Global entropy solutions to a variant of the compressible Euler equations

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Abstract


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1. Introduction

Let us consider the Cauchy problem for the nonlinear hyperbolic system

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ u_t + \left( \frac{1}{2} u^2 + P(\rho) \right)_x = 0 
\end{cases}$$

with bounded measurable initial data

$$(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)) \quad \rho_0(x) \geq 0,$$

where the nonlinear function $P(\rho) = \theta \rho^{\gamma-1}$, $\theta = \frac{\gamma-1}{2}$ and $\gamma \in (1, 3)$ is a constant.

System (1) was first derived by Earnshaw [2] in 1858 for isentropic flow and is also referred to as the Euler equations of one-dimensional, compressible fluid flow, where $\rho$ denotes the density, $u$ the velocity, and $P(\rho)$ the pressure of the fluid. System (1) has other different physical backgrounds. For instance, it is a scaling limit system of...
a Newtonian dynamics with long-range interaction for a continuous distribution of mass in \( R \) and also a hydrodynamic limit for the Vlasov equation (see [5]).

By simple calculations, two eigenvalues of system (1) are
\[
\lambda_1 = u - \theta \rho^\theta, \quad \lambda_2 = u + \theta \rho^\theta
\]
with corresponding right eigenvectors
\[
\mathbf{r}_1 = (1, -\theta \rho^{\theta-1})^T, \quad \mathbf{r}_1 = (1, \theta \rho^{\theta-1})^T;
\]
the two corresponding Riemann invariants are
\[
w = u + \rho^\theta, \quad z = u - \rho^\theta;
\]
and
\[
\nabla \lambda_1 \cdot \mathbf{r}_1 = -\theta (\theta + 1) \rho^{\theta-1}, \quad \nabla \lambda_2 \cdot \mathbf{r}_2 = \theta (\theta + 1) \rho^{\theta-1}.
\]

Thus both characteristic fields are linearly degenerate on \( \rho = \infty \), since \( 1 < \gamma < 3 \).

The study of the existence of global weak solutions for the Cauchy problem (1) and (2) was started by DiPerna [1] for the case of \( 1 < \gamma < 3 \) by using the Glimm’s scheme method, while in this work, we use the compensated compactness method and the kinetic formulation to get the existence of global entropy solutions for the Cauchy problem with a uniform amplitude bound. Namely, we assume the viscosity solutions to the following Cauchy problem (3) and (4) for the related parabolic system are uniformly bounded,

\[
\begin{cases}
\rho_t + (\rho u)_x = \varepsilon \rho_{xx} \\
u_t + \left( \frac{1}{2} u^2 + P(\rho) \right)_x = \varepsilon u_{xx}
\end{cases}
\tag{3}
\]

with initial data
\[
(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)),
\tag{4}
\]
where \((\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\rho_0(x) + \varepsilon, u_0(x)) * G^\varepsilon\), and \(G^\varepsilon\) is a mollifier.

**Theorem 1.** Let the initial data \((\rho_0(x), u_0(x))\) be bounded measurable and \(\rho_0(x) \geq 0\). Then the Cauchy problem (1) and (2) with a uniform amplitude bound has a global bounded entropy solution.

**Remark 1.** A pair of functions \((\rho(x, t), u(x, t))\) is called an entropy solution of the Cauchy problem (1) and (2) if

\[
\begin{align*}
\int_0^\infty \int_{-\infty}^\infty \rho \phi_t + \rho u \phi_x dx dt + \int_{-\infty}^\infty \rho_0(x) \phi(x, 0) dx &= 0 \\
\int_0^\infty \int_{-\infty}^\infty u \phi_t + \left( \frac{1}{2} u^2 + P(\rho) \right) \phi_x dx dt + \int_{-\infty}^\infty u_0(x) \phi(x, 0) dx &= 0
\end{align*}
\]

for any test function \(\phi(x, t) \in C_0^1(R \times R^+)\) and

\[
\eta(\rho(x, t), u(x, t))_t + q(\rho(x, t), u(x, t))_x \leq 0
\]

in the sense of distributions for any convex entropy \(\eta(\rho, u)\) of system (1), where \(q(\rho, u)\) is the entropy flux associated with \(\eta(\rho, u)\).

