Multi-dimensional backward stochastic differential equations with one reflecting lower barrier of Itô diffusion type

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Abstract

This paper investigates a class of multi-dimensional stochastic differential equations with one reflecting lower barrier (RBSDEs in short), where the random obstacle is described as an Itô diffusion type of stochastic differential equation. The existence and uniqueness results for adapted solutions to such RBSDEs are established based on a penalization scheme and some higher moment estimates for solutions to penalized BSDEs under the Lipschitz condition and a higher moment condition on the coefficients. Finally, two examples are given to illustrate our theory and their applications.

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Keywords: Reflected backward stochastic differential equations; Adapted solution; Penalization; Higher moment estimates

1. Introduction

In [1], El Karoui et al. discussed a class of reflected backward stochastic differential equations (RBSDEs in short) with one lower barrier. A solution of such an equation is forced to stay above a given random barrier, which is a continuous progressively measurable real-valued process. They proved the existence and uniqueness of the solution result both by the Snell envelope theory and by the penalization argument. Since then RBSDEs have been studied well by many authors.

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We here mention Matoussi [11], Ma and Zhang [9], Hamadène and Lepeltier [4], Lepeltier and Xu [7], Xu [12], and Li and Tang [8], who investigated one-dimensional RBSDEs driven by a Brownian motion. We also mention Hamadène and Ouknine [5], Hamadène and Hassani [6], Essaky [3], who studied one-dimensional RBSDEs with jumps, that is, the RBSDEs are driven by a Brownian motion and an independent Poisson point process.

In above papers, the authors only deal with one-dimensional RBSDEs. They, generally, using the penalization scheme or the Snell envelope theory, show the existence and uniqueness of adapted solutions for that RBSDE under appropriate assumptions. The penalization scheme is heavily based on the comparison theorem on BSDEs, while the Snell envelope theory is only used to study one-dimensional RBSDEs. However, for the multi-dimensional RBSDEs case, the comparison theorem and the Snell envelope theory do not work. So a natural question is: how to prove the existence of a solution to a multi-dimensional RBSDE with one lower barrier?

The aim of this paper is to study a class of multi-dimensional reflected backward stochastic differential equations driven by an \( r \)-dimensional Brownian motion. One of the components for these RBSDEs is reflected and the others are just standard BSDEs. The random obstacle is an Itô diffusion type of stochastic differential equation. By using a penalization scheme and higher moment estimates for solutions to penalized BSDEs, we show the existence and uniqueness result for adapted solutions to such RBSDEs with Lipschitzian coefficients. However, the terminal value and the coefficient in such an equation are imposed a bit restrictive conditions (see, (H2)–(H3)) in order to overcome the difficulty that we are unable to apply the comparison theorem on BSDEs.

This paper is organized as follows: Section 2 contains some notations and the definition of an adapted solution to an RBSDE with one lower barrier. Some assumptions on RBSDEs and the random barrier are also given. In Section 3, we give some necessary lemmas, which will be used to prove the existence and uniqueness of solutions to RBSDE (2.1). Among these, Lemma 3.2 gives a crucial estimate. Section 4 devotes to showing the existence and uniqueness of solutions to RBSDE (2.1) under Lipschitzian coefficient. The first theorem is about the existence of a solution to an RBSDE whose coefficient does not depend on \((Y, Z)\). Then we construct a contractive mapping to derive the existence result of a solution to Eq. (2.1). In the last section, we give two examples to illustrate our theory and their applications.

### 2. Reflected BSDEs with one lower barrier

Throughout this paper, we assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is a complete probability space on which an \( r \)-dimensional Brownian motion \( B = (B_t)_t \geq 0 \) is defined, where \( T \) is a positive constant. Let \( \mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T} \) be the natural filtration generated by \( B \), augmented by the \( \mathbb{P} \)-null sets of \( \mathcal{F} \), hence \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) satisfies the usual conditions. We denote by \( \mathcal{P} \) the \( \sigma \)-algebra of progressively measurable sets on \([0, T] \times \Omega\). We always use \(| \cdot |\) to denote the Euclidean norm for vectors or the trace norm for matrices. The following spaces will be used in this paper.

- **\( L^p \):** the set of \( \mathcal{F}_T \)-measurable variables \( \xi : \Omega \to \mathbb{R}^d \) with \( \mathbb{E}[|\xi|^p] < \infty \), \( p \geq 2 \);
- **\( H^2 \):** the set of \( \mathcal{P} \)-measurable processes \( \varphi : [0, T] \times \Omega \to \mathbb{R}^{d \otimes r} \) with \( \mathbb{E}\int_0^T |\varphi(t)|^2 \, dt < \infty \);
- **\( S^2 \):** the set of \( \mathcal{P} \)-measurable processes \( \psi : [0, T] \times \Omega \to \mathbb{R}^d \) with \( \mathbb{E}(|\sup_{0 \leq t \leq T} |\psi(t)|^2) < \infty \);
- **\( A^2 \):** the set of continuous \( \mathcal{P} \)-measurable increasing processes \( K : [0, T] \times \Omega \to \mathbb{R}_+ := [0, \infty) \) with \( K_0 = 0, \mathbb{E}[K_T]^2 < \infty \).
This paper will study a backward stochastic differential equation with one Itô diffusion type of reflecting lower barrier as follows:

\[
\begin{cases}
Y^i_t = \xi^i + \int_t^T f^i(s, Y_s, Z_s, \omega) \, ds - \sum_{j=1}^r \int_t^T Z^i_{sj} \, dB^j_s, & i = 1, \ldots, d - 1, \\
Y^d_t = \xi^d + \int_t^T f^d(s, Y_s, Z_s, \omega) \, ds - \sum_{j=1}^r \int_t^T Z^d_{sj} \, dB^j_s + K_T - K_t, \\
L_t \leq Y^d_t, & \int_0^T (Y^d_t - L_t) \, dK_t = 0, \quad 0 \leq t \leq T,
\end{cases}
\]  
(2.1)

