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# Embedding complete trees into the hypercube

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**Abstract**

We consider embeddings of the complete $t$-ary trees of depth $k$ (denotation $T^{k,t}$) as subgraphs into the hypercube of minimum dimension $n$. This $n$, denoted by $\dim(T^{k,t})$, is known if $\max\{k,t\} \leqslant 2$. First, we study the next open cases $t=3$ and $k=3$. We improve the known upper bound $\dim(T^{k,3}) \leqslant 2k+1$ up to $\lim_{k\to\infty}\dim(T^{k,3})/k \leqslant 5/3$ and show $\lim_{t\to\infty}\dim(T^{3,t})/t = 227/120$. As a co-result, we present an exact formula for the dimension of arbitrary trees of depth 2, as a function of their vertex degrees. These results and new techniques provide an improvement of the known upper bound for $\dim(T^{k,t})$ for arbitrary $k$ and $t$. 

*Keywords*: Embedding; Dilation; Complete tree; Hypercube

## 1. Introduction

Throughout the paper we mean by a graph an ordered pair $G=(V(G),E(G))$ where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. All the graph-theoretical concepts which are not defined here can be found in any introductory book on graph theory (e.g. [3]).

For $n \geqslant 1$ let $Q^n$ denote the graph of the $n$-dimensional *hypercube*. The vertex set of $Q^n$ is formed by the collection of all $n$-dimensional vectors with binary entries. Two vertices $x,y \in V(Q^n)$ are adjacent iff the corresponding vectors differ exactly in one entry. Denote by $\rho(x,y)$ the distance between the vertices of $Q^n$ and let $\tilde{0}=(0,\dots,0)$. For $0 \leqslant \ell \leqslant n$ the set $Q^n_\ell = \{x \in V(Q^n) \mid \rho(x,\tilde{0}) = \ell\}$ is called the $\ell$th *level* of $Q^n$.

Let $T=(V(T),E(T))$ be a tree. Assume there exists an injective mapping $f: V(T) \mapsto V(Q^n)$ such that $\rho(f(v),f(w))=1$ for all $(v,w)\in E(T)$. Then we call $f$ an *embedding* of $T$ into $Q^n$. In this case $T$ is a subgraph of $Q^n$. It is easily shown that for any tree

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$T$ and sufficiently large $n$ an embedding of $T$ into $Q^n$ does exist. The minimum $n$ satisfying this property is called the *dimension* of $T$ and denoted by $\dim(T)$.

Let $T$ be a rooted tree with the root $r$. For $\ell \geqslant 0$ we call the set $T_\ell = \{x \in V(T) \mid \operatorname{dist}(x,r) = \ell\}$ the $\ell$th *level* of $T$. The largest number $\ell$ that fulfills $T_\ell \neq \emptyset$ is called the *depth* of $T$.

Consider the problem of finding $\dim(T)$ for a given tree $T$. Such problems are important, for example, for the theory of parallel algorithms on multiprocessor computing systems [11]. As it is shown in [11], for given $T$ and $n$ the problem of recognizing whether a tree $T$ is a subgraph of $Q^n$ is NP-complete for general trees. On the other hand, if one has information on the depth of the tree $T$ and its maximum degree $t$, one possible practical approach is to embed this tree into the complete $t$-ary tree of the same depth, and thus we can restrict ourselves to study such complete trees only.

This leads us to the problem of finding the dimension of the complete $t$-ary tree of depth $k$, which we denote by $T^{k,t}$. Such a tree has $k+1$ levels, its root has degree $t$, and all the other vertices which are not leaves have degree $t+1$.

The dimension of $T^{k,t}$ was already studied in [6] (the lower bound), and [9] (the upper bound) where it is proved that

$$\frac{kt}{e} \leqslant \dim(T^{k,t}) \leqslant \frac{(k+1)t}{2} + k - 1. \tag{1}$$

The lower bound in (1) can be derived from the following argument, which even provide a better lower bound for concrete values of $k$ and $t$ (see [2,6]). Given an embedding $f$ of $T^{k,t}$ into $Q^n$, we can assume that the root of $T^{k,t}$ is mapped into the origin of $Q^n$. Now since $Q^n$ and $T^{k,t}$ are bipartite graphs, then for the image of any vertex $v \in T^{k,t}_{k-2i}$ with $i \in [0, \lfloor k/2 \rfloor]$ one has $f(v) \in \bigcup_{i=0}^{\lfloor k/2 \rfloor} Q^n_{k-2i}$. Moreover, assuming that the images of the vertices of $T^{k,t}_1$ have zeros in the last $n-t$ entries, the vertices of $Q^n_k$ having zeros in the first $t$ entries cannot be images of the tree vertices. These assertions imply

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2i} - \binom{n-t}{k} \geqslant \sum_{i=0}^{\lfloor k/2 \rfloor} t^{k-2i}. \tag{2}$$

The upper bound in (1) is based on a rather tricky construction. Both bounds differ in a multiplicative factor from a trivial upper bound $\dim(T^{k,t}) \leqslant kt$.

For small values of $k,t$ some exact results are known. Among them are the following two, derived from [5,6], respectively:

$$\dim(T^{k,2}) = k+2 \quad (k \geqslant 2),$$

$$\dim(T^{2,t}) = \left\lceil \frac{3t+1}{2} \right\rceil, \quad (t \geqslant 1). \tag{3}$$

Notice that if an embedding $f$ of $T^{k,2}$ into $Q^n$ exists, then considering $T^{k,2}$ as a bipartite graph $(V', V''; E)$ one gets $\max\{|V'|, |V''|\} \leqslant 2^{n-1}$. This implies $n \geqslant k+2$. On the other hand, one can embed even two copies of $T^{k,2}$ into $Q^{k+2}$. The corresponding construction is well known as embedding of a double-rooted complete binary tree [8] and can be done by induction on $k$. In the case $k=2$ the lower bound for $\dim(T^{2,t})$ follows from (2) and the upper bound is also proved by induction on $t$ by considering two cases depending on the parity of $t$.

We study the next two open cases $k=3$ and $t=3$. If $k=3$, then the only known lower and upper bounds, which follow from (2) and (1), respectively, are

$$(3+\sqrt{69})t/6 \leqslant \dim(T^{3,t}) \leqslant 2t+2.$$

Here $(3+\sqrt{69})/6 \approx 1.884$. Furthermore, for $t=3$ it is known [2] that

$$\log_2 3k \leqslant \dim(T^{k,3}) \leqslant 2k+1. \tag{4}$$

Here $\log_2 3 \approx 1.585$, the lower bound follows from (2) and the upper bound is provided by an inductive construction.

In our paper we introduce new techniques for dealing with embedding problems and prove that $\dim(T^{3,t}) = 227t/120 + \mathrm{O}(1)$ as $t \to \infty$ (here $227/120 \approx 1.892$). It is the first known case for the $t$-ary trees, when the dimension of them is asymptotically bigger than a simple lower bound, implied by the counting arguments based on (2). We also improve the upper bound (4) for ternary trees up to $\lim_{k\to\infty} \dim(T^{k,3})/k \leqslant 5/3 \approx 1.66$.

The next key result of our paper is Theorem 2, where we present a formula for the dimension of an arbitrary tree of depth 2. The only result we know in this direction is published in [11], concerning the tree $T$ whose root has degree $t$ and the vertex $w_i \in T_1$ has degree $t-i+1$, $i \in [1,t]$, where it is proved that $\dim(T)=t$.

The obtained results for $k=3$ and $t=3$ lead to an improvement of the general upper bound (1) asymptotically. We show that $\lim_{k,t\to\infty} \dim(T^{k,t})/kt \leqslant \frac{307}{640} \approx 0.48$.

Section 2 of the paper is devoted to the dimension of trees of depth 2. We show that Hall's theorem on distinct representatives provides a construction of the embedding and at the same time implies a lower bound for the dimension.

Section 3 is devoted to $\dim(T^{3,t})$. The proof of the lower bound in Section 3.1 uses the above mentioned result on trees of depth 2. The upper bound for $\dim(T^{3,t})$ in Section 3.2 is based on a construction of Turán and asymptotically equals the lower bound proved in Section 3.1.

Section 4 is devoted to the ternary trees. We modify one of the methods of [1] and apply it also in Section 5 to improve the upper bound (1) for the $t$-ary trees. Concluding remarks in Section 6 complete the paper.

