arXiv:hep-th/0509144.
[5] R. Emparan, T. Harmark, V. Niarchos, N.A. Obers, M.J. Rodriguez, JHEP 0710 (2007) 110, arXiv:0708.2181 [hep-th].
[6] R. Emparan, T. Harmark, V. Niarchos, N.A. Obers, Phys. Rev. Lett. 102 (2009) 191301, arXiv:0902.0427 [hep-th].
[7] R. Emparan, T. Harmark, V. Niarchos, N.A. Obers, JHEP 1003 (2010) 063, arXiv:0910.1601 [hep-th].
[8] M.M. Caldarelli, R. Emparan, B. Van Pol, JHEP 1104 (2011) 013, arXiv:1012.4517 [hep-th].
[9] B. Kleihaus, J. Kunz, E. Radu, Phys. Lett. B 678 (2009) 301, arXiv:0904.2723 [hep-th].
[10] B. Kleihaus, J. Kunz, E. Radu, M.J. Rodriguez, JHEP 1102 (2011) 058, arXiv:1010.2898 [gr-qc].
[11] B. Kleihaus, J. Kunz, E. Radu, Phys. Lett. B 718 (2013) 1073, arXiv:1205.5437 [hep-th].
[12] M. Ortaggio, JHEP 0505 (2005) 048, arXiv:gr-qc/0410048.
[13] T. Wiseman, Class. Quant. Grav. 20 (2003) 1137, arXiv:hep-th/0209051.
[14] R.C. Myers, M.J. Perry, Ann. Phys. 172 (1986) 304.
[15] R. Emparan, Nucl. Phys. B 610 (2001) 169, arXiv:hep-th/0105062.
[16] F. Dowker, J.P. Gauntlett, G.W. Gibbons, G.T. Horowitz, Phys. Rev. D 53 (1996) 7115, arXiv:hep-th/9512154.
[17] C. Herdeiro, B. Kleihaus, J. Kunz, E. Radu, Phys. Rev. D 81 (2010) 064013, arXiv:0912.3386 [gr-qc].
[18] C. Herdeiro, E. Radu, C. Rebelo, Phys. Rev. D 81 (2010) 104031, arXiv:1004.3959 [gr-qc].
[19] R.P. Geroch, J.B. Hartle, J. Math. Phys. 23 (1982) 680.
[20] R. Emparan, Phys. Rev. D 61 (2000) 104009, arXiv:hep-th/9906160.
[21] S.S. Yazadjiev, Phys. Rev. D 73 (2006) 064008, arXiv:gr-qc/0511114.
[22] S.S. Yazadjiev, Phys. Rev. D 77 (2008) 127501, arXiv:0802.0784 [hep-th].
[23] D.V. Gal'tsov, O.A. Rytchkov, Phys. Rev. D 58 (1998) 122001, arXiv:hep-th/9801160.
[24] M.A. Melvin, Phys. Lett. 8 (1964) 65.
[25] E. Radu, Mod. Phys. Lett. A 17 (2002) 2277, arXiv:gr-qc/0211035.

[12] The topological black holes in anti-de Sitter spacetime are qualitatively a different class of black objects.