and

\[
L_t = L_0 + \int_t^T b(s, L_s) \, ds + \int_0^T \sigma(s, L_s) \, dB_s, \quad L_0 \in \mathbb{R}, \quad 0 \leq t \leq T,
\]  
(2.2)

where \( f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d+1} \times \mathcal{F} \to \mathbb{R}^d \) is generally called a coefficient, which is \( \mathcal{F} \times \mathcal{B}(\mathbb{R}^{d+1}) \)-measurable. The process \( K \) is a continuous increasing processes. The state process \( Y^d \) is forced to stay above the lower barrier \( L \) with a minimal way in the sense of \( \int_0^T (Y^d_t - L_t) \, dK_t = 0 \). For simplicity, the RBSDE (2.1) can be rewritten as

\[
\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \omega) \, ds - \int_t^T Z_s \, dB_s + \alpha(K_T - K_t), \\
L_t \leq Y^d_t, & \int_0^T (Y^d_t - L_t) \, dK_t = 0, \quad 0 \leq t \leq T,
\end{cases}
\]  
(2.3)

where \( \alpha = (0, 0, \ldots, 0, 1) \in \mathbb{R}^d \).

Let us first give the definition of a solution to the RBSDE (2.1).

**Definition 2.1.** A triple \((Y, Z, K) = (Y_t, Z_t, K_t)_{0 \leq t \leq T}\) of processes with values in \(\mathbb{R}^d \times \mathbb{R}^{d+1} \times \mathbb{R}_+\) is called a solution to Eq. (2.1), if and only if \((Y, Z, K)\) belongs to \(\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{A}^2\) and satisfies the RBSDE (2.1).

Traditionally, an RBSDE with one lower barrier corresponds to the RBSDE (2.1) with \(d = 1\) case, that is, \(Y\) is a real-valued continuous process. In our framework, since \(Y\) is an \(\mathbb{R}^d\)-valued process, neither the comparison theorem on BSDEs nor the Snell envelope theory does work. In order to overcome these difficulties, we here assume that the lower barrier is described as the solution to an Itô diffusion type of SDE. We assume:

(H0) The functions \(b\) and \(\sigma\) satisfy the local Lipschitz condition and the linear growth condition, that is, for each \(k = 1, 2, \ldots\), there is a \(c_k > 0\) such that

\[
|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq c_k |x - y|,
\]
for all \( t \geq 0 \) and those \( x, y \in R \) with \(|x| \vee |y| \leq k\), where \( x \vee y = \max(x, y)\); for all \( t \geq 0 \) and \( x \in R \), there is a \( c > 0 \) such that
\[
|b(t, x)| \vee |\sigma(t, x)| \leq c(1 + |x|).
\]

It is well known (see, Mao [10]) that under the hypothesis of (H0), Eq. (2.2) has a unique strong solution. Moreover, for every \( p \geq 2 \) (or more generally, \( p > 0 \)),
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |L_t|^p \right] < \infty.
\] (2.4)

In order to prove the existence and uniqueness of solutions to the RBSDE (2.1), we shall impose some hypotheses on the coefficient \( f \) and the terminal value \( \xi \) as follows:

(H1) The coefficient \( f \) satisfies the Lipschitz condition. That is, for any \( t, Y_1, Y_2, Z_1, Z_2, \) there is a \( C > 0 \) such that
\[
|f(t, Y_1, Z_1, \omega) - f(t, Y_2, Z_2, \omega)| \leq C[|Y_1 - Y_2| + |Z_1 - Z_2|], \quad \mathbb{P}\text{-a.s.}
\]
(H2) The coefficient \( f \) satisfies a general linear growth condition. That is, for any \( t, Y, Z, \) there is a \( \hat{C} > 0 \) such that
\[
|f(t, Y, Z, \omega)| \leq \hat{C}[|f_0(t, \omega)| + |Y| + |Z|], \quad \mathbb{P}\text{-a.s.},
\]
where \( \{f_0(t, \omega)\} \) is \( \mathcal{F}\)-measurable and satisfies \( \mathbb{E}[\int_0^T |f_0(t, \omega)|^2 \, dt]^\frac{p}{2} < \infty \) for some \( p > 2 \).

We will denote this space by \( L^{2+\frac{p}{2}}(0, T) \).
(H3) There exists a \( p > 2 \) such that the terminal value \( \xi \) belongs to \( L^p \). We also assume \( \xi^d \geq L_T \), where \( \xi = (\xi^1, \ldots, \xi^d)' \).

**Remark 2.1.** For (H2) and (H3), without loss of generality, we can assume that there is a common \( p > 2 \) such that \( \mathbb{E}[\int_0^T |f_0(t, \omega)|^2 \, dt]^\frac{p}{2} < \infty \) and \( \xi \in L^p \).

3. Preliminaries: some lemmas

We will apply a penalization scheme introduced by El Karoui et al. [1] to prove the existence and uniqueness of solutions to RBSDE (2.1). As a preparation, we first consider an RBSDE whose coefficient does not depend on \((Y, Z)\), that is, \( f(t, Y, Z, \omega) = g(t, \omega) \). We now introduce the following BSDEs without reflection:
\[
Y^n_t = \xi + \int_t^T f(s, \omega) \, ds - \int_t^T Z^n_s \, dB_s + \alpha(K^n_T - K^n_t),
\] (3.1)
where \( K^n_t = \int_0^t n(Y^n_s, L_s)^- \, ds \) and \( Y^n_t = (Y^n_1, \ldots, Y^n_d)' \); \( g : [0, T] \times \Omega \to R^d \) is \( \mathcal{F}\)-measurable and belongs to \( L^{2+\frac{p}{2}}(0, T) \), where \( p > 2 \). We have:

**Lemma 3.1.** Suppose that \( g \) belongs to \( L^{2+\frac{p}{2}}(0, T) \) and (H3) holds with the same \( p \), then BSDE (3.1) has a unique solution \((Y^n, Z^n) \in S^2 \times L^2 \) for every \( n \in \mathbb{N} \), which also has the property that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y^n_t|^p + \int_0^T |Y^n_t|^{p-2} |Z^n_t|^2 \, dt + \left( \int_0^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} + (K^n_T)^p \right] \leq C_p, \tag{3.2}
\]

where \( C_p > 0 \) depends on \( p, T, \mathbb{E}|\xi|^p, \mathbb{E}[\int_0^T |g(t, \omega)|^2 \, dt]^\frac{p}{2} \) and \( \sup_{0 \leq t \leq T} |L_t|^p \) only.