**2. Proof of Theorem 1**

Since the viscosity solutions to the Cauchy problem (3) and (4) are uniformly bounded, there exists a subsequence of the viscosity solutions (still labelled) \((\rho^\varepsilon(x, t), u^\varepsilon(x, t))\) such that

\[
w^* - \lim(\rho^\varepsilon(x, t), u^\varepsilon(x, t)) = (\rho(x, t), u(x, t)).
\]

We shall show that \((\rho(x, t), u(x, t))\) is an entropy solution of the Cauchy problem (1) and (2). For simplicity, we will drop the superscript \(\varepsilon\).
Now we use the kinetic formulation to give three families of entropies and entropy fluxes of the system, and later we shall show that these entropy–entropy flux pairs satisfy the compactness in $H^{-1}$. Any entropy flux pair $(\eta(\rho, u), q(\rho, u))$ of system (1) satisfies the additional system

$$q_\rho = u\eta_\rho + \theta^2 \rho^{\gamma - 2} \eta_u, \quad q_u = \rho \eta_\rho + u \eta_u. \tag{5}$$

Eliminating $q$ from (5), we have the entropy equation $\eta_{\rho \rho} = \theta^2 \rho^{\gamma - 3} \eta_{uu}$.

One family of weak entropies of system (1) is given by

$$\eta^0(\rho, u) = \int_R g(\xi)G_0(\rho, \xi - u) d\xi,$$

and the weak entropy flux $q^0$ associated with $\eta^0$ is

$$q^0(\rho, u) = \int_R g(\xi)(\theta \xi + (1 - \theta)u)G_0(\rho, \xi - u) d\xi;$$

two families of strong entropies of system (1) are given as follows (see [3–5]):

$$\eta^\pm(\rho, u) = \int_R g(\xi)G_{\pm}(\rho, \xi - u) d\xi,$$

and the strong entropy fluxes $q^\pm$ associated with $\eta^\pm$ are

$$q^\pm(\rho, u) = \int_R g(\xi)(\theta \xi + (1 - \theta)u)G_{\pm}(\rho, \xi - u) d\xi,$$

where $g(\xi)$ is a smooth function with a compact support set in $(-\infty, \infty)$ and the fundamental solutions

$$\begin{align*}
\begin{cases}
G_0(\rho, \xi - u) = [(w - \xi)(\xi - z)]^\gamma, \\
G_+(\rho, \xi - u) = (\xi - z)^\gamma(\xi - w)^\gamma, \\
G_-(\rho, \xi - u) = (w - \xi)^\gamma(z - \xi)^\gamma
\end{cases}
\end{align*}$$

and $\lambda = \frac{3 - \gamma}{2(\gamma - 1)} > 0$. Here we use the notation $x_+ = \max\{0, x\}$.

Next, we verify the compactness of $\eta^0 + q^0$ in $H^{-1}$. However, in the case of $\gamma > 3$, the entropy–entropy flux pair given above does not satisfy the compactness in $H^{-1}$. Let $\tau = \xi - w$. Then

$$\begin{align*}
\eta^+ (\rho, u) &= \int_w^\infty g(\xi)(\xi - z)^\gamma(\xi - w)^\gamma d\xi = \int_0^\infty g(\tau + w)(\tau + 2\rho^\theta)^\gamma \tau^\gamma d\tau, \\
\eta^+ _\rho &= \theta \rho^{\theta - 1} \int_0^\infty g'(\tau + w)(\tau + 2\rho^\theta)^\gamma \tau^\gamma d\tau + 2\lambda \theta \rho^{\theta - 1} \int_0^\infty g(\tau + w)(\tau + 2\rho^\theta)^\gamma \tau^\gamma d\tau \\
&= \theta \rho^\theta - 1 I^1 + 2\lambda \theta \rho^{\theta - 1} I^2. \tag{6}
\end{align*}$$

Since $\lambda > 0$, the integrals $I^1$ and $I^2$ are convergent as $\rho \to 0$; it follows from (6) that $\eta^+ _\rho = O(\rho^{\theta - 1})$, as $\rho \to 0$.