## 2. Dimension of an arbitrary tree of depth 2

We need a slight generalization of Hall's theorem on a family of distinct representatives.

Let $\mathscr{F} = \{F_1,\ldots,F_p\}$ be a family of subsets of a finite set and let $s_1,\ldots,s_p$ be integers. A family $\mathscr{G} = \{G_1,\ldots,G_p\}$ where $G_i \subseteq F_i$, $|G_i| = s_i$, $1 \leqslant i \leqslant p$, and $G_i \cap G_j = \emptyset$, $i \neq j$, (if it exists) is called the system of representative subsets (*SRS*) of the family $\mathscr{F}$ with spectrum $(s_1,\ldots,s_p)$.

**Theorem 1** (Mirsky [7]). *The SRS of a family $\mathscr{F}$ with spectrum $(s_1,\ldots, s_p)$ exists iff for any subset $I \subseteq \{1,2,\ldots,p\}$ the following condition is satisfied*:

$$\left|\bigcup_{i\in I} F_i\right| \geqslant \sum_{i\in I} s_i. \tag{5}$$

Let $T$ be a rooted tree of depth 2 with the root $r$ and $\deg(r)=a$. Let $T_1=\{v_1,\ldots,v_a\}$ and denote $\deg(v_i)=b_i+1$ for $i\in[1,a]$. We assume that

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_a. \tag{6}$$

**Theorem 2.** *With the above notation, we have*

$$\dim(T)=\max\left\{a, \max_{1\leqslant m\leqslant a}\left\lceil \frac{m+1}{2}+\frac{1}{m}\sum_{i=1}^{m} b_i\right\rceil\right\}. \tag{7}$$

**Proof.** Consider an embedding $f$ of the tree $T$ into $Q^n$ for some $n$. Without loss of generality, we assume that the root $r$ of $T$ is mapped into the origin of $Q^n$. Then $f$ maps $T_\ell$ into $Q^n_\ell$ for $\ell=1,2$. Since $|Q^n_1|=n$ and $|T_1|=a$, we have

$$n \geqslant a. \tag{8}$$

For $i\in[1,a]$ define

$$F_i=\{z\in Q^n_2 \mid (z, f(v_i))\in E(Q^n)\},$$

$$G_i=\{f(w) \mid w\in T_2 \text{ and } (w,v_i)\in E(T)\}.$$

Since $f$ is an injection, then $G_i\cap G_j=\emptyset$ for $i\neq j$. Moreover, $G_i\subseteq F_i$, $|G_i|=b_i$ for $i\in[1,a]$. Therefore, the family $\{G_1,\ldots,G_a\}$ is the SRS of the family $\{F_1,\ldots,F_a\}$ with spectrum $(b_1,\ldots,b_a)$. Theorem 1 provides a necessary and sufficient condition for existence the SRS and thus the embedding $f$ of $T$ into $Q^n$. In our case (5) has to be applied with $p=a$ and $s_1=b_1,\ldots,s_p=b_a$. It is easily shown that for a subset $I\subseteq\{1,\ldots,a\}$ we have

$$\left|\bigcup_{i\in I} F_i\right| = (n-1)|I| - \binom{|I|}{2}.$$

This and (6) imply that (5) is satisfied for any $I\subseteq\{1,\ldots,a\}$ iff

$$(n-1)m-\binom{m}{2} \geqslant \sum_{i=1}^{m} b_i$$

for any $m\in[1,a]$, which is equivalent to

$$n \geqslant \max_{1\leqslant m\leqslant a}\left\lceil \frac{m+1}{2}+\frac{1}{m}\sum_{i=1}^{m} b_i\right\rceil. \tag{9}$$

Therefore, the embedding $f$ of $T$ into $Q^n$ exists iff (8) and (9) are satisfied, which completes the proof. $\square$

We will need two following technical corollaries of this theorem.

**Corollary 3.** *With the same notation as in Theorem* 2, *we have*

$$\dim(T) \geqslant \frac{a}{2} + \frac{1}{a}\sum_{i=1}^{a} b_i.$$

**Corollary 4.** *Let* $b_1 = \cdots = b_c = a$ *and* $b_{c+1} = \cdots = b_a = b$ *with* $a > b$, *and* $a > c \geqslant 2$. *Then*

$$\dim(T) = \max\left\{ \left\lceil \frac{c+1}{2} \right\rceil + a, \left\lceil \frac{a+1}{2} + \frac{ac + (a-c)b}{a} \right\rceil \right\}.$$

For the proof one just has to partition the segment $1 \leqslant m \leqslant a$ into two segments $1 \leqslant m \leqslant c$, and $c+1 \leqslant m \leqslant a$ and check that the right-hand side of (7) increases in both segments, reaching the corresponding maximum values of the statement.

## 3. The dimension of $T^{3,t}$

To simplify matters, throughout this section we denote $T = T^{3,t}$.

### 3.1. The lower bound

**Lemma 5.** *If* $T$ *is a subgraph of* $Q^n$, *then there exists an embedding of* $T$ *into* $Q^{n'}$ *with* $n \leqslant n' \leqslant n+1$, *such that*

$$f(T_\ell) \subseteq Q_\ell^{n'} \quad \textit{for } \ell = 0, 1, 2, 3. \tag{10}$$

**Proof.** Consider an embedding $f$ of $T$ into $Q^n$. Without loss of generality, we assume that the root of $T$ is embedded into the origin of $Q^n$. Thus $f(T_\ell) \subseteq Q_\ell^n$ for $\ell = 0, 1, 2$ and $f(T_3) \subseteq Q_3^n \cup Q_1^n$. Denote

$$U = \{v \in T_3 \mid f(v) \in Q_1^n\},$$

$$W = \{v \in T_3 \mid f(v) \in Q_3^n\}.$$

If $U = \emptyset$, then (10) holds for $n' = n$. Assume $U \neq \emptyset$.

We construct a new embedding $f'$ of $T$ into $Q^{n+1}$. For $z \in V(Q^n)$ and $\sigma \in \{0, 1\}$ denote by $z\sigma$ the vertex of $Q^{n+1}$ obtained from $z$ by adding the $(n+1)$th entry $\sigma$. For $v \in V(T)$ put

$$f'(v) = \begin{cases} f(v)0 & \text{for } v \in (T_0 \cup T_1 \cup T_2) \cup W, \\ f(u)1 & \text{for } v \in U \text{ and } (v, u) \in E(T). \end{cases}$$

Note that for any $z \in Q_2^n$ there exist exactly two vertices $y_1, y_2 \in Q_1^n$ with $\rho(z, y_1) = \rho(z, y_2) = 1$. Now for any $u \in T_2$ the parent of $u$ is mapped to $Q_1^n$ by $f$. Hence, at most one child of $u$ can be mapped to $Q_1^n$ by $f$. Thus, there is at most one vertex $u \in U$ with $(v, u) \in E(T)$. This guarantees that the mapping $f'$ is injective. Obviously, $f'$ satisfies (10) with $n' = n + 1$. $\square$

For $\ell < n$ and $A \subseteq Q_\ell^n$ denote $S(A) = \{x \in Q_{\ell+1}^n \mid \rho(x, A) = 1\}$.