**Proof.** By the standard theory of BSDEs (see, for example, Yin and Mao [13] for the BSDEs with jumps case), it is easy to know that Eq. (3.1) has a unique solution. We only notice that \( g(t, \omega) + n(L_t)^{+} \) belongs to \( L^2(0, T) \) according to (2.4) and the assumption on \( g \). It remains to show the inequality (3.2). We will first prove that

\[
\sup_{t \in [0, T]} \mathbb{E}|Y^n_t|^p + \mathbb{E} \left( \int_0^T |Z^n_t|^2 \, dt \right)^{\frac{p}{2}} + (K^n_T)^p < \infty. \tag{3.3}
\]

For this, we define a sequence of BSDEs for every \( n \in \mathbb{N} \) with the form of

\[
y^{(n)}_m(t) = \xi + \int_t^T f(s, \omega) \, ds - \int_t^T z^{(n)}_m(s) \, dB_s + \alpha \int_t^T n(y^{(n), d}_{m-1}(s) - L_s)^{+} \, ds,
\]

\[
(y^{(n)}_0(t), z^{(n)}_0(t)) = (0_d, 0_d \times r), \quad m = 1, 2, \ldots. \tag{3.4}
\]

Obviously, Eq. (3.4) admits a unique solution \((y^{(n)}_m, z^{(n)}_m) \in S^2 \times H^2 \) for each \( m \in \mathbb{N} \). When we take conditional expectations on both sides of (3.4) with respect to \( \mathcal{F}_t \) and apply Jensen’s inequality to obtain iteratively

\[
\mathbb{E}|y^{(n)}_m(t)|^p \leq K_1(p, n, T) \mathbb{E} \left[ |\xi|^p + \left( \int_0^T |g(t, \omega)|^2 \, dt \right)^{\frac{p}{2}} \right.
\]

\[
+ \int_0^T |y^{(n)}_{m-1}(t)|^p \, dt + \sup_{0 \leq t \leq T} |L_t|^p \left. \right] \leq C_1(p, n, T) \left[ 1 + \mathbb{E} \int_0^T |y^{(n)}_{m-1}|^p \, dt \right] \leq C_1(p, n, T) e^{C_1(p, n, T)T}. \tag{3.5}
\]

In above and what follows, \( K_i(p, n, T) \) denote some constants only depending on \((n, p, T)\), while \( C_i(p, n, T) \) denote some constants only depending on \( \mathbb{E}(|\xi|^p + [\int_0^T |g(t, \omega)| \, dt]^\frac{p}{2} + \sup_{0 \leq t \leq T} |L_t|^p) \) and \((n, p, T)\), \( i = 1, 2, \ldots \). Note that

\[
\lim_{m \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} |y^{(n)}_m(t) - Y^n_t|^2 = 0,
\]

since \( \{Y^n_t\} \) is the limit of \( \{y^{(n)}_m\}, m = 1, 2, \ldots, \) in \( S^2 \). So we can take a subsequence of \( \{y^{(n)}_m(t)\} \) denoted by \( \{y^{(n)}_{m_k}(t)\} \) and apply Fatou’s lemma to obtain

\[
\sup_{0 \leq t \leq T} \mathbb{E}|Y^n_t|^p \leq \sup_{0 \leq t \leq T} \lim_{k \to \infty} \mathbb{E}|y^{(n)}_{m_k}|^p \leq C_1(p, n, T) e^{C_1(p, n, T)T} := C_2(p, n, T), \tag{3.6}
\]
which, together with the Hölder inequality and $C_p$-inequality, yields that
\[
\mathbb{E}(K_T^n)^p \leq 2^{p-1}T^p n^p \left( C_2(p, n, T) + \mathbb{E} \sup_{0 \leq t \leq T} |L_t|^p \right). \tag{3.7}
\]
Furthermore, we take conditional expectations on both sides of (3.1) and use (3.7), Jesen’s inequality, $C_p$-inequality and Doob’s martingale inequality to get
\[
\mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p \leq C_3(p, n, T). \tag{3.8}
\]
Finally, for BSDE (3.1), it follows from the B-D-G inequality, (3.7) and (3.8) that
\[
\mathbb{E} \left[ \int_0^T |Z^n_t|^2 \, dt \right]^\frac{p}{2} \leq C_4(p, n, T). \tag{3.9}
\]
By Itô’s formula to $|Z^n_t|^p$, we have
\[
|Y^n_t|^p = |\xi|^p + \int_t^T pY^n_s |Y^n_s|^{p-2} g(s, \omega) \, ds - \int_t^T pY^n_s |Y^n_s|^{p-2} \, dB_s
+ \int_t^T pY^n_s, d |Y^n_s|^{p-2} \, dK^n_s
- \int_t^T \frac{p}{2} |Y^n_s|^{p-2} |Z^n_s|^2 \, ds \right.
- \int_t^T \left. \frac{p(p - 2)}{2} |Y^n_s|^{p-2} \frac{|Y^n_s \cdot Z^n_s|^2}{|Y^n_s|^2} \, ds. \tag{3.10}
\]
Note that $\int_0^t pY^n_s |Y^n_s|^{p-2} \, dB_s$ is a uniformly integrable martingale from the Burkholder–Davis–Gundy inequality, (3.8) and (3.9). Indeed, this is based on the fact that
\[
\mathbb{E} \left( \int_0^T |Y^n_t|^{2p-2} |Z^n_t|^2 \, dt \right)^\frac{1}{2} \leq \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y^n_t|^{p-1} \left( \int_0^T |Z^n_t|^2 \, dt \right)^\frac{1}{2} \right)
\leq \left( \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p \right)^{\frac{p-1}{p}} \mathbb{E} \left( \int_0^T |Z^n_t|^2 \, dt \right)^\frac{1}{2} \left( \mathbb{E} \left( \int_0^T |Z^n_t|^2 \, dt \right)^\frac{1}{2} \right)^\frac{1}{p}. \tag{3.11}
\]
By the definition of $K^n_t$, it is easy to see that
\[
\int_t^T pY^n_s, d |Y^n_s|^{p-2} \, dK^n_s \leq \int_t^T pL_s |Y^n_s|^{p-2} \, dK^n_s.
\]
Therefore, applying Yang’s inequality and Hölder’s inequality, we can derive
Making use of (3.13), (3.14) and the B-D-G inequality for Eq. (3.10), we obtain