Obviously, if $\gamma \neq 2$, system (1) has a strictly convex entropy $\eta^* = \frac{u^2}{2} + \frac{\gamma - 1}{4(\gamma - 2)} \rho^{\gamma - 1}$ and the corresponding entropy flux $q^* = \frac{(\gamma - 1)^2}{4(\gamma - 2)} \rho^{\gamma - 1} u + \frac{u^2}{3}$; if $\gamma = 2$, system (1) has a strictly convex entropy $\eta^* = \frac{u^2}{2} + \frac{1}{4} \rho(\ln \rho - 1)$ and the corresponding entropy flux $q^* = \frac{1}{4} u\rho \ln \rho + \frac{u^2}{3}$. We multiply system (3) by $\nabla \eta^*(\rho, u)$ to obtain that

$$\begin{align*}
\eta^+_x + q^+_x &= \varepsilon \eta^*_{xx} - \varepsilon (\eta^*_\rho\rho_x^2 + 2\eta^*_\rho u_x + \eta^*_{uu} u_x^2) = \varepsilon \eta^*_{xx} - \varepsilon (\theta^2 \rho^{\gamma - 3} \rho_x^2 + u_x^2), \\
\text{so } \varepsilon \rho^{\gamma - 3} \rho_x^2 \text{ and } \varepsilon u_x^2 \text{ are bounded in } L^1_{\text{loc}}.
\end{align*}$$

Multiplying system (3) by $\nabla \eta^+ (\rho, u)$, we have

$$\begin{align*}
\eta^+_x + q^+_x &= \varepsilon \eta^+_{xx} - \varepsilon (\eta^*_\rho \rho_x^2 + 2\eta^+_\rho u_x + \eta^+_{uu} u_x^2) = I_1 + I_2.
\end{align*}$$

It is easy to see that $\eta^+ (\rho, u)$ is smooth on the variable $u$, so

$$|\eta^+ _{\rho \rho}| = |\theta^2 \rho^{\gamma - 3} \eta^+_{uu}| \leq M_1 \rho^{\gamma - 3}.$$
Since \( \eta^+_\rho = O(\rho^{-1}) \) as \( \rho \to 0 \), we have

\[
|\eta^+_{\rho\rho}\rho_x u_x| \leq M_2 \rho^{-1}|\rho_x u_x| \leq \frac{M_2}{2}(\rho^\gamma - \rho_x^2 + u_x^2),
\]

and hence

\[
|\varepsilon (\eta^+_{\rho\rho}\rho_x^2 + 2\eta^+_{\rho\rho}\rho_x u_x + \eta^+_u u_x^2)| \leq C(\varepsilon \rho^\gamma - \rho_x^2 + \varepsilon u_x^2).
\]

In view of the boundedness of \( \varepsilon \rho_x^2 \) and \( \varepsilon u_x^2 \) in \( L^1_{\text{loc}}(\xi) \), the part \( J_2 \) is bounded in \( L^1_{\text{loc}}(\xi) \) and hence compact in \( W^{-1,\alpha} \) for a constant \( \alpha \in (1, 2) \). The part \( \varepsilon \eta^+_u \) is compact in \( H^{-1} \), since

\[
|\eta^+_u| = |\eta^+_{\rho}\rho_x + \eta^+_u u_x| \leq C(\rho^{-1}|\rho_x| + |u_x|).
\]

Noticing the boundedness of \( \varepsilon^{1,\infty} \) in \( W^{-1,\infty} \), we get the compactness of \( \eta^+_\rho + q^+_x \) in \( H^{-1} \) by Murat’s Lemma (see [6,7]). A similar treatment gives the proof for \( \eta^- \). Since \( \eta^0(\rho, u, \rho) = \int_{-1}^1 (g(u + \rho s) + \theta \rho^2 g'(u + \rho s))(1 - s^2) \, ds \) (see [5]), we can easily get the compactness of \( \eta^0 + q^0 \) in \( H^{-1} \) by a similar treatment.