**Lemma 6.** *Let $t \geqslant 3$ and $A \subseteq Q_2^t$. Then $|S(A)| \leqslant t|A| - (4/3t)|A|^2$.*

**Proof.** We consider $A$ as the edge set of some graph $G$ with $t$ vertices. In these terms $|S(A)|$ is the number of triples of vertices in $G$ having at least one of their 3 unordered pairs as an edge in $G$. Denote by $\sigma_1, \ldots, \sigma_t$ the degrees of vertices of $G$ and let $\Delta(G)$ be the number of cycles of length 3 in $G$. Applying the inclusion–exclusion principle, one has

$$|S(A)| = (t-2)|A| - \sum_{i=1}^{t} \binom{\sigma_i}{2} + \Delta(G). \tag{11}$$

In order to estimate $\Delta(G)$ remove all the edges from $G$ which do not belong to some cycle of length 3 in $G$. This operation results in a graph $G'$ with $\Delta(G') = \Delta(G)$ and if $\sigma_i'$ is the degree of the corresponding vertex in $G'$, then $\sigma_i' \leqslant \sigma_i$ for $i \in [1, t]$. Now each pair of incident edges of $G'$ (if such exist) belongs to a cycle of length 3 in $G'$. Thus,

$$\Delta(G) = \Delta(G') = \frac{1}{3} \sum_{i=1}^{t} \binom{\sigma_i'}{2} \leqslant \frac{1}{3} \sum_{i=1}^{t} \binom{\sigma_i}{2}.$$

Substituting this upper bound into (11) and taking into account

$$\sum_{i=1}^{t} \sigma_i = 2|A|, \tag{12}$$

one has

$$|S(A)| \leqslant (t - 4/3)|A| - \frac{1}{3} \sum_{i=1}^{t} \sigma_i^2.$$

To complete the proof, note that (12) implies $\sum_{i=1}^{t} \sigma_i^2 \geqslant 4|A|^2/t$. $\square$

**Remark 7.** It follows from Lemma 6 that the maximum size of $S(A)$ for a set $A \subseteq Q_2^t$ is strictly less than $\binom{t}{3}$ if $|A| < t^2/4(1 - 6/t + 4/t^2)$. This agrees with a theorem of Turán [10], by which for a subset $A \subseteq Q_2^t$ with $S(A) = Q_3^t$ one has $|A| \geqslant \lceil t^2/4 \rceil$.

**Lemma 8.** *If $t$ is large enough, then* $\dim(T) \geqslant \frac{227}{120}t - 1$.

**Proof.** We consider only embeddings $f$ that satisfy (10) and denote by $n$ the minimum dimension of the hypercube for which such an embedding exists. By Lemma 5, $\dim(T) \geqslant n - 1$.

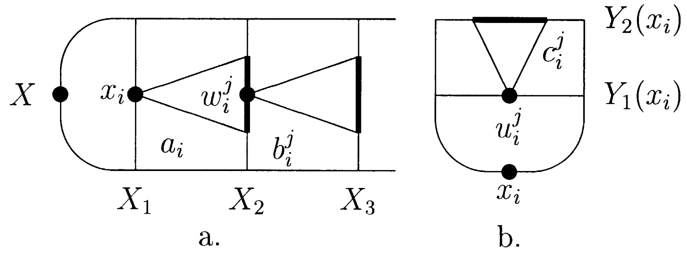


Fig. 1. The subcube $X$ (a) and the tree $T(v_i)$ (b).

Let $x=(\alpha_1,\ldots,\alpha_n)\in V(Q^n)$. We introduce the subcubes $X$ and $Y(x)$ of $Q^n$ as induced subgraphs by the vertex sets $\{(\beta_1,\ldots,\beta_n)\in V(Q^n)\mid \beta_{t+1}=\cdots=\beta_n=0\}$, and $\{(\beta_1,\ldots,\beta_n)\in V(Q^n)\mid \beta_i=\alpha_i,\ i\in[1,t]\}$, respectively. Furthermore for $\tilde{0}=(0,\ldots,0)\in V(Q^n)$ and $\ell\geqslant 0$ denote

$$X_\ell=\{z\in V(X)\mid \rho(\tilde{0},z)=\ell\},$$

$$Y_\ell(x)=\{z\in V(Y(x))\mid \rho(x,z)=\ell\}.$$

Now, we consider the number of vertices of $T$ mapped by $f$ into the sets $X_\ell$ and $Y_\ell(x)$ for $x\in V(X)$, introducing the parameters $a_i, b_i^j$ and $c_i^j$. Estimation of these parameters will lead us to the desired lower bound.

Let $T_1=\{v_1,\ldots,v_t\}$. Without loss of generality, we assume that $f(T_1)=X_1$ and denote $x_i=f(v_i)$, $i\in[1,t]$. Consider the vertices of $T_2$ adjacent with $v_i$. Some of these vertices are mapped into $X_2$ in the embedding and we denote by $w_i^1,\ldots,w_i^{a_i}$ their images (cf. Fig. 1a). Because of (10), the remaining $t-a_i$ vertices of $T_2$ that are adjacent with $v_i$ must be mapped into $Y_1(x_i)$. We denote the images of these vertices by $u_i^1,\ldots,u_i^{t-a_i}$ (cf. Fig. 1b). Furthermore, denote by $b_i^j$ (resp. by $c_i^j$), $j\in[1,a_i]$, the number of vertices of $T_3$ adjacent with $f^{-1}(w_i^j)$ (resp. with $f^{-1}(u_i^j)$) which are mapped by $f$ into $X_3$ (resp. into $Y_2(x_i)$), $i\in[1,t]$.

Therefore, for $i\in[1,t]$ the subcube $Y(x_i)$ contains the image of some subtree of $T$ of depth 2 rooted in $v_i$. We denote this subtree by $T(v_i)$ and apply to it Corollary 3. One has

$$n\geqslant t+\dim(T(v_i))\geqslant t+(t-a_i)/2+\sum_{j=1}^{t-a_i} c_i^j/(t-a_i)$$

or

$$2(n-t)(t-a_i)\geqslant (t-a_i)^2+2\sum_{j=1}^{t-a_i} c_i^j. \tag{13}$$

Summing (13) for $i\in[1,t]$, one gets

$$2(n-t)\sum_{i=1}^{t}(t-a_i)\geqslant \sum_{i=1}^{t}(t-a_i)^2+2\sum_{i=1}^{t}\sum_{j=1}^{t-a_i} c_i^j. \tag{14}$$

We proceed by deriving a lower bound for $\sum\sum c_i^j$ in (14). Remember that for the vertex $f^{-1}(u_i^j)\in T_2$ there are $c_i^j$ vertices of $T_3$ adjacent with $f^{-1}(u_i^j)$, which are mapped by $f$ into $Y_2(x_i)$. Since $f$ satisfies (10), the remaining $t-c_i^j$ vertices of $T_3$, which are adjacent with $f^{-1}(u_i^j)$, have to be mapped into $\bigcup_{x\in X_2} Y_1(x)$. On the other hand, for each vertex $f^{-1}(w_i^j)\in T_2$ there are $b_i^j$ vertices of $T_3$ adjacent with $f^{-1}(w_i^j)$, which are mapped by $f$ into $X_3$, and due to the same reason mentioned above the remaining $t-b_i^j$ vertices of $T_3$, which are adjacent with $f^{-1}(w_i^j)$, have to be mapped into $\bigcup_{x\in X_2} Y_1(x)$. Since $f$ is a injection, the images of all the mentioned vertices have to be distinct and the number of them does not exceed the size of $\bigcup_{x\in X_2} Y_1(x)$, i.e.

$$\sum_{i=1}^{t}\sum_{j=1}^{t-a_i}(t-c_i^j)+\sum_{i=1}^{t}\sum_{j=1}^{a_i}(t-b_i^j)\leqslant\left|\bigcup_{x\in X_2} Y_1(x)\right|=(n-t)\binom{t}{2}.$$

This provides a lower bound for the double sum in (14):

$$\sum_{i=1}^{t}\sum_{j=1}^{t-a_i}c_i^j\geqslant t^3-(n-t)\binom{t}{2}-\sum_{i=1}^{t}\sum_{j=1}^{a_i}b_i^j.$$

Substituting this bound into (14), one has

$$2(n-t)\left(\frac{3t^2}{2}-\sum_{i=1}^{t}a_i\right)\geqslant 3t^3-2t\sum_{i=1}^{t}a_i+\sum_{i=1}^{t}a_i^2-2\sum_{i=1}^{t}\sum_{j=1}^{a_i}b_i^j. \tag{15}$$

Applying Lemma 6 to the subcube $X$ of dimension $t$ and the subset $A=f(T_2)\cap X_2$ of cardinality $\sum_{i=1}^t a_i$, we get

$$\sum_{i=1}^{t}\sum_{j=1}^{a_i}b_i^j\leqslant|S(A)|\leqslant\min\left\{t\sum_{i=1}^{t}a_i-\frac{4}{3t}\left(\sum_{i=1}^{t}a_i\right)^2,\ \binom{t}{3}\right\}. \tag{16}$$

Denote $z=\sum_{i=1}^t a_i/t^2$. Then $0\leqslant z\leqslant 1/2$ and by simple optimization $\sum_{i=1}^t a_i^2\geqslant(\sum_{i=1}^t a_i/t)^2 t=z^2t^3$. Substituting this and (16) into (15) and taking into account Remark 7, one has

$$\frac{2(n-t)}{t}\geqslant\begin{cases}F_1(z)=\frac{11z^2/3-4z+3}{3/2-z} & \text{for } 0\leqslant z\leqslant 1/4,\\ F_2(z)=\frac{z^2-2z+8/3}{3/2-z} & \text{for } 1/4\leqslant z\leqslant 1/2.\end{cases}$$

It is easily shown that the function $F_1(z)$ decreases in its domain and the function $F_2(z)$ increases. Therefore,

$$2(n-t)/t\geqslant\min\{F_1(1/4),F_2(1/4)\}=107/60$$

and the lemma follows. $\square$

### 3.2. The upper bound

Throughout this section we use the notations introduced in the proof of the lower bound. To avoid writing the integer parts in the terms below we assume first that $t$ is a multiple of 12. For the upper bound we need two auxiliary lemmas.