\[ \mathbb{E}|Y^n_t|^p + \mathbb{E} \int_t^T \frac{p}{2} |Y^n_s|^p |Z^n_s|^2 \, ds \]

\[ \leq \mathbb{E} \left[ |\xi|^p + (p-1)^{-1} T \epsilon (p-1) \right] \left[ \int_0^T |g(s, \omega)|^2 \, ds \right]^{p/2} \]

\[ + \epsilon^{-(p-1)} (p-1)^{p-1} \sup_{0 \leq t \leq T} |L_t|^p + \epsilon \sup_{0 \leq t \leq T} |K^n_T|^p + 2 \epsilon \sup_{0 \leq t \leq T} |Y^n_t|^p \]

where \( \epsilon \) is a sufficiently small positive constant determined later and may vary from line to line for conciseness. Thus we have

\[ \sup_{0 \leq t \leq T} \mathbb{E}|Y^n_t|^p \leq C_1(p, \epsilon, T) + \epsilon \mathbb{E}|K^n_T|^p + \epsilon \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p \quad (3.13) \]

and

\[ \mathbb{E} \int_0^T |Y^n_t|^p |Z^n_t|^2 \, dt \leq C_2(p, \epsilon, T) + \epsilon \mathbb{E}|K^n_T|^p + \epsilon \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p, \quad (3.14) \]

where \( C_i(p, \epsilon, T) > 0, i = 1, 2, \ldots \) are similar to \( C_i(p, n, T) \) except independence on \( n \). We now use the Burkholder–Davis–Gundy inequality to get

\[ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_t^T p Y^n_s |Y^n_s|^p |Z^n_s|^2 \, dB_s \right| \]

\[ \leq 2 \left( \frac{p^{p+1}}{2(p-1)^{p-1}} \right)^{p/2} \mathbb{E} \left( \int_0^T |Y^n_t|^{2(p-1)} |Z^n_t|^2 \, dt \right)^{1/2} \]

\[ \leq 2 \left( \frac{p^{p+1}}{2(p-1)^{p-1}} \right)^{p/2} p \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y^n_t|^{p/2} \left( \int_0^T |Y^n_t|^p |Z^n_t|^2 \, dt \right)^{1/2} \right) \]

\[ \leq \frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p + 2p^2 \left( \frac{p^{p+1}}{2(p-1)^{p-1}} \right)^p \mathbb{E} \int_0^T |Y^n_t|^p |Z^n_t|^2 \, dt. \quad (3.15) \]

Making use of (3.13), (3.14) and the B-D-G inequality for Eq. (3.10), we obtain

\[ \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p \leq C_3(p, \epsilon, T) + \epsilon \mathbb{E}|K^n_T|^p + \epsilon \mathbb{E} \sup_{0 \leq t \leq T} |Y^n_t|^p. \quad (3.16) \]

On the other hand, by the chain rule, it is easily derived that

\[ \mathbb{E}|K^n_T|^p = \mathbb{E} \int_0^T (K^n_s)^{p-1} d K^n_s \leq p \mathbb{E} \int_0^T (K^n_s)^{p-1} d K^{n,d}_s + p \mathbb{E} \int_0^T (K^n_s)^{p-1} |g^d(s, \omega)| \, ds \]

\[ \leq p \mathbb{E}[(K^n_T)^{p-1}|Y^n_{T,d}|] - p(p-1) \mathbb{E} \int_0^T Y^{n,d}_s (K^n_s)^{p-2} \, dK^n_s \]
\[ + p \mathbb{E} \int_{0}^{T} (K_{n}^{p})^{p-1} |g^d(s, \omega)| \, ds \]

\[ \leq \frac{1}{2} \mathbb{E}(K_{n}^{p})^{p} + C(p, T) \left( 1 + \mathbb{E} \sup_{0 \leq t \leq T} |Y_{t}^{n}|^{p} \right), \tag{3.17} \]

where \( C(p, T) > 0 \) is a constant which does not depend on \( \varepsilon \). Substituting (3.16) into (3.15) gives

\[ \mathbb{E} \sup_{0 \leq t \leq T} |Y_{t}^{n}|^{p} \leq C_{4}(p, \varepsilon, T) + \varepsilon \mathbb{E} \sup_{0 \leq t \leq T} |Y_{t}^{n}|^{p}. \tag{3.18} \]

The required conclusion follows from taking a sufficiently small \( \varepsilon \). \( \Box \)

We also have the following crucial estimate.