Finally, we use a new technique to reduce the Young measure. We apply the measure equation to obtain

\[
\int g(\xi_1)G_i(\xi_1) \, d\xi_1 \int h(\xi_2)[\theta\xi_2 + (1 - \theta)u]G_j(\xi_2) \, d\xi_2
\]

\[- \int h(\xi_2)G_j(\xi_2) \int h(\xi_1)[\theta\xi_1 + (1 - \theta)u]G_i(\xi_1) \, d\xi_1
\]

\[
= \int_{\mathcal{R}} g(\xi_1)h(\xi_2)G_i(\xi_1)[\theta\xi_2 + (1 - \theta)u]G_j(\xi_2) \, d\xi_1 \, d\xi_2
\]

\[- \int_{\mathcal{R}} g(\xi_1)h(\xi_2)G_j(\xi_1)[\theta\xi_1 + (1 - \theta)u]G_j(\xi_2) \, d\xi_1 \, d\xi_2,
\]

(7)

where \( G_i \) is any one of the three fundamental solutions. Here and below we use the overbar to indicate the usual integration with respect to the young measure; for instance \( \overline{G}(\xi) = \int_{\mathcal{R}} G(\rho, \xi - u) \, dv_{\xi, t}(\rho, u) \).

The equality (7) holds for any smooth functions \( g, h \) with compact support sets and this yields

\[
G_i(\xi_1)[\theta\xi_2 + (1 - \theta)u]G_j(\xi_2) - G_j(\xi_2)[\theta\xi_1 + (1 - \theta)u]G_i(\xi_1)
\]

\[
= G_i(\xi_1)[\theta\xi_2 + (1 - \theta)u]G_j(\xi_2) - G_i(\xi_1)[\theta\xi_1 + (1 - \theta)u]G_j(\xi_2)
\]

\[
= \theta(\xi_2 - \xi_1) \overline{G_i(\xi_1)G_j(\xi_2)}.
\]

(8)

Let

\[
\begin{align*}
    z_- &= \inf_{(\rho, u) \in \text{supp}_{\nu_{\xi, t}}} z(\rho, u), & z_+ &= \sup_{(\rho, u) \in \text{supp}_{\nu_{\xi, t}}} z(\rho, u), \\
    w_- &= \inf_{(\rho, u) \in \text{supp}_{\nu_{\xi, t}}} w(\rho, u), & w_+ &= \sup_{(\rho, u) \in \text{supp}_{\nu_{\xi, t}}} w(\rho, u).
\end{align*}
\]

If we choose \( G_i = G_j = G_+ \) and \( \xi_1, \xi_2 \in (w_-, +\infty) \), then we may rewrite (8) as

\[
\frac{\theta}{1 - \theta} \left[ \frac{G_+(\xi_1)G_+(\xi_2)}{G_+(\xi_1)G_+(\xi_2)} - 1 \right] = \frac{1}{\xi_2 - \xi_1} \left[ \frac{uG_+(\xi_2) - uG_+(\xi_1)}{G_+(\xi_2)} \right].
\]

(9)

Similarly, choosing \( G_i = G_j = G_- \) and \( \xi_1, \xi_2 \in (\infty, z_+) \), we have

\[
\frac{\theta}{1 - \theta} \left[ \frac{G_-(\xi_1)G_-(\xi_2)}{G_-(\xi_1)G_-(\xi_2)} - 1 \right] = \frac{1}{\xi_2 - \xi_1} \left[ \frac{uG_-(\xi_2) - uG_-(\xi_1)}{G_-(\xi_2)} \right].
\]