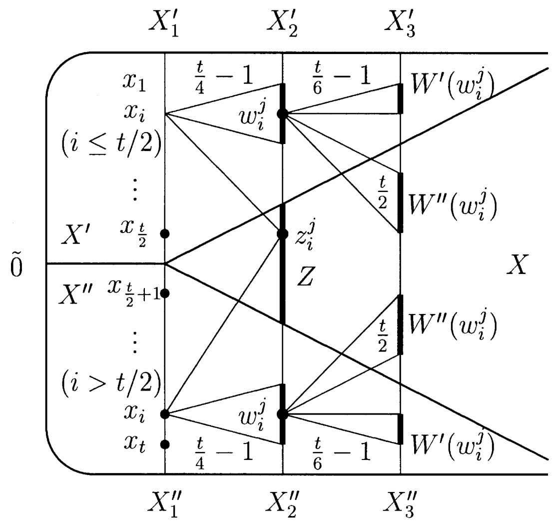


Fig. 2. The image of $T$ in the subcube $X$.

**Lemma 9.** *Let $\ell < \lfloor n/2 \rfloor$, and $\mathscr{F} = \{S(x) \mid x \in Q_\ell^n\}$. Then for $p = \lfloor (n-\ell)/(\ell+1) \rfloor$ there exists the SRS of the family $\mathscr{F}$ with spectrum $(p, p, \ldots, p)$.*

**Proof.** Indeed, let $A \subseteq Q_\ell^n$. Counting by two ways the number $d$ of edges of $Q^n$ connecting the vertex sets $A$ and $S(A)$, one has $|A|(n-\ell) = d \leqslant |S(A)|(\ell+1)$. Therefore,

$$|S(A)|/|A| \geqslant (n-\ell)/(\ell+1)$$

and the lemma follows from Theorem 1. $\square$

It follows from the proof of the lower bound, that the dimension of the hypercube containing $T$ as a subgraph is minimum, if $\sum_{i=1}^t a_i \sim t^2/4$ and, respectively, $\sum_{i=1}^t \sum_{i=1}^{a_i} b_i^j \sim t^3/6$. In other words, the set of the tree vertices mapped into $X_2$ should have cardinality around $t^2/4$ and (asymptotically) cover $X_3$ in the subcube $X$. This forces to use the Turán's construction [10] for the corresponding covering set.

Consider the subcubes $X'$ and $X''$ of $X$ of dimension $t/2$ which contain the origin of $X$ (cf. Fig. 2). Without loss of generality we assume that $x_i$ has a 1 in its $i$th entry and 0's in all other entries. Then we can view the vertex sets of these subcubes as $\{(\beta_1, \ldots, \beta_t) \mid \beta_i = 0 \text{ for } i \geqslant 1 + t/2\}$ and $\{(\beta_1, \ldots, \beta_t) \mid \beta_i = 0 \text{ for } i \leqslant t/2\}$, respectively. The subcubes $X'$ and $X''$ partition the set $X_1$ into two equal parts $X_1' = \{x_1, \ldots, x_{t/2}\}$ and $X_1'' = \{x_{t/2+1}, \ldots, x_t\}$. Let $Z = X_2 \setminus (X_2' \cup X_2'')$. Thus each vertex of $Z$ has exactly two entries which are 1, one of these being among the first $t/2$ entries and the other being among the second set of $t/2$ entries. We denote the vertices of $Z$ by $z_i^j$ $(i, j \in [1, t/2])$, assuming that $x_i \in X_1'$ is adjacent with $z_i^1, \ldots, z_i^{t/2}$ $(i \in [1, t/2])$ and $x_i \in X_1''$ is adjacent with $z_1^{i-t/2}, \ldots, z_{t/2}^{i-t/2}$, $i \in [t/2+1, t]$. Obviously, $|Z| = t^2/4$.

**Lemma 10.** *For any $i \in [1,n]$ and $x_i \in X_1$ there exists a subset $R_i \subseteq S(x_i) \cap Z$ with $|R_i| = t/4$ so that the family $\{R_i \mid i \in [1,t]\}$ forms a partition of the set $Z$.*

**Proof.** To do so just put

$$R_i = \begin{cases} \{z_i^j \mid j \in [1,t/4]\} & \text{for } i \in [1,t/4], \\ \{z_i^j \mid j \in [t/4+1,t/2]\} & \text{for } i \in [t/4+1,t/2], \\ \{z_j^{i-t/2} \mid j \in [t/4+1,t/2]\} & \text{for } i \in [t/2+1,3t/4], \\ \{z_j^{i-t/2} \mid j \in [1,t/4]\} & \text{for } i \in [3t/4+1,t]. \end{cases} \qquad \square$$

Now, we are ready to construct an embedding of $T$ which satisfies (10).

**Lemma 11.** *If $t = 0 \pmod{12}$ and $t$ is large enough, then* $\dim(T) \leqslant \frac{227}{120}t + 5$.

**Proof.** Embed the root of $T$ into the origin of $Q^n$, and embed $T_1$ into $X_1$. First we describe the embedding of $T_2$. Below we introduce some subsets in the subcubes $X'$ and $X''$ which are schematically shown in Fig. 2.

Applying Lemma 9 to the subcubes $X'$ and $X''$ with $n = t/2$ and $\ell = 1$, we obtain that for each $x_i \in X_1'$ (resp. for each $x_i \in X_1''$) it is possible to choose $\lfloor (t/2-1)/2 \rfloor = t/4 - 1$ vertices $w_i^1, \ldots, w_i^{t/4-1} \in S(x_i) \cap X_2'$ (resp. of $S(x_i) \cap X_2''$) so that all the vertices $w_i^j$ ($i \in [1,t]$, $j \in [1,t/4-1]$) are distinct. In accordance with this, for each $v_i \in T_1$ we embed some $t/4-1$ vertices of $T_2$, which are adjacent with $v_i$, into $X_2' \cup X_2''$, $i \in [1,t]$.

Now, consider the subcubes $Y(x)$ with $x \in V(X)$, whose dimension we denote by $n'$. The origins of these subcubes (considered as vertices of $Q^n$) are vertices of the subcube $X$ and so they have zeros in the last $n - t = n'$ entries. Let $y' \in Y(x')$ and $y'' \in Y(x'')$ for some $x', x'' \in V(X)$. We call the vertices $y'$ and $y''$ *complementary* if they agree in the last $n'$ entries. Obviously, if the vertices $x'$ and $x''$ are adjacent, then the complementary vertices $y'$ and $y''$ are also adjacent. For $x \in V(X)$ we denote the vertices of $Y_1(x)$ by $y_1(x), \ldots, y_{n'}(x)$, assuming that the vertices with the same index corresponding to different $x$ are complementary.

For each $v_i \in T_1$ we have to embed into $Y(x_i)$ a subtree $T(v_i)$ of $T$ of depth 2. The first level of $T(v_i)$ consists of $3t/4+1$ vertices of $T_2$ that have not been embedded so far. We now embed them into the set $U_i$, $i \in [1,t]$, defined by

$$U_i = \begin{cases} \{y_1(x_i), \ldots, y_{m_3}(x_i)\} & \text{for } i \in [1,t/2], \\ \{y_1(x_i), \ldots, y_{m_1}(x_i)\} \cup \{y_{m_2+1}(x_i), \ldots, y_{n'}(x_i)\} & \text{for } i \in [t/2+1,t], \end{cases}$$

where $m_1 = n' - t/3 + 1$, $m_2 = 2n' - 13t/12$, $m_3 = 3t/4 + 1$.