**Lemma 3.2.** Assume that \( g \) belongs to \( L^{2+\frac{p}{2}} \) and (H0) holds. Then there is a positive constant \( K_{0} \) which only depends on \( p, T \) and (2.4) such that

\[ \mathbb{E} \int_{0}^{T} n \left[ (Y_{t}^{n,d} - L_{t})^{-} \right]^{p} dt \leq K_{0} n^{-\frac{r}{p}}, \quad 0 \leq 2r \leq p - 2. \]

**Proof.** Applying Itô’s formula to \( (Y_{t}^{n,d} - L_{t})^{-} \), we have

\[ \left[ (Y_{t}^{n,d} - L_{t})^{-} \right]^{p} + \frac{p(p-1)}{2} \int_{t}^{T} \left[ (Y_{s}^{n,d} - L_{s})^{-} \right]^{p-2} \left( |Z_{s}^{n,d}|^{2} + |\sigma(s, L_{s})|^{2} \right) ds \]

\[ + p \int_{t}^{T} n \left[ (Y_{s}^{n,d} - L_{s})^{-} \right]^{p} ds \]

\[ = p \int_{t}^{T} \left[ (Y_{s}^{n,d} - L_{s})^{-} \right]^{p-1} (g^d(s, \omega) - b(s, L_{s})) ds \]

\[ + p(p-1) \int_{t}^{T} \left[ (Y_{s}^{n,d} - L_{s})^{-} \right]^{p-2} (Z_{s}^{n,d}, \sigma(s, L_{s})) ds + M_{T}^{p} - M_{t}^{p} \]

\[ := \sum_{i=1}^{4} I_{i}, \tag{3.19} \]

where

\[ I_{4} = M_{t}^{p} = p \int_{0}^{t} \left[ (Y_{s}^{n,d} - L_{s})^{-} \right]^{p-1} (-Z_{s}^{n,d} + \sigma(s, L_{s})) dB_{s}. \]
We can easily verify that

\[
\mathbb{E} \left[ \int_0^T \left( (Y_{t}^{n,d} - L_t)^- \right)^{2(p-1)} \left( |Z_{t}^{n,d}|^2 + |\sigma(t, L_t)|^2 \right) dt \right]^{\frac{1}{2}} \\
\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( (Y_{t}^{n,d} - L_t)^- \right)^{p-1} \left( \int_0^T \left( |Z_{t}^{n,d}|^2 + |\sigma(t, L_t)|^2 \right) dt \right)^{\frac{1}{2}} \right] \\
\leq \frac{p-1}{p} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( (Y_{t}^{n,d} - L_t)^- \right)^p \right] \\
+ \frac{1}{p} \mathbb{E} \left( \int_0^T \left( |Z_{t}^{n,d}|^2 + |\sigma(t, L_t)|^2 \right) dt \right)^{\frac{p}{2}} < \infty \quad (3.20)
\]

from Lemma 3.1, (H0) and (2.4). Hence \( M^n_t \) is a martingale. We now compute

\[
\mathbb{E} I_1 = \mathbb{E} p \sup_{0 \leq t \leq T} \left[ (Y_{t}^{n,d} - L_t)^- \right]^\frac{p}{2} \int_0^T \left[ \left( Y_{s}^{n,d} - L_s \right)^- \right]^\frac{p-2}{2} \left( |g^d(s, \omega)| + |b(s, L_s)| \right) ds \\
\leq \frac{p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left[ (Y_{t}^{n,d} - L_t)^- \right]^p}{4 \varepsilon n^{-\frac{r}{p}}} + \varepsilon n^{-\frac{r}{p}} \mathbb{E} \left[ \left( \int_0^T \left( Y_{s}^{n,d} - L_s \right)^- \right)^{p-2} ds \right] \\
\times \left( \int_0^T 2 |g(s, \omega)|^2 ds + \int_0^T 2 |g^d(s, \omega)|^2 ds \right) \\
\leq \frac{p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left[ (Y_{t}^{n,d} - L_t)^- \right]^p}{4 \varepsilon n^{-\frac{r}{p}}} + 4 \varepsilon T^{-\frac{2}{p-2}} \mathbb{E} \int_0^T \left[ \left( Y_{s}^{n,d} - L_s \right)^- \right]^p ds \\
+ \frac{4 \varepsilon}{pn^{-\frac{r}{2}}} \mathbb{E} \left( \int_0^T |g^d(s, \omega)|^2 ds \right)^{\frac{p}{2}} + \frac{4 \varepsilon}{pn^{-\frac{r}{2}}} \mathbb{E} \left( \int_0^T |b(s, L_s)|^2 ds \right)^{\frac{p}{2}} \\
\leq \hat{K}_0 \left[ \varepsilon^{-1} n^{-\frac{r}{p}} + \varepsilon \mathbb{E} \int_0^T \left( Y_{s}^{n,d} - L_s \right)^-^p ds + \varepsilon n^{-\frac{r}{2}} \right], \quad (3.21)
\]

where \( \hat{K}_0 > 0 \) is a constant, \( \varepsilon > 0 \) is an arbitrary constant determined later, \( r > 0 \) is a constant such that \( p - 2 \geq 2r \) and we have used Yang’s inequality, Hölder’s inequality, Lemma 3.1 and (2.4). On the other hand, we have

\[
EI_2 \leq \frac{p(p-1)}{2} \mathbb{E} \int_0^T \left[ (Y_{s}^{n,d} - L_s)^- \right]^{p-2} \left[ |Z_{s}^{n,d}|^2 + |\sigma(s, L_s)|^2 \right] ds. \quad (3.22)
\]
We now take expectations on both sides of (3.19) and use (3.20)–(3.22) to get

\[ p \mathbb{E} \int_t^T n [(Y_{s-}^{n,d} - L_s)^-]^p \, ds \leq \hat{K}_0 \left[ \frac{1}{\varepsilon n^p} + \varepsilon \mathbb{E} \int_t^T n [(Y_{s-}^{n,d} - L_s)^-]^p \, ds + \frac{1}{n^p} \right]. \]

If we take a sufficiently small \( \varepsilon > 0 \), then it is easy to derive

\[ \mathbb{E} \int_0^T n [(Y_{t-}^{n,d} - L_t)^-]^p \, dt \leq K_0 n^{-\frac{p}{2}}. \]

**Remark 3.1.** From the proof of Lemma 3.2, we observe that, we cannot derive a similar lemma to Lemma 3.2 when \( p = 2 \). This is why we impose (H2) on the coefficient \( f \).