(10)
As was done in [3], let \( f_0^\pm(\xi) \) abbreviate \( f_0^\pm(\xi) = \frac{G_{\pm}-G_{\pm}(\xi)}{G_{\pm}(\xi)} \), so that (9) and (10) take the equivalent form

\[
\frac{\theta}{1-\theta} f_0^\pm(\xi_1) f_0^\pm(\xi_2) = \frac{1}{\xi_2 - \xi_1} \left[ \frac{u G_{\pm}(\xi_2)}{G_{\pm}(\xi_2)} - \frac{u G_{\pm}(\xi_1)}{G_{\pm}(\xi_1)} \right].
\]

Let \( I_x(\xi) \) be a nonnegative, smooth function with compact support set in \((-\frac{1}{\alpha}, \frac{1}{\alpha})\) and \( I_x(\xi) \to 1 \) as \( \alpha \to 0^+ \), \( \psi_\alpha(\xi) \geq 0 \) be a unit mass mollifier, and define \( f_\alpha^\pm = (f_0^\pm I_\alpha) \ast \psi_\alpha \). Then we have from (11) that

\[
\frac{\theta}{1-\theta} f_\alpha^\pm(\xi_1) f_\alpha^\pm(\xi_2) = \frac{1}{\xi_2 - \xi_1} \left[ \frac{u G_{\pm}(\xi_2)}{G_{\pm}(\xi_2)} - \frac{u G_{\pm}(\xi_1)}{G_{\pm}(\xi_1)} \right] I_\alpha(\xi_1) I_\alpha(\xi_2) \ast \psi_\alpha(\xi_1) \ast \psi_\alpha(\xi_2).
\]

Thanks to the boundedness of the left-hand side and the smoothness of the right-hand side, we may now take \( \xi_2 = \xi_1 = \xi \) to find out that

\[
\frac{\theta}{1-\theta} (f_\alpha^\pm(\xi))^2 = \frac{1}{\xi_2 - \xi_1} \left[ \frac{u G_{\pm}(\xi_2)}{G_{\pm}(\xi_2)} - \frac{u G_{\pm}(\xi_1)}{G_{\pm}(\xi_1)} \right] I_\alpha(\xi_1) I_\alpha(\xi_2) \ast \psi_\alpha(\xi_1) \ast \psi_\alpha(\xi_2)|_{\xi_2=\xi_1=\xi}.
\]

If we now let \( \alpha \to 0^+ \), then the left-hand side of (12) yields a positive measure, since \( 0 < \theta < 1 \), whereas the right-hand side tends to \( \frac{\partial}{\partial \xi} \frac{u G_{\pm}(\xi)}{G_{\pm}(\xi)} \). Therefore \( \frac{u G_{+}(\xi)}{G_{+}(\xi)} \) and \( \frac{u G_{-}(\xi)}{G_{-}(\xi)} \) are nondecreasing respectively in \((w_-, \infty)\) and \((-\infty, z_+)\). By the same treatment, we have that \( \frac{u G_{0}(\xi)}{G_{0}(\xi)} \) is nondecreasing in \((z_-, w_+)\).

**Case I: \( z_+ \leq w_- \).**

If \( z_+ \leq w_- \), then we choose \( G_i = G_+ \), \( G_j = G_0 \) and \( \xi_2 = \xi_1 = \xi \) in (8) to obtain \( \frac{u G_{+}(\xi)}{G_{+}(\xi)} \to \frac{u G_{0}(\xi)}{G_{0}(\xi)} \). Hence

\[
\frac{u G_{+}(\xi)}{G_{+}(\xi)} = \frac{u G_{0}(\xi)}{G_{0}(\xi)}
\]

for \( \xi \in (w_-, w_+) \). In particular,

\[
\lim_{\xi \to w_-^+} \frac{u G_{+}(\xi)}{G_{+}(\xi)} = \frac{u G_{0}(w_-)}{G_{0}(w_-)}.
\]