The choice of the sets $U_i$ is graphically shown in Figs. 3a and b for $i \in [1,t/2]$ and $i \in [t/2+1,t]$, respectively. In this figure we schematically show the vertices of $Y_1(x_i)$, numbered by $1, \ldots, n'$. The solid segments represent the vertices of $U_i$.

As we will show that $n' = \frac{107}{120}t + O(1)$, then $0 < m_1 < m_2 < m_3 < n'$ for $t$ sufficiently large and $m_3 = m_1 + (n' - m_2) = |U_i|$. This ensures the correctness of the embedding of $T_2$.

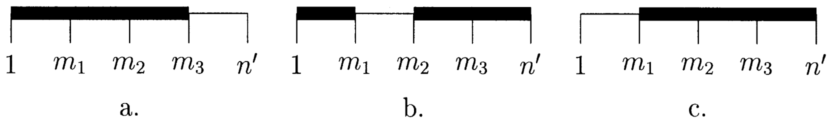


Fig. 3. The image of $T_2$ in $Y_1(x_i)$ (a), (b) and the image of $T_3$ in $Y_1(w_i^j)$ (c).

Now let us turn to the embedding of $T_3$. Applying Lemma 9 to the subcube $X'$ with $n=t/2$ and $\ell=2$, we obtain that for each $w_i^j \in S(x_i) \cap X_2'$ ($i \in [1,t/2]$, $j \in [1,t/4-1]$) it is possible to choose $\lfloor (t/2-2)/3 \rfloor = t/6-1$ vertices of $S(w_i^j) \cap X_3'$ so that these subsets for different $w_i^j$ are disjoint. A similar fact is also valid for the subcube $X''$. Denote by $W'(w_i^j)$ the subset related with the vertex $w_i^j$ (cf. Fig. 2).

Furthermore, since for any $z' \in X_2'$ and any $z'' \in X_2''$ we have $\rho(z',z'')=4$, then $S(z') \cap S(z'')=\emptyset$. For $z \in X_2' \cup X_2''$ denote $W''(z)=S(z) \cap (X_3 \setminus (X_3' \cup X_3''))$ (cf. Fig. 2). Therefore, the subsets $W''(w_i^j)$ for different $w_i^j$ are disjoint. It is easily shown that $|W''(w_i^j)|=t/2$. In accordance with this we embed for each $f^{-1}(w_i^j) \in T_2$ some $(t/6-1)+t/2=2t/3-1$ vertices of $T_3$, which are adjacent with $f^{-1}(w_i^j)$, into the subset $W'(w_i^j) \cup W''(w_i^j) \subseteq X_3$, $i \in [1,t]$, $j \in [1,t/4-1]$. We embed the remaining $t-(2t/3-1)=t/3+1$ vertices of $T_3$, which are adjacent with $f^{-1}(w_i^j)$, into $Y_1(w_i^j)$, using for this purpose the vertices $y_{m_1+1}(w_i^j),\dots,y_{n'}(w_i^j)$. These $t/3+1$ vertices are schematically shown in Fig. 3c by the solid line.

To complete the embedding we have to embed for each $u_i^j \in U_i$ $t$ vertices of $T_3$, adjacent with $f^{-1}(u_i^j)$ into $Y_2(x_i) \cup (\bigcup_{z \in X_2} Y_1(z))$. Our goal is to use as many vertices of $\bigcup_{z \in X_2} Y_1(z)$ as possible in order to decrease the dimension of the tree $T(x_i)$ according to Theorem 2. We describe for each $u_i^j$ the choice of the complementary vertices in $\bigcup_{z \in X_2} Y_1(z)$. First, for $i \in [1,t]$ and $j \in [1,m_1]$, we choose $t/4-1$ complementary vertices in $\bigcup_{l=1}^{t/4-1} Y_1(w_i^l)$. Thus, (cf. Figs. 2 and 3c) all the vertices of $\bigcup_{l=1}^{t/4-1} Y_1(w_i^l)$ are used for the embedding of $T_3$.

Now consider the vertices of $\bigcup_{z \in Z} Y_1(z)$. Remember that for each $x_i \in X_1$ we have $|S(x_i) \cap Z|=t/2$. In accordance with this we choose for $i \in [1,t/2]$ and $j \in [m_1+1,m_2]$ and for $i \in [t/2+1,t]$ and $j \in [m_3+1,n']$ all the $t/2$ complementary (to $u_i^j$) vertices in $\bigcup_{z \in Z} Y_1(z)$. Since the ranges for $j$ for these vertices do not intersect, all the selected vertices are distinct. Finally, for $i \in [1,t]$ and $j \in [1,m_1] \cup [m_2+1,m_3]$, we choose $t/4$ complementary (to $u_i^j$) vertices in $\bigcup_{z \in R_i} Y_1(z)$. Lemma 10 ensures that all the selected vertices are distinct.

Therefore, for each $i \in [1,t]$ and $j \in [1,m_3]$ we have embedded some vertices of $T_3$ adjacent with $f^{-1}(u_i^j)$ into $\bigcup_{z \in X_2} Y_1(z)$. The number of these vertices is $t/2-1$ for $j \in [1,m_1]$, $t/2$ for $j \in [m_1+1,m_2]$ and $t/4$ for $j \in [m_2+1,m_3]$. We embed the remaining vertices of $T_3$ adjacent with $f^{-1}(u_i^j)$ into $Y_2(x_i)$. In other words, for each $i \in [1,t]$ we have to embed a tree $T(x_i)$ of depth 2 rooted in $x_i$ into the subcube $Y(x_i)$. One has: $T_1(x_i)=\{u_i^j \mid j \in [1,m_3]\}$. Furthermore, the degrees of vertices of $T_1(x_i)$ are complementary (relative to $t+1$) to the number of vertices already embedded and are equal to $t/2+2$, $t/2+1$ and $3t/4+1$, respectively.

Table 1
The dimension of ternary trees of small depth

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\dim(T^{k,3})$ | 3 | 5 | 7 | 8 | 10 |

Applying Corollary 4 with $a=3t/4$, $b=t/2+1$ and $c=m_3-m_2=11t/6-2n'+1$, we get that the sufficient dimension of a hypercube for embedding $T(x_i)$ is

$$n' = \max\left\{\frac{5t}{3}-n'+1,\quad \left\lceil \frac{3t+4}{8} + \frac{(11t/6-2n'+1)\cdot 3t/4+(2n'-13t/12)(t/2+1)}{3t/4}\right\rceil\right\}.$$

This implies $n' \leqslant \frac{107}{120}t+5$ for $t$ large enough and the upper bound $\dim(T) \leqslant n = n'+t$ follows. $\square$

If $t$ is not a multiple of 12, then the described construction provides an embedding of $T^{3,t}$ into a hypercube of dimension $227t/120+O(1)$. Therefore, Lemmas 8 and 11 imply the following result:

**Theorem 12.** $\lim_{n\to\infty}\dim(T^{3,t})/t = \frac{227}{120}$.

## 4. Embedding ternary trees

In this section we prove that $\lim_{k\to\infty}\dim(T^{k,3})/k \leqslant 5/3$. Obviously, if $T^{p,t}$ is a subgraph of $Q^q$ and $T^{r,t}$ is a subgraph of $Q^s$ for some $p,r \geqslant 1$ and a fixed $t$, then $T^{p+r,t}$ is a subgraph of $Q^{q+s}$. A standard way to get an upper bound for $\dim(T^{k,t})$ is to find a clever embedding of $T^{k_0,t}$ into $Q^{n_0}$ for some $n_0$, which would imply the upper bound $\lim_{k\to\infty}\dim(T^{k,t})/k \leqslant n_0/k_0$ (cf. e.g. [4]).

Following this idea let us consider the function $\dim(T^{k,3})$ for small values of $k$ stored in Table 1. The entries of this table are equal to the corresponding lower bounds implied by counting arguments and they are supported by constructing embeddings with the help of a computer [2]. From this table it follows that in order to get an upper bound, necessarily smaller than $2k$, one has to consider hypercubes of a relatively large dimension (at least 11), and so to find a satisfactory bound in this way is technically difficult. In the next theorem we introduce a new approach.