Now we can prove the following

**Lemma 3.3.** Under the same assumptions as Lemma 3.2, we have

\[ \lim_{n \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} [(Y_{t-}^{n,d} - L_t)^-]^p = 0, \quad \forall m \in \mathbb{N}. \]

**Proof.** Note that

\[ Y_{t-}^{n,d} = \xi_d + \int_t^T f^d(s, \omega) \, ds - \int_t^T Z_{s-}^{n,d} \, dB_s + \int_t^T n (Y_{s-}^{n,d} - L_s)^- \, ds, \]

then by the comparison theorem on BSDEs (see, for example, Yin and Mao [13]), we have \( Y_{t-}^{n,d} \leq Y_{t-}^{n+1,d} \) for each \( t \in [0, T] \) a.s. By this and (3.2), there exists a measurable process \( \{Y^d_t\} \) such that a.s.

\[ Y_{t-}^{n,d} \uparrow Y^d_t, \quad 0 \leq t \leq T. \]

We now use the comparison theorem again to have that \( Y_{t-}^{n,d} \geq \tilde{Y}_{t-}^{n,d} \), \( 0 \leq t \leq T \), a.s., where \( \{\tilde{Y}_{t-}^{n,d}, \tilde{Z}_{t-}^{n,d}; 0 \leq t \leq T\} \) is the unique solution of the BSDE

\[ \tilde{Y}_{t-}^{n,d} = \xi_d + \int_t^T f^d(s, \omega) \, ds - \int_t^T \tilde{Z}_{s-}^{n,d} \, dB_s + \int_t^T n (L_s - \tilde{Y}_{s-}^{n,d}) \, ds. \]

Let \( \nu \) be a stopping time such that \( 0 \leq \nu \leq T \). Next, we can mimic the proof of Lemma 6.1 in [1, p. 723] to obtain that \( Y^d_t \geq L^\nu_t \) a.s. It then follows from the section theorem that

\[ Y^d_t \geq L_t, \quad 0 \leq t \leq T. \]

Hence \( [(Y_{t-}^{n,d} - L_t)^-]^p \downarrow 0 \) as \( n \to \infty \) a.s. We recall that \( \{Y_{t-}^{n,d}\} \) and \( \{L_t\} \) are all continuous processes, then

\[ \lim_{n \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} [(Y_{t-}^{n,d} - L_t)^-]^p = 0. \]
follows from the Dini theorem and the dominated convergence theorem since \((Y_t^{n,d} - L_t)^- \leq |Y_t^1| + |L_t|\). This and Lemma 3.1 imply that

\[
0 \leq E \int_0^T (Y_t^{n,d} - L_t)^- \, dK_t^m \leq E \left[ \sup_{0 \leq t \leq T} (Y_t^{n,d} - L_t)^- K_T^m \right] \\
\leq E \sup_{0 \leq t \leq T} \left[ (Y_t^{n,d} - L_t)^- \right] \frac{1}{2} E (K_T^m) \frac{p}{p-1} \frac{p-1}{p} \to 0.
\]

The proof is therefore complete. 

4. Main results

This section devotes to proving the existence and uniqueness of solutions to RBSDE (2.1). The first result is Theorem 4.1, which shows that there exists a unique solution to an RBSDE associated with the coefficient \(g\) and the terminal value \(\xi\). We then apply the contractive mapping principle to prove the existence result of a solution to Eq. (2.1).

**Theorem 4.1.** If there is \(p > 2\) such that \(g \in L_2^{2+\frac{d}{2}}\) and \(\xi \in L^p\), then there exists a triple \((Y, Z, K) \in S^2 \times H^2 \times A^2\) of processes, which satisfies

\[
\begin{align*}
Y_t &= \xi + \int_0^T g(s, \omega) \, ds - \int_0^T Z_s \, dB_s + \alpha (K_T - K_t), \\
L_t &\leq Y_t^d, \quad \int_0^T (Y_s^d - L_s) \, dK_s = 0, \quad 0 \leq t \leq T.
\end{align*}
\]

**Proof.** Existence: applying Itô’s formula to BSDE (3.1), one has that

\[
\mathbb{E} \left[ |Y_t^n - Y_t^m|^2 + \int_0^T |Z_s^n - Z_s^m|^2 \, ds \right] \\
= 2 \mathbb{E} \int_0^T (Y_s^{n,d} - Y_s^{m,d}) (dK_s^n - dK_s^m) \\
\leq 2 \mathbb{E} \int_0^T (Y_s^{n,d} - L_s)^- \, dK_s^m + 2 \mathbb{E} \int_0^T (Y_s^{m,d} - L_s)^- \, dK_s^n \to 0
\]

as \(n, m \to \infty\) from Lemma 3.3. Applying Itô’s formula to \(|Y_t^n - Y_t^m|^2\) on \([0, T]\) again, by the B-D-G inequality, it is not hard to get

\[
\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 \to 0, \quad \text{as } n, m \to \infty.
\]

Therefore there exists a tuple of processes \((Y, Z) \in S^2 \times H^2\) such that

\[
\lim_{n \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} |Y_t^n - Y_t^1|^2 = 0
\]
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T |Z^n_t - Z_t|^2 \, dt = 0. \quad (4.4)
\]

By these, it is easily seen that there exists a \( \{K_t\} \in \mathcal{A}^2 \) such that as \( n \to \infty \),
\[
\mathbb{E} \sup_{0 \leq t \leq T} |K^n_t - K_t|^2 \to 0. \quad (4.5)
\]

By (4.3) and (4.5), there exists a subsequence of \((Y^n_t, K^n_t)\), denoted by \((Y^n_t, K^n_t)\) again, which uniformly tends to \((Y_t, K_t)\) in \( t \in [0, T] \) with probability one. Obviously, \( \{K_t\} \) is a continuous and increasing process with \( K_0 = 0 \). By means of the uniform convergence of \( \{K^n_t\} \) to \( \{K_t\} \) in \( t \in [0, T] \), for any \( \omega \in \{\lim_{n \to \infty} \sup_{0 \leq t \leq T} |Y^n_t - Y_t| = 0\} \), the measure sequence \( dK^n_t \) converges to \( dK_t \) on any open set, closed set even arbitrary Borel set \( A \) in \( [0, T] \), that is,
\[
\int_0^T 1_A \, dK^n_t \to \int_0^T 1_A \, dK_t, \quad \mathbb{P}\text{-a.s.}
\]