Similarly, if choosing \( G_i = G_- \), \( G_j = G_0 \) and \( \xi_1 = \xi_2 = \xi \), we have

\[
\frac{u G_{-}(\xi)}{G_{-}(\xi)} = \frac{u G_{0}(\xi)}{G_{0}(\xi)}
\]

for \( \xi \in (z_-, z_+) \). In particular,

\[
\lim_{\xi \to z_-^0} \frac{u G_{-}(\xi)}{G_{-}(\xi)} = \frac{u G_{0}(z_+)}{G_{0}(z_+)}.
\]

Therefore,

\[
\overline{u} = \lim_{\xi \to \infty} \frac{u G_{+}(\xi)}{G_{+}(\xi)} \geq \lim_{\xi \to w_-^+} \frac{u G_{+}(\xi)}{G_{+}(\xi)} = \frac{u G_{0}(w_-)}{G_{0}(w_-)} \geq \frac{u G_{0}(z_+)}{G_{0}(z_+)} \geq \lim_{\xi \to z_-^0} \frac{u G_{-}(\xi)}{G_{-}(\xi)} \geq \lim_{\xi \to -\infty} \frac{u G_{-}(\xi)}{G_{-}(\xi)} = \underline{u}
\]

and hence \( \frac{u G_{+}(\xi)}{G_{+}(\xi)} \), \( \frac{u G_{-}(\xi)}{G_{-}(\xi)} \) are constant respectively in \((w_-, \infty)\) and \((-\infty, z_+)\) by the monotonicity of the two functions.

Using the equality (12), we have \((f_\alpha^\pm(\xi))^2 = 0\). Hence \( f_\alpha^\pm(\xi) \) vanishes on the support of \( v \) and, in particular, by letting \( \alpha \to 0 \), so does \( f_0^\pm(\xi) \).

\[
f_0^\pm(\xi) = \frac{G(\rho, u - \xi)}{G(\xi)} - 1 = 0, \quad (\rho, u) \in \text{supp } v.
\]
This shows that the Young measure is reduced to a Dirac mass.

Case II: \( z_+ > w_- \).

If \( z_+ > w_- \), then we choose \( G_i = G_+ \), \( G_j = G_- \) and \( \xi_2 = \xi_1 = \xi \) in (8) to obtain \( u_{G_+}(\xi) \, G_+(\xi) = u_{G_-}(\xi) \, G_-(\xi) \).

Hence
\[
\frac{u_{G_+}(\xi)}{G_+(\xi)} = \frac{u_{G_-}(\xi)}{G_-(\xi)}
\]
for \( \xi \in (w_-, z_+) \). In particular,
\[
\lim_{\xi \to w_- + 0} \frac{u_{G_+}(\xi)}{G_+(\xi)} = \frac{u_{G_-}(w_-)}{G_-(w_-)}, \quad \lim_{\xi \to z_+ - 0} \frac{u_{G_-}(\xi)}{G_-(\xi)} = \frac{u_{G_+}(z_+)}{G_+(z_+)}.\]

Therefore,
\[
\bar{u} = \lim_{\xi \to \infty} \frac{u_{G_+}(\xi)}{G_+(\xi)} \geq \lim_{\xi \to z_+ - 0} \frac{u_{G_-}(\xi)}{G_-(\xi)} \geq \lim_{\xi \to -\infty} \frac{u_{G_-}(\xi)}{G_-(\xi)} = \bar{u}
\]
and hence \( \frac{u_{G_+}(\xi)}{G_+(\xi)}, \frac{u_{G_-}(\xi)}{G_-(\xi)} \) are constant respectively in \((w_-, \infty)\) and \((-\infty, z_+)\) by the monotonicity of the two functions. Hence the Young measure \( \nu \) is also a Dirac mass from the proof in Case I. This is contrary to the assumption \( z_+ > w_- \) since \( w_- \geq z \). Thus only Case I, i.e., \( z_+ \leq w_- \), is permitted, and \( \nu \) is a Dirac mass. According to the compensated compactness method (see [8]), \((\rho(x, t), u(x, t))\) is a global entropy solution of system (1). So we end the proof of Theorem 1.

References