**Theorem 13.** $\lim_{k\to\infty}\dim(T^{k,3})/k \leqslant \frac{5}{3}$.

**Proof.** Using an embedding of $T^{k_0,3}$ into $Q^{n_0}$ for some $k_0,n_0 \geqslant 1$, assume for a moment that one can extend this embedding up to an embedding of $T^{k_0+3,t}$ into $Q^{n_0+5}$. Then applying this construction recursively, at the $i$th step of this process we obtain an

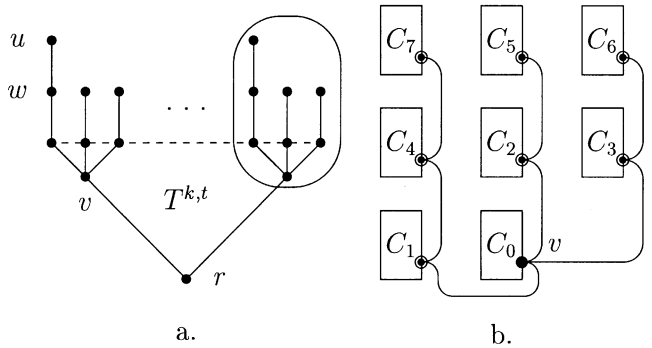


Fig. 4. The tree $\hat{T}^k$ (a) and its fragment used in the construction (b).

embedding of $T^{k_0+3i,3}$ into $Q^{n_0+5i}$. This would lead to the upper bound

$$\lim_{k\to\infty} \frac{\dim(T^{k,3})}{k} \leqslant \lim_{i\to\infty} \frac{n_0+5i}{k_0+3i} = \frac{5}{3}. \tag{17}$$

A straightforward realization of this idea would be to embed $T^{3,3}$ into $Q^5$ and apply the standard arguments mentioned above. However, Table 1 shows that $\dim(T^{3,3}) = 7$, so we need a deeper insight on the embedding of $T^{k_0,3}$ into $Q^{n_0}$.

To reduce the number of vertices considered under this approach we use a stronger inductive hypothesis, extending the tree $T^{k,3}$ up to the tree $\hat{T}^k$ for $k \geqslant 1$. To define this tree we first introduce the tree $\tilde{T}^k$ as one obtained from $T^{k,3}$ by joining each leaf with one new vertex. Thus each of the $3^k$ new vertices is a leaf of $\tilde{T}^k$ and each leaf of $T^{k,3}$ transforms into a vertex of degree 2 in $\tilde{T}^k$. Let the root of $T^{k,3}$ be the root of $\tilde{T}^k$ (cf. Fig. 4a).

Now let $v$ be a vertex at distance $k-1$ from the root of $\tilde{T}^k$ and let $w$ be a leaf of $\tilde{T}^k$ at distance 2 from $v$. The tree $\hat{T}^k$ is obtained from the tree $\tilde{T}^k$ by adding for each such $v$ one new vertex $u$ adjacent with $w$. In this construction we assume that all the new vertices are distinct (cf. Fig. 4a). Thus, the tree $\hat{T}^k$ also has $3^k$ leaves and $3^{k-1} + 3^k$ more vertices than the tree $T^{k,3}$.

We represent $Q^{n_0+5}$ as the cartesian product $Q^{n_0} \times Q^5$. The simplified graph of $Q^5$ is shown in Fig. 5a (some edges parallel to the edge $(x,y)$ are omitted) and we further reduce it to the one shown in Fig. 5b leaving only the vertices of $Q^5$ in the same positions as in Fig. 5a. The origins are shown by larger circles.

Now, our goal is to extend an embedding of $\hat{T}^{k_0}$ into $Q^{n_0}$ up to an embedding of $\hat{T}^{k_0+3}$ into $Q^{n_0+5}$. Since $\dim(T^{k,3}) \leqslant \dim(\hat{T}^k)$ then by using similar arguments as in the proof of (17), we will get the theorem. Note that since the values $k_0$ and $n_0$ are fixed, they do not affect the limit in (17).

We start with any embedding of $\hat{T}^{k_0}$ into $Q^{n_0}$ and show how to embed the rest of the vertices of $\hat{T}^{k_0+3}$ by using 5 new dimensions. For each vertex $v$ of $\hat{T}^{k_0}$ at distance

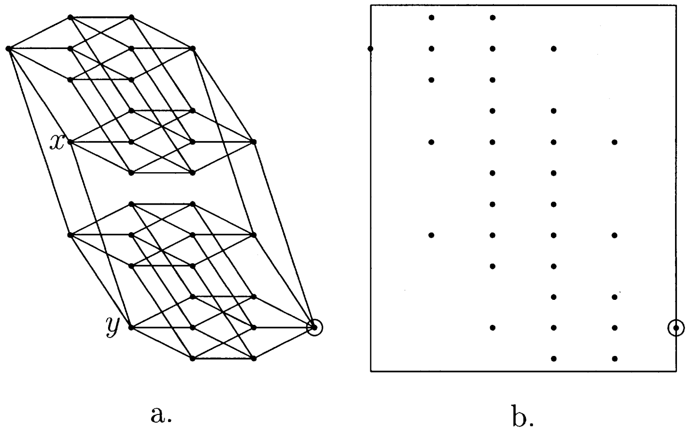


Fig. 5. $Q^5$ (a) and its simplified image (b).

$k_0 - 1$ from its root, we consider a subtree, which is rooted in $v$ and shown in the oval in Fig. 4a. Thus we obtain the structure shown in Fig. 4b where each rectangle $C_0, \ldots, C_7$ represents $Q^5$ formed by the 5 extra dimensions. We assume that the vertices shown in Fig. 4b are the origins of the five-dimensional hypercubes. The definition of the cartesian product implies that if the origins of $C_i$ and $C_j$ $(i, j \in [0, 7])$ are adjacent in Fig. 4b, then the remaining corresponding vertices of $C_i$ and $C_j$ are also adjacent. Thus the embedding of $\hat{T}^{k_0}$ into $Q^{n_0}$ provides many edges between the subcubes $C_i$, which we will use in the construction below.

The embedding is shown in Fig. 6, where we depicted the tree edges only. The vertices at distance 1 from $v$ correspond to the leaves of $T^{k_0,3}$. The vertices at distance 4 from $v$ (corresponding to the leaves of $T^{k_0+3,3}$) are shown as larger circles. The vertices of distance 5 and 6 from $v$ (corresponding to the leaves of $\tilde{T}^{k_0+3}$) are shown as the endpoints of vectors. $\square$

**Remark 14.** If one compares this result in the light of the old techniques, it becomes apparent that to prove Theorem 13 by using the old approach, one would have to prove that $T^{3r}$ can be embedded into $Q^{5r}$ for some $r \geqslant 6$. To demonstrate this, we computed the function $n(k)$ defined by (2) for $k \in [1, 18]$ and found out that the ratio $n(k)/k$ reaches 5/3 for the first time just when $k = 18$.

**Remark 15.** With help of our method further constructive improvements of the upper bound for $\dim(T^{k,3})$, involving consideration of subcubes of relatively small dimensions, are possible if one succeeds to embed 5 extra levels of the tree using 8 extra dimensions of the hypercube. Then one would get the multiplicative constant 1.60 instead of $5/3 \approx 1.66$ as in our case.