Hence
\[
\int_0^T (Y^n_t - L_t) \, dK^n_t \to \int_0^T (Y_t - L_t) \, dK_t
\]
as \( n \to \infty \) with probability one. Note that
\[
\int_0^T (Y^n_t - L_t) \, dK^n_t \leq 0, \quad \forall n \in \mathbb{N},
\]

hence it follows that
\[
\int_0^T (Y_t - L_t) \, dK_t \leq 0.
\]

However, in Lemma 3.3 we have proved that
\[
\lim_{n \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} \left[ (Y^n_t - L_t)^- \right]^p = 0
\]
which implies that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left[ (Y_t - L_t)^- \right]^p = 0
\]
and
\[
Y_t^d \geq L_t, \quad \forall t \in [0, T].
\]

Therefore
\[
\int_0^T (Y_t^d - L_t) \, dK_t = 0.
\]
Uniqueness: Let \((Y', Z', K')\) be another solution to RBSDE (4.1). Itô’s formula to \(|Y_t - Y_t'|^2\) yields that
\[
\mathbb{E}|Y_t - Y_t'|^2 + \mathbb{E}\int_t^T |Z_s - Z_s'|^2 \, ds = 2\mathbb{E}\int_t^T (Y^d_s - Y^d_s')(dK_s - dK_s') \leq 0.
\]
By this and the B-D-G inequality, we further have
\[
\mathbb{E}\sup_{0 \leq t \leq T} |Y_t - Y_t'|^2 = 0.
\]
Finally, it is very easy to derive that
\[
\mathbb{E}\sup_{0 \leq t \leq T} |K_t - K_t'|^2 = 0.
\]
The proof is therefore complete.

\textbf{Remark 4.1.} For the solution \((Y, Z, K)\) to RBSDE (4.1), by Lemma 3.1 and Fatou’s lemma, it is very easy to get \(\mathbb{E}\sup_{0 \leq t \leq T} \|Y_t - Y_t\|^p \leq C_p\) and \(\mathbb{E}(K_T)^p \leq C_p\). We further apply the B-D-G inequality for Eq. (4.1) to obtain
\[
\mathbb{E}\left[\int_0^T |Z|_t^p \, dt\right]^{\frac{p}{2}} \leq K_1 C_p, \text{ where } K_1 > 0 \text{ is a constant.}
\]

\textbf{Theorem 4.2.} Under the hypotheses of \((H_0)-(H_3)\), the RBSDE (2.1) has a unique solution \((Y, Z, K) \in S^2 \times H^2 \times A^2\).

\textbf{Proof.} Uniqueness: Let \((Y^1, Z^1, K^1)\) and \((Y^2, Z^2, K^2)\) be two solutions to the RBSDE (2.1). By Itô’s formula, we have
\[
\mathbb{E}|Y_t^1 - Y_t^2|^2 + \mathbb{E}\int_t^T |Z_s^1 - Z_s^2|^2 \, ds
\]
\[
= 2\mathbb{E}\int_t^T (Y^1_s - Y^2_s)[f(s, Y^1_s, Z_s^1, \omega) - f(s, Y^2_s, Z_s^2, \omega)] \, ds
\]
\[
+ 2\mathbb{E}\int_t^T (Y^1_s^d - Y^2_s^d) (K_s^1 - K_s^2)
\]
\[
\leq (2C + 2C^2)\mathbb{E}\int_t^T |Y_s^1 - Y_s^2|^2 \, ds + \frac{1}{2} \mathbb{E}\int_t^T |Z_s^1 - Z_s^2|^2 \, ds.
\]
By this and Gronwall’s inequality, we have \(\mathbb{E}|Y_t^1 - Y_t^2|^2 = 0\) and then \(\mathbb{E}\int_0^T |Z_t^1 - Z_t^2|^2 \, dt = 0\).
We now proceed the proof of Theorem 4.1, the required conclusion follows.
Existence: For $n = 1, 2, \ldots$, we iteratively solve the following RBSEEs:

\[
\begin{cases}
Y^n_t = \xi + \int_t^T f(s, Y^{n-1}_s, Z^{n-1}_s, \omega) \, ds - \int_t^T Z^n_s \, dB_s + \alpha(K^n_T - K^n_t), \\
L_t \leq Y^{n,d}_t, \quad \forall t \in [0, T], \quad \int_0^T (Y^{n,d}_t - L_t) \, dK^n_t = 0, \quad (Y^0_t, Z^0_t) = (0, 0).
\end{cases}
\] (4.6)

We notice that, for RBSDEs (4.6), when $n = 1$, the corresponding RBSDE has a unique solution $(Y^1, Z^1, K^1)$ from Theorem 4.1 since (H2) and (H3) hold. Moreover, by Remark 4.1, we have $\mathbb{E} \sup_{0 \leq t \leq T} |Y^1_t|^p < \infty$ and $\mathbb{E} [\int_0^T |Z^1_t|^2 \, dt]^\frac{p}{2} < \infty$. These and the assumption (H2) ensure a unique solution to the corresponding RBSDE in (4.6) with $n = 2$. So we can proceed this process, and obviously, the RBSDEs (4.6) have a unique solution for each $n \in \mathbb{N}$. By Itô’s formula to $e^{\beta t} |Y^{n+1}_t - Y^n_t|^2$, we have

\[
\mathbb{E} \left[ e^{\beta t} |Y^{n+1}_t - Y^n_t|^2 + \int_t^T \beta e^{\beta s} |Y^{n+1}_s - Y^n_s|^2 \, ds + \int_t^T |Z^{n+1}_s - Z^n_s|^2 \, ds \right] \\
\leq \mathbb{E} \int_t^T 2e^{\beta s} |Y^{n+1}_s - Y^n_s| |f(s, Y^n_s, Z^n_s) - f(s, Y^{n-1}_s, Z^{n-1}_s)| \, ds \\
+ \mathbb{E} \int_t^T 2e^{\beta s} (Y^{n+1,d}_s - Y^{n,d}_s)(dK^{n+1}_s - K^n_s) \\
\leq \mathbb{E} \int_t^T 4C^2 e^{\beta s} |Y^{n+1}_s - Y^n_s|^2 \\
+ \frac{1}{2} \mathbb{E} \left[ \int_t^T e^{\beta s} |Y^n_s - Y^{n-1}_s|^2 \, ds + \int_t^T e^{\beta s} |Z^n_s - Z^{n-1}_s|^2 \, ds \right].
\] (4.7)