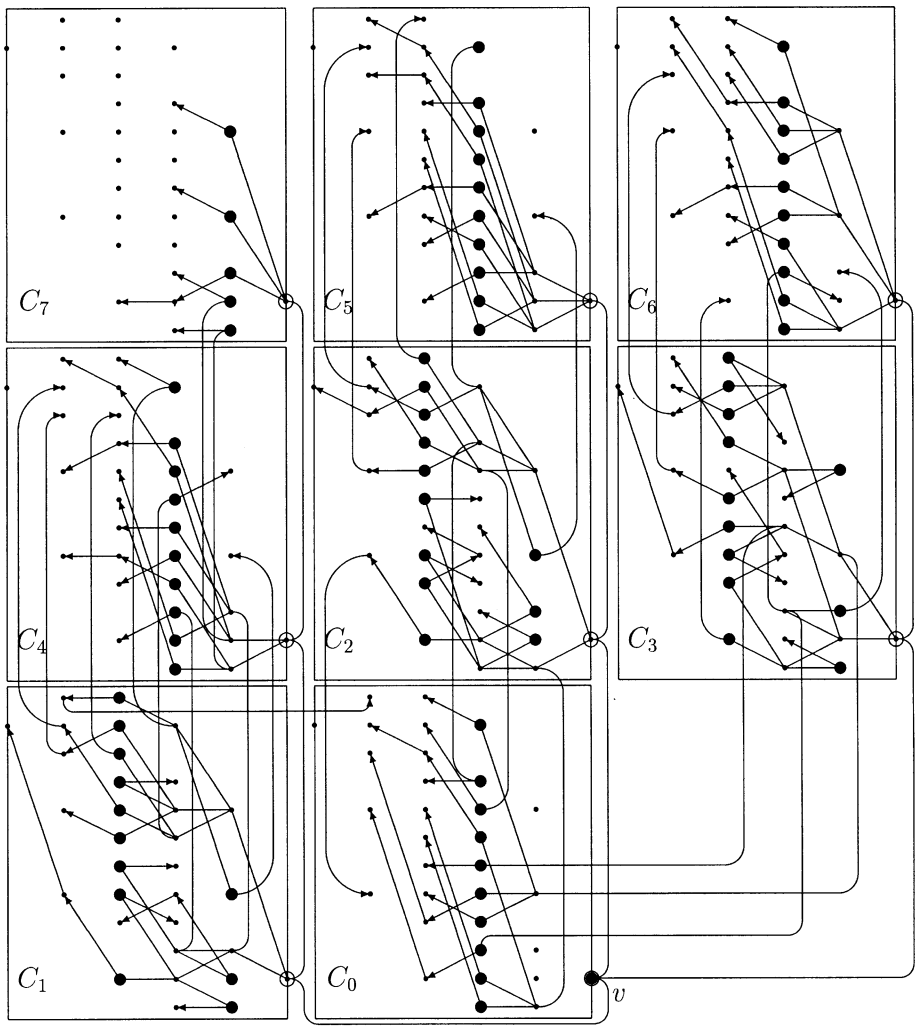


Fig. 6. Embedding of three extra levels of the tree $\hat{T}$.

## 5. The general case

In this section we improve the upper bound (1) for $\dim(T^{k,t})$ asymptotically.

**Theorem 16.** $\lim_{k,t\to\infty}\dim(T^{k,t})/kt \leqslant \frac{307}{640} \approx 0.48$.

**Proof.** Denote by $T^{k,t}(\ell)$ the tree which is obtained from $T^{k,t}$ by adding to each its leaf $\ell$ new vertices adjacent with the leaf. Thus the tree $T^{k,t}(\ell)$ has $k+2$ levels and $\ell t^k$ leaves.

Consider the embedding of $T^{3,t}$ into $Q^n$ with $n=\frac{227}{120}t+O(1)$, which was constructed in Section 3. From Lemma 9 it follows that for each vertex $x\in Q_3^n$ one can choose a set $L(x)\subseteq S(x)$ with $|L(x)|=n/4+O(1)$ so that the sets $L(x)$ for distinct $x$ are disjoint. In other words, there exists an embedding of $T^{3,t}(\ell)$ into $Q^n$ with $n$ defined above and $\ell=n/4+O(1)=\frac{227}{480}t+O(1)$ as $t\to\infty$.

For our purposes, however, it is necessary to be able to embed the tree $T^{3,t}(\lceil t/2\rceil)$ into some hypercube. To do so, we simply add $\lceil t/2\rceil-\ell$ extra dimensions to $Q^n$. This provides an embedding of $T^{3,t}(\lceil t/2\rceil)$ into $Q^{n_0}$ with $n_0=n+(\lceil t/2\rceil-\ell)=307t/160+O(1)$. Note that (10) is satisfied for this embedding.

Now, we introduce an inductive procedure similar to one in the proof of Theorem 13. We start with embedding of $T^{k_0,t}(\lceil t/2\rceil)$ into $Q^{n_0}$ for $k_0=3$. Assuming that $T^{k_i,t}(\lceil t/2\rceil)$ is a subgraph of $Q^{n_i}$, we show that $T^{k_{i+1},t}(\lceil t/2\rceil)$ can be embedded into $Q^{n_{i+1}}$ with $k_{i+1}$ and $n_{i+1}$ defined by $k_{i+1}=2k_i+1$ and $n_{i+1}=2n_i$, $i=0,1,2,\dots$ . Thus we got the sequences

$$n_i=2^in_0 \quad\text{and}\quad k_i=2^ik_0+2^i-1. \tag{18}$$

This would imply

$$\frac{\dim(T^{k,t})}{k}\leqslant\lim_{i\to\infty}\frac{2^in_0}{2^ik_0+2^i-1}=\frac{n_0}{k_0+1}=\frac{307}{640}t+O(1) \tag{19}$$

and hence $\dim(T^{k,t})\leqslant\frac{307}{640}kt(1+o(1))$ as $k,t\to\infty$.

To prove the inductive step consider $Q^{2n_i}$ and represent it as $Q'\times Q''$, where $Q'$ and $Q''$ are hypercubes of dimension $n_i$. For $x=(\alpha_1,\dots,\alpha_{2n_i})\in Q^{2n_i}$ introduce the subcubes $Q'(x)$ and $Q''(x)$ with vertex sets

$$\{(\beta_1,\dots,\beta_{2n_i})\in Q^{2n_i}\mid \beta_j=\alpha_j,\ j\in[1,n_i]\},$$
$$\{(\beta_1,\dots,\beta_{2n_i})\in Q^{2n_i}\mid \beta_j=\alpha_j,\ j\in[n_i+1,2n_i]\}$$

and origins in $(\alpha_1,\dots,\alpha_{n_i},0,\dots,0)$ and $(0,\dots,0,\alpha_{n_i+1},\dots,\alpha_{2n_i})$, respectively.

Let $f'$ be an embedding of $T^{k_i,t}(\lceil t/2\rceil)$ into the subcube $Q'(\tilde 0)$ of dimension $n_i$. We additionally claim that the image of the $\ell$th level of this tree is embedded into the $\ell$th level of $Q'(\tilde 0)$ for $\ell\in[0,k+2]$. This embedding induces an embedding of $T^{k_i,t}$ into $Q'(\tilde 0)$ with the similar property. Let $x$ be the image of a leaf of $T^{k_i,t}$ in this embedding. Construct for each $x$ the isomorphic embedding $f''$ of $T^{k_i,t}(\lceil t/2\rceil)$ (and thus $T^{k_i,t}$) into the subcube $Q''(x)$. This procedure results in an embedding $f$ of $T^{2k_i,t}(\lceil t/2\rceil)$ into $Q^{2n_i}$.

Let $y\in Q''(x)$ be the image of a leaf of $T^{k_i,t}$ in embedding $f''$ (cf. Fig. 7a). Since according to our assumption $T^{k_i,t}(\lceil t/2\rceil)$ can be embedded into $Q'(\tilde 0)$, then one can choose a subset $R'(y)\subseteq S(y)\cap Q'(y)$ with $|R'(y)|=\lceil t/2\rceil$, so that these subsets taken for different $y$ are disjoint. Similarly, since $T^{k_i,t}(\lceil t/2\rceil)$ can be embedded into $Q''(x)$, one can choose a subset $R''(y)\subseteq S(y)\cap Q''(x)$ with $|R''(y)|=\lceil t/2\rceil$, so that these subsets taken for different $y$ are disjoint (cf. Fig. 7a). This means that one can embed the tree $T^{2k_i+1,t}$ into $Q^{2n_i}$ and it remains to show that this embedding can be extended up to an embedding of $T^{2k_i+1,t}(\lceil t/2\rceil)$ into $Q^{2n_i}$.

Consider the subgraphs $G'$ and $G''$ of $Q^{2n_i}$ induced by the vertex sets $R'(y)\cup\{y\}$ and $R''(y)\cup\{y\}$, respectively. Both subgraphs are isomorphic to the star shown in Fig. 7b.