We now choose

\[ \beta = 1 + 4C^2 \]

in above inequality to obtain that

\[
\mathbb{E} \int_0^T e^{\beta t} |Y^{n+1}_t - Y^n_t|^2 \, dt + \int_0^T e^{\beta t} |Z^{n+1}_t - Z^n_t|^2 \, dt \\
\leq \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{\beta t} |Y^n_t - Y^{n-1}_t|^2 \, dt + \int_0^T e^{\beta t} |Z^n_t - Z^{n-1}_t|^2 \, dt \right],
\]

which implies that there exists a tuple $(Y, Z)$ such that as $n \to \infty$
\[ E \int_0^T |Y^n_t - Y^m_t|^2 \, dt \to 0 \]

and

\[ E \int_0^T |Z^n_t - Z^m_t|^2 \, dt \to 0. \]

By these and B-D-G’s inequality, Itô’s formula to \(|Y^n_t - Y^m_t|^2\) yields that, as \(n, m \to \infty\),

\[ E \sup_{0 \leq t \leq T} |Y^n_t - Y^m_t|^2 = 0 \]

and

\[ E \sup_{0 \leq t \leq T} |K^n_t - K^m_t|^2 = 0. \]

The rest of the proof is similar to Theorem 4.1, so we omit it. The proof is complete. \(\square\)

5. Some examples

In this section, we will give some examples to illustrate our theory. Obviously, in our framework, if we set \(d = 1\), then the RBSDE (2.1) reduces to a traditional reflected BSDE with one lower barrier.

**Example 5.1.** Assume that \(\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d\) is non-random such that \(0 \leq \xi^d\). We consider the following reflected backward ordinary differential equation:

\[
\begin{aligned}
Y^i_t &= \xi^i + \int_t^T f^i(s, Y^1_s, \ldots, Y^d_s) \, ds, \quad i = 1, \ldots, d - 1, \\
Y^d_t &= \xi^d + \int_t^T f^d(s, Y^1_s, \ldots, Y^d_s) \, ds + K_T - K_t, \\
0 \leq Y^d_t, & \quad \forall t \in [0, T], \quad \int_0^T Y^d_t \, dK_t = 0.
\end{aligned}
\]  

(5.1)

If \(f = (f^1, \ldots, f^d)\) satisfies the Lipschitz condition with respect to \(Y\), and also satisfies

\[ \int_0^T |f(t, 0, \ldots, 0)|^2 \, dt < \infty, \]

(5.2)

then Eq. (4.1) has a unique solution \((Y, K)\) which satisfies \(Y^d_t \geq 0, K_t \geq 0, K_0 = 0\) and \(\int_0^T Y^d_t \, dK_t = 0\) from Theorem 4.2. A trivial example for (4.2) is that \(f\) satisfies the linear growth condition.
Example 5.2. For simplicity, we assume $d = 2$ and $r = 1$ and consider an RBSDE with the form of

$$
\begin{align*}
Y^1_t &= \xi^1 + \int_t^T f^1(s, Y^1_s, Y^2_s, Z^1_s, Z^2_s) \, ds - \int_t^T Z^1_s \, dB_s, \\
Y^2_t &= \xi^2 + \int_t^T f^2(s, Y^1_s, Y^2_s, Z^1_s, Z^2_s) \, ds - \int_t^T Z^2_s \, dB_s + K_T - K_t, \\
L_t &\leq Y^2_t, \quad \forall t \in [0, T], \quad \int_0^T (Y^2_t - L_t) \, dK_t = 0,
\end{align*}
$$

(5.3)

where the terminal value and the random obstacle satisfy

$$
\begin{align*}
\xi^2 &= \xi + L_0 + \int_0^T b(s, L_s) \, ds + \int_0^T \sigma(s, L_s) \, dB_s, \quad \xi \geq 0 \text{ a.s.,} \\
L_t &= L_0 + \int_0^t b(s, L_s) \, ds + \int_0^t \sigma(s, L_s) \, dB_s, \quad t \in [0, T].
\end{align*}
$$

(5.4)

(5.5)

If $b$ and $\sigma$ satisfy the local Lipschitz condition and the linear growth condition, then Eq. (4.4) has a unique solution, moreover, $\xi^2 \geq L_T$. Suppose that $f$ satisfies the Lipschitz condition. We also assume that

$$
\mathbb{E} \left[ \int_0^T |f(t, 0, 0, 0, 0)|^2 \, dt \right]^\frac{p}{2} < \infty, \quad \mathbb{E}|\xi^1|^p < \infty, \quad \mathbb{E}|\xi|^p < \infty,
$$

(5.6)

for some $p > 2$, then by Theorem 4.2, RBSDE (4.3) has a unique solution $(Y, Z, K)$. In this case, we can imagine that $L_t$ represents the price of a stock in a financial market, $Y^1_t$ represents the price of a European option and $Y^2_t$ represents the price of an American call option but with the payoff $L_t$ (see, El Karoui, Pardoux and Quenez [2]). RBSDE (4.3) means that one price of an option influences the other. Besides, if $L_t \equiv 0$ for any $t \in [0, T]$, then RBSDE (4.3) reduces to an $R^2$-dimensional RBSDE in a half-space.

References