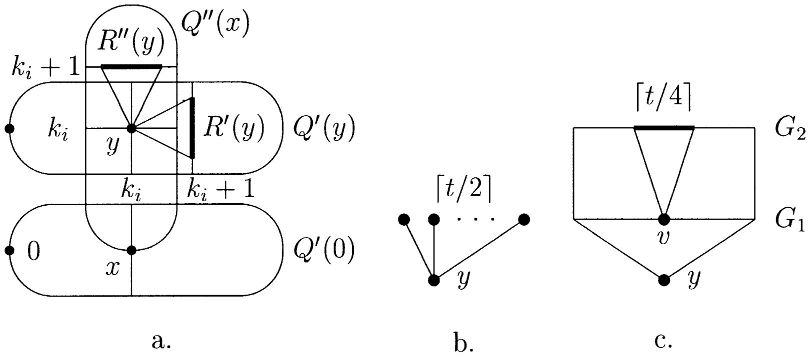


Fig. 7. Constructions for embedding $T^{k,t}$.

Clearly, $Q^{2n_i}$ contains for each $y$ the graph $G = G' \times G''$ as a subgraph and these graphs for different $y$ are disjoint. The graph $G$ is schematically shown in Fig. 7c. In this figure we denote by $G_1$ and $G_2$ the sets of vertices of distance 1 and 2 from the vertex $y$, respectively. It is easily shown that in $G$ each vertex $v \in G_1$ is adjacent with exactly $\lceil t/2 \rceil$ vertices of $G_2$ and each vertex $w \in G_2$ is adjacent with exactly 2 vertices of $G_1$. Thus, applying similar arguments as in the proof of Lemma 9, we conclude that for each vertex $v \in G_1$ one can choose $\lceil t/4 \rceil$ vertices of $G_2$ adjacent with $v$ so that all such sets considered for distinct $v$ are disjoint. Note that $G_2 \cap V(Q'(y)) = G_2 \cap V(Q''(x)) = \emptyset$.

Finally, Lemma 9 applied to the hypercubes $Q'$ and $Q''$ with $n = n_i$ and $\ell = k_i + 1$ ($n_i$ and $k_i$ are determined by (18)) implies that for each $z \in Q'_{k_i+1}$ (resp. $z \in Q''_{k_i+1}$) one can choose $\lceil t/4 \rceil$ vertices of $S(z) \cap Q'_{k_i+2}$ (resp. $S(z) \cap Q''_{k_i+2}$) in such a way that these sets considered for distinct $z$ are disjoint. The choice of these vertices applied to $z \in R'(y) \cup R''(y)$ in combination with the $\lceil t/4 \rceil$ vertices chosen above results in the required embedding of $T^{k_{i+1},t}(\lceil t/2 \rceil)$ into $Q^{n_{i+1}}$. □

**Remark 17.** Let us mention that one cannot get an improvement of the upper bound (1) in our method, using only trees of depth 2.

## 6. Concluding remarks

Let us call an embedding $f$ of $T^{k,t}$ into $Q^n$ *oriented* if $f(T^{k,t}_\ell) \subseteq Q^n_\ell$ for $\ell \in [0,k]$ (cf. (10) for $\ell = 3$). We denote the minimum $n$ for which there exists an oriented embedding of $T^{k,t}$ into $Q^n$ by $\overrightarrow{\dim}(T^{k,t})$.

Oriented embeddings are easier for analysis, because one may restrict oneself to two consecutive levels of the hypercube. Oriented embeddings of binary trees are studied in [1,4], where it is proved that

$$1.29 \leqslant \lim_{k\to\infty} \overrightarrow{\dim}(T^{k,2})/k \leqslant 4/3.$$

Note that the upper bounds (1) and (19) are obtained by using oriented embeddings.

Table 2
Some asymptotic upper bounds

| $k$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $\frac{\dim(T^{k,t})}{t}$ | $\frac{3}{2}=1.5$ | $\frac{227}{120}\approx 1.89$ | $\frac{387}{160}\approx 2.42$ | $\frac{467}{160}\approx 2.92$ | $\frac{13}{4}=3.25$ | $\frac{307}{80}\approx 3.84$ |

Remember that the embedding of $T^{3,t}$ in the construction of Theorem 12 is also oriented. To get a lower bound for $\dim(T^{3,t})$ we have proved in Lemma 5 that one can construct an oriented embedding of $T^{3,t}$ into the hypercube of a dimension, which is at most one more than the minimum one. Now we extend Lemma 5 to a more general case.

**Theorem 18.** *Let $k,t\to\infty$ and $k=\mathrm{o}(\sqrt{t})$. Then $\dim(T^{k,t})\sim\overrightarrow{\dim}(T^{k,t})$.*

**Proof.** Consider an embedding $f$ of $T=T^{k,t}$ into $Q^n$ and assume that $f$ is not oriented. Now, we describe a procedure to obtain an oriented embedding $g$ of $T$ into $Q^{n'}$ with $n\leqslant n'\leqslant n+(k-2)(k-1)/2$.

Without loss of generality, we assume $T_\ell\subseteq Q^n_\ell$ for $\ell=0,1,2$. Assume that this holds for $\ell\in[0,p]$ for some $p\geqslant 2$ and does not hold for $\ell=p+1\leqslant k$. Then there exist vertices $v\in T_{p+1}$ and $u\in T_p$ with $(u,v)\in E(T)$ such that

$$f(v)\in Q^n_{p-1}\quad\text{and}\quad f(u)\in Q^n_p. \tag{20}$$

For $w\in T_i$ denote by $T(w)$ the subtree of $T$ isomorphic to $T^{k-i,t}$, which has its root in $w$ and $V(T(w))\subseteq\{w\}\cup T_{i+1}\cup\cdots\cup T_k$. Since $V(T(v))\subseteq V(T(u))$, then $f(T(v))\subseteq f(T(u))$.

Split $Q^{n+1}$ into 2 subcubes $Q'$ and $Q''$ defined by $x_{n+1}=0$ and $x_{n+1}=1$, respectively. We assume that the origins of $Q$ and $Q'$ are the same. For $A\subseteq V(Q')$ denote by $\pi(A)$ its projection into the subcube $Q''$, i.e. the set obtained from $A$ by replacing the $(n+1)$th entry of each of its vertices with 1.

We embed $T$ into the subcube $Q'$ using the embedding $f$ and denote by $T'(u)$ the subtree of $T(u)$, which is rooted in $u$ and is isomorphic to $T^{k-p-1,t}$. Now for all edges $(u,v)\in E(T)$ satisfying (20) replace $f(T(v))$ with $\pi(f(T'(u)))$. This provides an embedding $g$ of $T$ into $Q^{n+1}$. Since $f(T(w))\cap f(T(w'))=\emptyset$ for distinct $w,w'\in T_p$, then $g$ is an injective mapping and $g(T_\ell)\subseteq Q^{n+1}_\ell$ for $\ell\in[0,p+1]$.

Repeating this process for $p=3,\ldots,k$ results in an oriented embedding of $T$ into $Q^{n'}$ with $n'\leqslant n+(k-2)(k-1)/2$. Since (cf. (1)) $\overrightarrow{\dim}(T^{k,t})=\mathrm{O}(kt)$ as $k,t\to\infty$, then for $k=\mathrm{o}(\sqrt{t})$ we constructed an oriented embedding of $T^{k,t}$ into the hypercube of asymptotically the same dimension as $\dim(T^{k,t})$. $\square$

Theorem 18 gives one more motivation for studying the oriented embeddings. It would be of interest to know if the condition $k=\mathrm{o}(\sqrt{t})$ can be weakened.

The techniques which we demonstrated here and in [1] gives a way to obtain better asymptotic upper bounds for $T^{k,t}$ than (1), particularly if one of the parameters $k,t$ is fixed. In Table 2 we present without proof some asymptotic upper bounds for $\dim(T^{k,t})$ for initial values of $k$, which may easily be established by combining the techniques used in the proof of Theorem 16.

Concerning the asymptotic lower bounds as $k, t \to \infty$, we distinguish the two following cases. The first case is $k = \Omega(t)$. We conjecture that in this case $\dim(T^{k,t}) \sim kt/e$. In contrast to this the second case is $k = o(\sqrt{t})$ (condition of Theorem 18). We conjecture that then $\lim_{k,t\to\infty} \dim(T^{k,t})/kt > 1/e$.

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